DEMONSTRATIO MATHEMATICA
Vol. XXXIII No 4 2000

Mohamed Akkouchi

COMMON FIXED POINT THEOREMS BY ALTERING THE DISTANCES BETWEEN THE POINTS IN BOUNDED COMPLETE METRIC SPACES

Abstract. The aim of this note is to prove some common fixed point theorems in bounded complete metric spaces for self-maps verifying generalized contractive conditions obtained by altering the distances between the points.

1. Introduction

In the paper [4], the authors gave some fixed point theorems generalizing and unifying many fixed point theorems obtained by Delbosco in [1], Skof in [8], Rakotch in [5], Reich in [7], and Fisher in [3]. Precisely in [4] the following theorem was established:

1.1. THEOREM. *Let T be a self-map of a complete metric space (X, d) and let ϕ be a function verifying:*

(P_1) $\phi : [0, \infty[\to [0, \infty[$ *is continuous and increasing in* $[0, \infty[$, *and*

(P_2) $\phi(t) = 0 \iff t = 0$.

We suppose that T satisfies the following condition:

$$(K)\quad \phi(d(Tx, Ty)) \leq a(d(x, y))\phi(d(x, y)) + b(d(x, y))[\phi(d(x, Tx)) + \phi(d(y, Ty))] + c(d(x, y)) \min\{\phi(d(x, Ty)), \phi(d(y, Tx))\} \quad \forall x, y \in X \text{ with } x \neq y,$$

where a, b, c are three decreasing functions from $]0, \infty[$ into $[0, 1[$ such that $a(t) + 2b(t) + c(t) < 1$, for every $t > 0$. Then T has a unique point.

1991 *Mathematics Subject Classification*: 47H10.

Key words and phrases: common fixed point theorem in bounded complete metric spaces, generalized contractive conditions.

In this note we work in a bounded complete metric space (X,d). In the second section we prove some common fixed point theorems for sets of self-mappings verifying contractive conditions close to the relation (K) but using only continuous functions ϕ satisfying (P_2). Thus, in this case the assumption "ϕ is increasing" becomes superfluous and can be removed. Furthermore, our main theorem is an improvement upon some other fixed point theorems (see [3], [5], [6], [7]). In section 3, we establish a common fixed point theorem in compact metric spaces generalizing Theorem 4 of the paper [4] which itself was considered by the authors as a generalization of a theorem given by B. Fisher in [3].

2. Main result

Many authors (see the references) were interested by fixed point theorems by altering the distances between the points with the use of functions. The purpose of this section is to contribute in this field of investigations. We start by our first main result.

2.1. THEOREM. *Let (X,d) be a bounded complete metric space. Let $\phi : [0,\infty[\to [0,\infty[$ be a continuous function verifying property (P_2). Let S,T be two self-maps of (X,d) such that*

$$(A) \qquad \begin{aligned}\phi(d(Sx,Ty)) \le\, & a(d(x,y))\phi(d(x,y)) + b(d(x,y))\phi(d(x,Sx)) \\ & + c(d(x,y))\phi(d(y,Ty)) \\ & + e(d(x,y))\min\{\phi(d(x,Ty)),\phi(d(y,Sx))\} \\ & \qquad \forall x,y \in X \text{ with } x \neq y,\end{aligned}$$

where a,b,c,e are four decreasing functions from $]0,+\infty[$ into $[0,1[$ such that $a(t)+\gamma[b(t)+c(t)]+e(t) < 1$, for every $t > 0$, where γ is a fixed constant in $]1,+\infty[$. Then S and T have a unique common fixed point $z \in X$. Moreover $Fix(S) = Fix(T) = \{z\}$.

Proof. (I) We shall prove that the pair $\{S,T\}$ has a common fixed point. Let x_0 be some point in X. We define

$$x_{2n} = Sx_{2n-1}, \quad n = 1,2,\dots$$
$$x_{2n+1} = Tx_{2n}, \quad n = 0,1,2,\dots$$

We put $t_n := d(x_n, x_{n+1})$ for all integer n. (I) is proved if $t_{n_0} = 0$ for some integer n_0. Therefore, we may assume that $t_n > 0$ for all integer n. We see that for an even integer n, we have

$$\phi(t_n) = \phi(d(Sx_{n-1}, Tx_n)) \le \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4,$$

where

$$\Psi_1 = a(d(x_{n-1}, x_n))\phi(d(x_{n-1}, x_n)),$$
$$\Psi_2 = b(d(x_{n-1}, x_n))\phi(d(x_{n-1}, Sx_{n-1})),$$
$$\Psi_3 = c(d(x_{n-1}, x_n))\phi(d(x_n, Tx_n)),$$
$$\Psi_4 = e(d(x_{n-1}, x_n))\phi(d(x_n, Sx_{n-1})).$$

So

$$\phi(t_n) \leq a(t_{n-1})\phi(t_{n-1}) + b(t_{n-1})\phi(t_{n-1}) + c(t_{n-1})\phi(t_n).$$

Hence we get

$$\phi(t_n) \leq \frac{a(t_{n-1}) + b(t_{n-1})}{1 - c(t_{n-1})}\phi(t_{n-1}) < \phi(t_{n-1}). \tag{1}$$

In a similar manner, one can prove, (for the same even integer) that

$$\phi(t_{n-1}) \leq \frac{a(t_{n-2}) + c(t_{n-2})}{1 - b(t_{n-2})}\phi(t_{n-2}) < \phi(t_{n-2}). \tag{1'}$$

The inequalities (1) and (1') show that the sequence $(\phi(t_n))_n$ is decreasing. Let θ be the limit of $(\phi(t_n))_n$. Let us suppose that $\theta > 0$. Since (X, d) is bounded there exists a subsequence $(t_{n(k)})_k$ converging to some element t. By the continuity of ϕ, we have $\phi(t) = \theta > 0$. By the property (P$_2$) of ϕ, we must have $t > 0$. In this case, since a, b, c are decreasing on $]0, \infty[$, then by using (1) and (1'), we get

$$\phi(t_{n(k)}) \leq \max\left\{\frac{a(t) + b(t)}{1 - c(t)}, \frac{a(t) + c(t)}{1 - b(t)}\right\}\phi(t_{n(k)-1}).$$

Now, by letting $k \to \infty$, and using the continuity of ϕ, we obtain

$$\phi(t) \leq \max\left\{\frac{a(t) + b(t)}{1 - c(t)}, \frac{a(t) + c(t)}{1 - b(t)}\right\}\phi(t) < \phi(t),$$

which is a contradiction. Hence $t = \theta = 0$. Now, from the considerations made above, we may deduce that the whole sequence $(t_n)_n$ is converging to zero.

(II) Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. Since $t = 0$ one needs only to see that $\{x_{2n}\}$ is a Cauchy sequence. To get a contradiction, let us suppose that there is a number $\epsilon > 0$ and two sequences $\{2n(k)\}$, $\{2m(k)\}$ with $2k \leq 2m(k) < 2n(k)$, $(k \in \mathbb{N})$ verifying

$$d(x_{2n(k)}, x_{2m(k)}) > \epsilon. \tag{2}$$

For each integer k, we shall denote $2n(k)$ the least even integer exceeding $2m(k)$ for which (2) holds. Then

$$d(x_{2m(k)}, x_{2n(k)-2}) \leq \epsilon \quad \text{and} \quad d(x_{2m(k)}, x_{2n(k)}) > \epsilon.$$

For each integer k, we shall put

$$p_k := d(x_{2m(k)}, x_{2n(k)}), \qquad s_k := d(x_{2m(k)}, x_{2n(k)+1}),$$
$$q_k := d(x_{2m(k)+1}, x_{2n(k)+1}), \quad \text{and} \quad r_k := d(x_{2m(k)+1}, x_{2n(k)+2}),$$

then by using triangular inequalities, we obtain

$$(3) \qquad \begin{aligned} \epsilon < p_k &\leq \epsilon + t_{2n(k)-2} + t_{2n(k)-1} \\ |s_k - p_k| &\leq t_{2n(k)}, \\ |q_k - s_k| &\leq t_{2m(k)}, \\ |r_k - s_k| &\leq t_{2n(k)+1}. \end{aligned}$$

Since the sequence $\{t_n\}$ converges to 0, we deduce from (3) that the sequences $\{p_k\}, \{s_k\}, \{q_k\}$ and $\{r_k\}$ are converging to ϵ. From (2) and these facts, one can deduce that there exists an integer k_0 such that $d(x_{2n(k)+1}, x_{2m(k)}) > 0$, and $\frac{\epsilon}{2} \leq p_k - t_{2k} \leq d(x_{2n(k)+1}, x_{2m(k)})$, for each integer $k \geq k_0$. Therefore, (for all $k \geq k_0$) we have

$$\begin{aligned} \phi(r_k) = \phi(d(x_{2n(k)+2}, x_{2m(k)+1})) &= \phi(d(Sx_{2n(k)+1}, Tx_{2m(k)})) \\ &\leq \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \leq G_1 + G_2 + G_3 + G_4, \end{aligned}$$

where

$$\begin{aligned} \Gamma_1 &= a(d(x_{2n(k)+1}, x_{2m(k)}))\phi(d(x_{2n(k)+1}, x_{2m(k)})), \\ \Gamma_2 &= b(d(x_{2n(k)+1}, x_{2m(k)}))\phi(d(x_{2n(k)+1}, x_{2n(k)+2})), \\ \Gamma_3 &= c(d(x_{2n(k)+1}, x_{2m(k)}))\phi(d(x_{2m(k)}, x_{2m(k)+1})), \\ \Gamma_4 &= e(d(x_{2n(k)+1}, x_{2m(k)})) \\ &\quad \times \min\left\{\phi(d(x_{2n(k)+1}, x_{2m(k)+1})), \phi(d(x_{2m(k)}, x_{2n(k)+2}))\right\} \end{aligned}$$

and

$$\begin{aligned} G_1 &= a(p_k - t_{2n(k)})\phi(s_k); \\ G_2 &= \phi(t_{2n(k)+1}); \\ G_3 &= \phi(t_{2m(k)}); \\ G_4 &= e(p_k - t_{2n(k)})\phi(q_k). \end{aligned}$$

We let $k \to \infty$, then by using the continuity of ϕ and the fact that a, b, c, e are decreasing on $]0, +\infty[$, we obtain

$$\phi(\epsilon) \leq \left[a\left(\frac{\epsilon}{2}\right) + e\left(\frac{\epsilon}{2}\right)\right] < \phi(\epsilon),$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence in the complete metric space (X, d), then one may find a point $z = z(S, T) \in X$ such that $x_n \longrightarrow z$ as $n \longrightarrow \infty$. Next, we shall prove that z is a common fixed point for S and T.

(III) Since $t_n > 0$ for all integer n, we see that both subseqences $(x_{2n})_n$ and $(x_{2n+1})_n$ are not stationary. Therefore, we may find a subsequence $(x_{n(k)})_k$ such that $x_{2n(k)+1} \neq z$ for every integer k. Let us suppose that $Tz \neq z$. In this case we are allowed to apply the inequality (A) and obtain, for all $k \in \mathbb{N}$, the relations

$$\begin{aligned} \phi(d(x_{2n(k)+2}, Tz)) &= \phi(d(Sx_{2n(k)+1}, Tz)) \\ &\leq a(d(x_{2n(k)+1}, z))\phi(d(x_{2n(k)+1}, z)) \\ &\quad + b(d(x_{2n(k)+1}, z))\phi(t_{2n(k)+1}) \\ &\quad + c(d(x_{2n(k)+1}, z))\phi(d(z, Tz)) \\ &\quad + e(d(x_{2n(k)+1}, z))\phi(d(x_{2n(k)+2}, z). \end{aligned} \tag{4}$$

We deduce then that

$$\begin{aligned} \phi(d(x_{2n(k)+2}, Tz)) \leq \phi(d(x_{2n(k)+1}, z)) \\ + \phi(t_{2n(k)+1}) + \phi(d(x_{2n(k)+2}, z) + \frac{1}{\gamma}\phi(d(z, Tz)). \end{aligned} \tag{5}$$

After letting $k \longrightarrow \infty$, it gives

$$\phi(d(z, Tz)) \leq \frac{1}{\gamma}\phi(d(z, Tz)) < \phi(d(z, Tz)), \tag{6}$$

which is a contradiction. Hence $z = Tz$, and in a similar way, it can be shown that $z = Sz$.

(IV) Suppose that there exists another point $\xi \neq z$ fixed, for instance, by S. Then by the property (A), we have

$$\begin{aligned} \phi(d(\xi, z)) &= \phi(d(S\xi, Tz)) \\ &\leq [a(d(\xi, z)) + e(d(\xi, z))]\, \phi(d(\xi, z)) < \phi(d(\xi, z)). \end{aligned}$$

a contradiction. Therefore, we deduce that there exists a unique point $z \in X$ such that $Fix(S) = \{z\} = Fix(T) = Fix(\{S, T\})$. This completes the proof of our theorem. ■

The following result is an easy consequence of our Theorem 2.1.

2.2. THEOREM. *Let (X, d) a bounded complete metric space, $\mathcal{A}$ a (finite or infinite) set of self-maps of X and ϕ as in Theorem 2.1. We suppose that for all $S, T \in \mathcal{A}$ the following generalized contractive condition holds true*

$$\begin{aligned} \phi(d(Sx, Ty)) \leq\ & a(d(x, y))\phi(d(x, y)) \\ & + b(d(x, y)) \min\{\phi(d(x, Sx)), \phi(d(y, Ty))\} \\ & + c(d(x, y)) \min\{\phi(d(x, Ty)), \phi(d(y, Sx))\} \\ & \qquad \forall x, y \in X \ \textit{with} \ x \neq y, \end{aligned} \tag{B}$$

where a, b, c are three decreasing functions from $]0, +\infty[$ into $[0, 1[$ such that $a(t)+\gamma b(t)+c(t) < 1$, for every $t > 0$, where γ is a fixed constant in $]1, +\infty[$. Then

(i) *there exists a unique point $z \in X$ such that $z \in Fix(S)$ for all $S \in \mathcal{A}$.*

(ii) *$Fix(S) = \{z\}$ for all $S \in \mathcal{A}$.*

(iii) *If $\mathcal{A}$ contains more than two elements, then* (i) *and* (ii) *are still valid whenever (B) holds true only for all $S, T \in \mathcal{A}$, with $S \neq T$.*

2.3. REMARKS.

(a) We point out that Theorems 2.1 and 2.2 are still valid when (X, d) is not bounded under the assumptions (P_1) and (P_2) (of 1.1) upon the function ϕ.

(b) If we take $b = c$ in 2.1, then we obtain the result established by R. A. Rashwan and A. M. Sadeek in [6].

(c) If we take $b = c$ and $S = T$ in 2.1, then we obtain one of the main results established by M. S. Khan et al. in the paper [4].

(d) If $e = 0$, $S = T$ and the functions a, b and c are constants then we get the results obtained by D. Delbosco in [1] and by F. Skof in the paper [8].

(e) We give here an example discussing the validity of the assumptions of Theorem 2.1. We take $X = \{1, 2, 3, 4\}$ and define a metric d on X by setting $d(1, 2) = 1$, and $d(1, 3) = d(1, 4) = d(2, 3) = d(2, 4) = d(3, 4) = 2$. We consider the maps S and T defined by, $S1 = S2 = S3 = 1$, $S4 = 2$; and $T1 = T2 = T4 = 1$, $T3 = 2$. For all $t \geq 0$, we put $a(t) = 2/5$, $b(t) = 1/20$, $c(t) = 7/20$, $e(t) = 1/6$, and $\phi(t) = t^2$. Then all the conditions of Theorem 2.1 are satisfied and the maps S and T have 1 as unique common fixed point.

3. A fixed point theorem in compact metric spaces

The purpose of this section is to generalize Theorem 4 of the paper [4] by the following

3.1. THEOREM. *Let S, T be two self-maps of a compact metric space (X, d) and let $\phi : [0, \infty[\to [0, \infty[$ be a continuous function verifying property (P_2). We suppose that $T, S \circ T$ are continuous and that S, T verify for all distinct x, y in X the inequality*

$$(C) \quad \phi(d(Sx, Ty)) < [1 - b - c]\phi(d(x, y)) + b\phi(d(Sx, x)) + c\phi(d(y, Ty))$$

where $b, c \in [0, 1]$ are two fixed constants such that $b + c \leq 1$. Then S and T have a unique common fixed point $z \in X$. Moreover $Fix(S) = Fix(T) = \{z\}$.

Proof. Let x_0 be an element in X, and associate to it the sequence $(x_n)_n$ given by

$$x_{2n} = Sx_{2n-1}, \quad n = 1, 2, \ldots$$
$$x_{2n+1} = Tx_{2n}, \quad n = 0, 1, 2, \ldots.$$

Without loss of generality, we may assume that $t_n \neq 0$ for every integer n. In this case, it is easy to see that the sequence $(\phi(t_n))$ is decreasing and therefore it converges. Since X is compact, we may find a subsequence $(x_{2n(k)})_k$ converging to some element $z \in X$. Then by using the continuity of the map T and the function ϕ, we get

$$\begin{aligned} \phi(d(z, Tz)) &= \lim_{k \to +\infty} \phi(t_{2n(k)}) = \lim_{k \to +\infty} \phi(t_{2n(k)+1}) \\ &= \lim_{k \to +\infty} \phi(d(x_{2n(k)+1}, x_{2n(k)+2})) \\ &= \lim_{k \to +\infty} \phi(d(Tx_{2n(k)}, (S \circ T)x_{2n(k)})) \\ &= \phi(d(Tz, (S \circ T)z)). \end{aligned} \tag{7}$$

Suppose that $z \neq Tz$, then by applying the inequality (C) to $x = Tz$ and $y = z$, and using (7), we obtain

$$\begin{aligned} \phi(d(z, Tz)) &= \phi(d(S(Tz), Tz)) \\ &< [1 - b - c]\phi(d(Tz, z)) + b\phi(d(Tz, STz)) + c\phi(d(z, Tz)) \\ &\leq \phi(d(Tz, z)). \end{aligned} \tag{8}$$

This is a contradiction. Therefore we must have $Tz = z$. The relation (7) will imply that $Sz = z$. To end the proof, let us suppose that there exists a point $\xi \neq z$ fixed, for instance, by S. Then by applying the inequality (C), we get

$$\phi(d(\xi, z)) = \phi(d(S\xi, Tz)) < [1 - b - c]\phi(d(\xi, z)) \leq \phi(d(\xi, z)), \tag{9}$$

which is a contradiction. ∎

References

[1] D. Delbosco, *Un estensione di un teorema sul punto fisso di S. Reich*, Rend. Sem. Mat. Univers. Politec. Torino, 35 (1976-77), 233–239.

[2] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. 37 (1962), 74–79.

[3] B. Fisher *A fixed point mapping*, Bull. Calcutta Math. Soc. 68 (1976), 265–266.

[4] M. S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distance between the points*, Bull. Austral. Math. Soc. 30 (1984), 1–9.

[5] E. Rakotch, *A note on contractive mappings*, Proc. Amer. Math. Soc. 13 (1962), 459–465.
[6] R. A. Rashwan and A. M. Sadeek, *A common fixed point theorem in complete metric spaces*, Electronic Journal: Southwest. J. Pure Appl. Math. Vol. No. 01 (1996), 6–10.
[7] S. Reich, *Kannan's fixed point theorem*, Boll. U. M. I. (4) 4 (1971), 1–11.
[8] F. Skof, *Teorema di punti fisso per applicazioni negli spazi metrici*, Atti. Accad. Aci. Torino. 111 (1977), 323–329.

DÉPARTEMENT DE MATHÉMATIQUES
FACULTÉ DES SCIENCES-SEMLALIA
UNIVERSITÉ CADI AYYAD
Avenue du prince My. Abdellah. B.P. 2390
MARRAKECH, MAROC (MOROCCO)

Received November 29, 1999.