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Irina Ana Calinov

ON PLURIHARMONIC MORPHISMS

Abstract. In this paper we first remind some results regarding pluriharmonic morphisms. We define similar concepts between Kähler or Sasaki manifolds, i.e. maps which pull back local pluriharmonic or ϕ-pluriharmonic functions to local pluriharmonic or ϕ-pluriharmonic functions.

1. Introduction

Harmonic morphisms between Riemannian manifolds are maps which pull back local harmonic functions to local harmonic functions. In his work, [7] E. Loubeau generalises the idea of harmonic morphisms on complex manifolds to maps which pull back pluriharmonic functions to pluriharmonic functions. These mappings are called pluriharmonic morphisms and are exactly $\pm$holomorphic maps. We find similar results on Kähler or Sasaki manifolds, considering maps which pull back local pluriharmonic or ϕ-pluriharmonic functions to local pluriharmonic or ϕ-pluriharmonic functions. The main results of this paper are pointed in the following table:

M	N	$f : M \to N$		$f : M \to N$	Remark
complex	complex	pluri-harmonic morphism	$\Leftrightarrow$	$\pm$ holomorphic	If M, N are Hermitian, the following conditions are equivalent: (1) all f $\pm$holomorphic are pluriharmonic (2) N is Kähler
Sasaki	Kähler	(ϕ, J)-pluri-harmonic morphism	$\Leftrightarrow$	constant	If a (ϕ, J)-holomorphic map is ϕ-pluriharmonic then it is constant
Kähler	Sasaki	(J, ϕ)-pluri-harmonic morphism	$\Leftrightarrow$	$\pm(J, \phi)$-holomorphic	$\pm(J, \phi)$-holomorphic $\Rightarrow$ pluriharmonic
Sasaki	Sasaki	ϕ-pluri-harmonic morphism	$\Leftrightarrow$	$\pm\phi$-holomorphic ϕ-pluriharmonic	If f is $\pm\phi$-holomorphic, then: ϕ-pluriharmonic $\Leftrightarrow$ isometric immersion

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DEFINITION 1.1. Let $f : M \to N$ be a smooth map between the almost Hermitian manifold (M, J, g) and the Riemannian manifold (N, h). We say that f is $(1,1)$-geodesic if its second fundamental form ∇df satisfies:

$$(\nabla df)(X,Y) + (\nabla df)(JX, JY) = 0, \tag{1.1}$$

for any X and Y vector fields in M.

REMARK 1.1. Any $(1,1)$-geodesic map is harmonic [4].

Indeed, consider $\{e_1, J(e_1), \ldots, e_n, J(e_n)\}$ a local orthonormal frame of TM adapted to the almost Hermitian structure of M. We have then the tension field of f:

$$\tau(f) = \sum_{i=1}^{n} (\nabla df)(e_i, e_i) + \sum_{i=1}^{n} (\nabla df)(J(e_i), J(e_i)) = 0$$

since f is $(1,1)$-geodesic.

If we suppose that M is a Hermitian manifold with complex coordinates $\{y^1, y^{\overline{1}}, \ldots, y^n, y^{\overline{n}}\}$, then if f is (1,1)-geodesic it satisfies locally:

$$\frac{\partial^2 f^A}{\partial y^i \partial y^{\overline{j}}} - {}^M\Gamma^K_{i\overline{j}} \frac{\partial f^A}{\partial y^K} + {}^N\Gamma^A_{BC} \frac{\partial f^B}{\partial y^i} \frac{\partial f^C}{\partial y^{\overline{j}}} = 0 \tag{1.2}$$

with $A, B, C = 1, \ldots, \dim N$, $i, j = 1, \ldots, n$ and $K = 1, \overline{1}, \ldots, n, \overline{n}$.

If we suppose M to be a Kähler manifold, then (1.2) becomes:

$$\frac{\partial^2 f^A}{\partial y^i \partial y^{\overline{j}}} + {}^N\Gamma^A_{BC} \frac{\partial f^B}{\partial y^i} \frac{\partial f^C}{\partial y^{\overline{j}}} = 0 \tag{1.3}$$

with $A, B, C = 1, \ldots, dim N$, $i, j = 1, \ldots, n$. Let $d'f = \frac{\partial f}{\partial y^i} dy^i$ and $d"f = \frac{\partial f}{\partial y^{\overline{j}}} dy^{\overline{j}}$.

But (1.3) is equivalent with [10]:

$$(\nabla^{(0,1)} d'f)(\overline{Z}, W) = 0$$

for any $Z, W \in T^{(1,0)}M$, where:

$$(\nabla^{(0,1)} d'f)(\overline{Z}, W) = \nabla_{\overline{Z}}(d'f(W)) - d'f(\nabla_{\overline{Z}} W)$$

which means that f is a pluriharmonic map.

Hence, when M is Kähler, the notions of $(1,1)$-geodesic map and pluriharmonic map coincide.

DEFINITION 1.2. Let $f : M \to N$ be a smooth map between complex manifolds M and N. Then f is called pluriharmonic morphism if it pulls back local pluriharmonic functions to local pluriharmonic functions, i.e. for any open set $U \subset N$, such that $f^{-1}(U)$ not empty, and any $a : U \to \mathbb{C}$ pluriharmonic function, we have that $a \circ f$ is a pluriharmonic function.

We have [7]:

THEOREM 1.1. *Let $f : M \to N$ be a smooth map between complex manifolds M and N. Then f is a pluriharmonic morphism if and only if f is $\pm$holomorphic.*

An immediate proof of this theorem yields from the following three lemmas (due to some discussions with Radu Pantilie). Consider D be a simply connected domain in $\mathbb{C}^n$.

LEMMA 1.1. *$f : D \to \mathbb{C}$ is pluriharmonic if and only if it is the sum of a holomorphic function g and an antiholomorphic function h.*

Proof. We have:

$$d(d'f) = (d' + d")(d'f) = (d')^2 f + d"d'f = 0.$$

Hence there exists $g : D \to \mathbb{C}$ such that $d'f = dg$. Moreover:

$$d'(f - g) = d"g = 0,$$

which means that g is holomorphic and $f - g = h$ is antiholomorphic. ∎

LEMMA 1.2. *If $f : D \to \mathbb{C}$ is holomorphic, $g : D \to \mathbb{C}$ is antiholomorphic, and $f \cdot g$ is pluriharmonic, then f is constant or g is constant.*

Proof. We have

$$0 = (d'd")(f \cdot g) = d'(fd"g) = d'f \wedge d"g + f \wedge d'd"g =$$
$$= d'f \wedge d"g - f \wedge d"d'g = d'f \wedge d"g.$$

It follows that $D = \{Z|(df)_Z = 0\} \cup \{Z|(dg)_Z = 0\}$ and hence at least one of the two sets has the interior not empty. Since f and g are $\pm$holomorphic, at least one of them is constant. ∎

LEMMA 1.3. *Let $f : D \to \mathbb{C}^n$ be a pluriharmonic morphism. Then f is $\pm$holomorphic.*

Proof. Induction by n:

Suppose first $n = 1$. If f is a pluriharmonic morphism, then f is a pluriharmonic map and, by Lemma 1.1, $f = g + h$, with g holomorphic and h antiholomorphic. Moreover, since f is a pluriharmonic morphism, then $f^2 = g^2 + 2\,g \cdot h + h^2$ is pluriharmonic, and then $g \cdot h$ is pluriharmonic. It follows, by Lemma 1.2 that g is constant or h is constant, and hence f is $\pm$holomorphic.

If $(f^1, \dots, f^{n+1})$ is a pluriharmonic morphism, then $(f^1, \dots, f^n)$ and f^{n+1} are $\pm$ holomorphic. Since $f^1 \cdot f^{n+1}$ is pluriharmonic and $\pm$holomorphic, it yields, by Lemma 1.2, that either both are holomorphic or both are antiholomorphic. It yields that $f^1, f^2, \dots, f^{n+1}$ are all holomorphic or all antiholomorphic. ∎

REMARK 1.2. Any pluriharmonic morphism between Kähler manifolds is a pluriharmonic map. Indeed, we know [7] that any ±holomorphic map between the Hermitian manifolds M and N is pluriharmonic if and only if the target N is a Kähler manifold. Then, using Theorem 1.1, the statement is proved.

2. Similar results on Kähler or Sasaki manifolds

DEFINITION 2.1. Let $(M, \varphi, \eta, \xi, g)$ be an almost contact manifold and (N, h) a Riemannian manifold. A smooth map $f : M \to N$ is said to be $\varphi - (1,1)$-geodesic if its second fundamental form ∇df satisfies:

$$(\nabla df)(X, Y) + (\nabla df)(\varphi X, \varphi Y) = 0 \tag{2.4}$$

for X, Y vector fields on M.

If M is Sasaki, then we also call f satisfying (2.1) a φ-pluriharmonic map. This second terminology is used to emphasize the analogy with $(1,1)$-geodesic maps $f : M \to N$ (M almost Hermitian manifold), which are pluriharmonic when M is Kähler.

DEFINITION 2.2. Let $f : M \to N$ be a smooth map between the Sasaki manifold $(M, \varphi, \eta, \xi, g)$ and the Kähler manifold (N, J, h). Then f is called (φ, J)-pluriharmonic morphism if it pulls back local pluriharmonic functions on N to local φ-pluriharmonic functions on M. It means that, if a is a pluriharmonic function defined on the open set $V \subset N$, with $f^{-1}(V)$ not empty, then $a \circ f$ is a φ-pluriharmonic function on $f^{-1}(V)$.

DEFINITION 2.3. A smooth map $f : M \to N$ between the almost contact metric manifold $(M, \varphi, \eta, \xi, g)$ and the almost Hermitian manifold (N, J, h) is called (φ, J)-holomorphic (or $+(\varphi, J)$-holomorphic) if $df \circ \varphi = J \circ df$, and it is called (φ, J)-antiholomorphic (or $-(\varphi, J)$-holomorphic) if $df \circ \varphi = -J \circ df$.

REMARK 2.1. If f is a $\pm(\varphi, J)$-holomorphic map as before, then

$$df(\xi) = 0.$$

We prove now:

THEOREM 2.1. *Let $f : M \to N$ be a smooth map between the Sasaki manifold $(M, \varphi, \eta, \xi, g)$ and the Kähler manifold (N, J, h). Then f is a (φ, J)-pluriharmonic morphism if and only if it is constant.*

Proof. Consider on the manifold $M \times \mathbb{R}$ the almost complex structure J' given by:

$$J'\Big(X, \alpha\frac{d}{dt}\Big) = \Big(\varphi X - \alpha\xi, \eta(X)\frac{d}{dt}\Big) \tag{2.5}$$

where X is a tangent vector field on M, t is the coordinate on $\mathbb{R}$ and α a C^∞ function on $\mathbb{R}$. Moreover, this is integrable, since we have assumed M to be Sasaki.

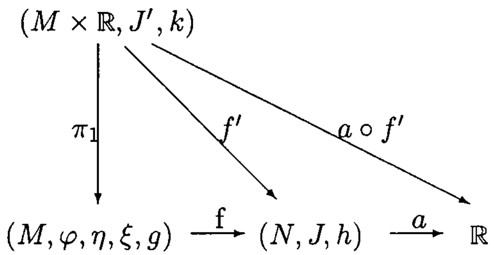
Let k be a Riemannian metric on $M \times \mathbb{R}$ given by:

$$k\Big(\Big(X, \alpha\frac{d}{dt}\Big), \Big(Y, \beta\frac{d}{dt}\Big)\Big) = g(X,Y) + \alpha\beta \tag{2.6}$$

for X, Y vector fields on M and α, β smooth functions on $\mathbb{R}$.

This way, we obtain on $M \times \mathbb{R}$ a Hermitian structure. Moreover, since M is Sasaki, then $(M \times \mathbb{R}, J', k)$ is Kähler. Denote by $\pi_1 : M \times \mathbb{R} \to M$ the canonical projection, and $f' : M \times \mathbb{R} \to N$ the composition $f' = f \circ \pi_1$.

Now, if $f : M \to N$ is a (φ, J)-pluriharmonic morphism, then for any local pluriharmonic function a on N, the composition $a \circ f$ is φ-pluriharmonic. Let $a' = a \circ f$.



We can see that since a' is φ-pluriharmonic, then $a' \circ \pi_1$ is pluriharmonic. Indeed, remind that [6] if a' is φ-pluriharmonic, then $(\nabla da')(\xi, \xi) = 0$ and $(\nabla da')(X, \xi) = 0$, for any X vector field on M. Hence, for X , Y vector fields on M and α, β smooth functions on $\mathbb{R}$ we have

$$\begin{aligned}
&(\nabla d(a' \circ \pi_1))\Big(\Big(X, \alpha\frac{d}{dt}\Big), \Big(Y, \beta\frac{d}{dt}\Big)\Big) + \\
&\qquad + (\nabla d(a' \circ \pi_1))\Big(J'\Big(X, \alpha\frac{d}{dt}\Big), J'\Big(Y, \beta\frac{d}{dt}\Big)\Big) \\
&= da'(\nabla d\pi_1)\Big(\Big(X, \alpha\frac{d}{dt}\Big), \Big(Y, \beta\frac{d}{dt}\Big)\Big) + \\
&\quad + \nabla da'\Big(d\pi_1\Big(X, \alpha\frac{d}{dt}\Big), \Big(d\pi_1\Big(Y, \beta\frac{d}{dt}\Big)\Big) + \\
&\quad + da'(\nabla d\pi_1)\Big(\Big(\varphi X - \alpha\xi, \eta(X)\frac{d}{dt}\Big), \Big(\varphi Y - \beta\xi, \eta(Y)\frac{d}{dt}\Big)\Big) + \\
&\quad + \nabla da'\Big(d\pi_1\Big(\varphi X - \alpha\xi, \eta(X)\frac{d}{dt}\Big), \Big(d\pi_1\Big(\varphi Y - \beta\xi, \eta(Y)\frac{d}{dt}\Big)\Big) = \\
&= (\nabla da')(X,Y) + (\nabla da')(\varphi X - \alpha\xi, \varphi Y - \beta\xi) =
\end{aligned}$$

$$= (\nabla da')(X,Y) + (\nabla da')(\varphi X, \varphi Y) - \alpha(\nabla da')(\xi, \varphi Y) -$$
$$- \beta(\nabla da')(\varphi X, Y) + \alpha\beta(\nabla da')(\xi,\xi) = 0$$

since a' is φ-pluriharmonic. Therefore φ-pluriharmonicity of a' implies pluriharmonicity of $a' \circ \pi_1$, for any local pluriharmonic function a on N. It means that $f \circ \pi_1 = f'$ is a pluriharmonic morphism on $M \times \mathbb{R}$, hence it is a $\pm$-holomorphic map (from Theorem 1.1).

Since f' is $\pm$holomorphic, we have $df' \circ J' = \pm J \circ df'$. Then, for any X vector field on M:

$$\begin{aligned}(df \circ \varphi)(X) &= (df \circ d\pi_1 \circ J')(X,0) = \\ &= (df' \circ J')(X,0) = \pm(J \circ df')(X,0) = \\ &= \pm(J \circ df \circ d\pi_1)(X,0) = \pm(J \circ df)(X)\end{aligned}$$

and therefore f is $\pm(\varphi, J)$-holomorphic.

Since f is a pluriharmonic morphism, for any local pluriharmonic map a on N, $a \circ f$ is φ-pluriharmonic, i.e.

$$\begin{aligned}&\nabla d(a \circ f)(X,Y) + \nabla d(a \circ f)(\varphi X, \varphi Y) = \\ &\quad = (\nabla da)(dfX, dfY) + (\nabla da)(df\varphi X, \varphi Y) + \\ &\qquad + da((\nabla df)(X,Y) + (\nabla df)(\varphi X, \varphi Y)) = 0,\end{aligned}$$

for any vector fields X and Y on M. But:

$$\begin{aligned}&(\nabla da)(dfX, dfY) + (\nabla da)(df\varphi X, \varphi Y) = \\ &\quad = (\nabla da)(dfX, dfY) + (\nabla da)(J \circ dfX, J \circ dfY) = 0\end{aligned}$$

since f is $\pm(\varphi, J)$-holomorphic and a is a pluriharmonic map.

Hence for any (local) pluriharmonic map a on N, we have:

$$da((\nabla df)(X,Y) + (\nabla df)(\varphi X, \varphi Y)) = 0,$$

for any vector fields X and Y on M. It yields [7] that

$$(\nabla df)(X,Y) + (\nabla df)(\varphi X, \varphi Y) = 0,$$

for any vector fields X and Y on M and therefore f is φ-pluriharmonic.

But if a (φ, J)-holomorphic map is φ-pluriharmonic then it is constant. Indeed, since f is φ-pluriharmonic, we have

$$(\nabla df)(X,Y) + (\nabla df)(\varphi X, \varphi Y)) = 0,$$

for any X, Y vector fields on M. Let's take $X \in D$, and $Y = \xi$ where D is the distribution orthogonal to ξ. Then $(\nabla df)(X, \xi) = 0$, i.e. $\nabla_X df\xi - df\nabla_X\xi = 0$. But f is (φ, J)-holomorphic, which means $J \circ df = df \circ \varphi$, and hence $J \circ df(\xi) = 0$ and finally $df(\xi) = 0$. Therefore we get $df(\nabla_X\xi) = 0$.

On the other hand, M being Sasaki, we have $\nabla_X\xi = -\varphi X$, hence $df(\varphi X) = 0$.

But f is (φ, J)-holomorphic, so it means $J \circ df(X) = 0$ or $df(X) = 0$, for any $X \in D$.

Also $df(\xi) = 0$, so we find that f must be constant. ■

REMARK 2.2. Let f be a smooth map from the Hermitian manifold M to the Hermitian manifold N, and P a normal almost contact metric manifold. Then $M \times P$ has a canonical normal almost contact metric structure [2]. Denote by $\pi : M \times P \to M$ the natural projection and let $f' : M \times P \to N$ be the composition $f' = f \circ \pi$. If f is $\pm$holomorphic, then f' is $\pm(\varphi, J)$-holomorphic. Suppose that M is Kähler and P is Sasaki. Hence $M \times P$ has a Sasaki structure in a natural way. We have [2] that f is a pluriharmonic map if and only if f' is a φ-pluriharmonic map. Suppose N is Kähler. Hence, if f is a pluriharmonic morphism, then f' is constant (see Theorem 1.1, Remark 1.2).

REMARK 2.3. A smooth map f between the Sasaki manifold $(M, \varphi, \eta, \xi, g)$ and a Kähler manifold (N, J, h) pulls back local pluriharmonic maps on N to local φ-pluriharmonic maps on M if and only if it is constant.

DEFINITION 2.4. Let $f : M \to N$ be a smooth map between the Kähler manifold (M, g, J) and the Sasaki manifold $(N, \varphi, \eta, \xi, h)$. We say that f is a (J, φ)-pluriharmonic morphism if it pulls back local φ-pluriharmonic functions on N to local pluriharmonic functions on M. It means that f is a (J, φ)-pluriharmonic morphism if and only if for any a real φ-pluriharmonic function defined on the open set $V \subset N$, with $f^{-1}(V)$ not empty, we have $a \circ f$ pluriharmonic function.

Consider on $N \times \mathbb{R}$ the Kähler structure given by (2.2) and (2.3), induced in the natural way by the Sasaki structure of $(N, \varphi, \eta, \xi, h)$. Let $a : N \to P$ be a smooth map, P any Riemannian manifold, and $b : N \times \mathbb{R} \to P$ given by $b(x, y) = a(x)$. Hence $b = a \circ \pi$, where $\pi : N \times \mathbb{R} \to N$ is the canonical projection.

LEMMA 2.1. *b is a pluriharmonic map if and only if a is a φ-pluri-harmonic map.*

Proof. We have, for any X and Y vector fields on N and α, β smooth functions on $\mathbb{R}$:

$$(\nabla db)\Big(\Big(X, \alpha\frac{d}{dt}\Big), \Big(Y, \beta\frac{d}{dt}\Big)\Big) =$$

$$= \nabla da\Big(d\pi\Big(X, \alpha\frac{d}{dt}\Big), d\pi\Big(Y, \beta\frac{d}{dt}\Big)\Big) + da((\nabla d\pi)\Big(\Big(X, \alpha\frac{d}{dt}\Big), \Big(Y, \beta\frac{d}{dt}\Big)\Big) =$$

$$= \nabla da(X, Y).$$

Hence:

$$(\nabla db)\Big(\Big(X, \alpha\frac{d}{dt}\Big), \Big(Y, \beta\frac{d}{dt}\Big)\Big) + (\nabla db)\Big(J\Big(X, \alpha\frac{d}{dt}\Big), J\Big(Y, \beta\frac{d}{dt}\Big)\Big) =$$
$$= \nabla da(X,Y) + \nabla da(\varphi X - \alpha\xi, \varphi Y - \beta\xi) =$$
$$= \nabla da(X,Y) + \nabla da(\varphi X, \varphi Y) -$$
$$-\alpha(\nabla da)(\xi, \varphi Y) - \beta(\nabla da)(\varphi X, \xi) + \alpha\beta(\nabla da)(\xi,\xi).$$

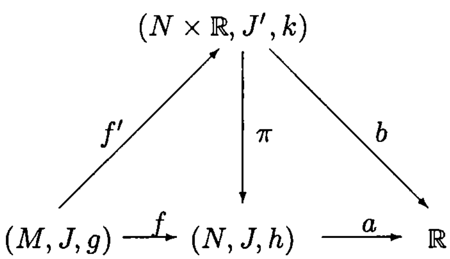
If a is φ-pluriharmonic, then $(\nabla da)(X,Y)+(\nabla da)(\varphi X, \varphi Y) = 0$, $(\nabla da)(\xi,\xi) = 0$ and $(\nabla da)(X,\xi) = 0$ for any X and Y vector fields on M. Therefore b is pluriharmonic.

If b is pluriharmonic, takink $\alpha = \beta = 0$ above, we obtain that a is a φ-pluriharmonic map. ∎

DEFINITION 2.5. A smooth map $f : M \to N$ between the almost Hermitian manifold (M, J, g) and the almost contact metric manifold $(N, \varphi, \eta, \xi, h)$ is called (J,φ)-holomorphic (or $+(J,\varphi)$-holomorphic) if $df \circ J = \varphi \circ df$, and it is called (J,φ)-antiholomorphic (or $-(J,\varphi)$- holomorphic) if $df \circ J = -\varphi \circ df$.

THEOREM 2.2. *Let $f : (M, J, g) \to (N, \varphi, \eta, \xi, h)$ be a smooth map between the Kähler manifold M and the Sasaki manifold N. Then f is a (J,φ)-pluriharmonic morphism if and only if it is $\pm(J,\varphi)$-holomorphic and pluriharmonic map.*

Proof. Suppose f is a (J,φ)-pluriharmonic morphism. Consider on $N \times \mathbb{R}$ the canonical Kähler structure given by the almost complex structure J' and the metric k, as above. It yields that for any local pluriharmonic function a on N, the composition $a \circ f$ is pluriharmonic. For any $b : N \times \mathbb{R} \to \mathbb{R}$ smooth function, we can consider $a : N \to \mathbb{R}$ given by $a(x) = b(x, 0)$. Moreover b is a pluriharmonic function if and only if a is a φ-pluriharmonic function (see Lemma 2.1).



On the other hand, let $f' : M \to N \times \mathbb{R}$ be the smooth function defined by $f'(x) = (f(x), 0)$. Hence $f = \pi \circ f'$. We get that for any local pluriharmonic function b on $N \times \mathbb{R}$, the composition $a \circ f = a \circ \pi \circ f'$ is pluriharmonic.

Hence f' is a pluriharmonic morphism. Since M and $N \times \mathbb{R}$ are both Kähler manifolds, it means that f' is $\pm$holomorphic map. We have then

$$\begin{aligned}(df \circ J)(X) &= (d\pi \circ df' \circ J)(X) = \pm d\pi(J' \circ df'(X)) \\ &= \pm d\pi \circ J'(dfX, 0) = \pm d\pi(\varphi df(X), \eta(df(X))\frac{d}{dt}) = \pm \varphi df(X)\end{aligned}$$

hence f is (J, φ)-holomorphic.

Now, since for any local φ-pluriharmonic function a on N, the composition $a \circ f$ is pluriharmonic, we have

$$\begin{aligned}0 &= \nabla d(a \circ f)(X,Y) + \nabla d(a \circ f)(JX, JY) = \\ &= (\nabla da)(dfX, dfY) + (\nabla da)(dfJX, dfJY) + \\ &\quad + da((\nabla df)(X,Y) + (\nabla df)(JX, JY)) = \\ &= (\nabla da)(dfX, dfY) + (\nabla da)(\varphi dfX, \varphi dfY) + \\ &\quad + da((\nabla df)(X,Y) + (\nabla df)(JX, JY)) = \\ &= da((\nabla df)(X,Y) + (\nabla df)(JX, JY)).\end{aligned}$$

It yields that f is pluriharmonic.

The converse comes easily from:

$$\begin{aligned}&\nabla d(a \circ f)(X,Y) + \nabla d(a \circ f)(JX, JY) = \\ &\quad = (\nabla da)(dfX, dfY) + (\nabla da)(dfJX, dfJY) + \\ &\quad\quad + da((\nabla df)(X,Y) + (\nabla df)(JX, JY))\end{aligned}$$

which vanishes when f is pluriharmonic and $\pm(J, \varphi)$-holomorphic. ∎

PROPOSITION 2.1. *Let $f : (M, J, g) \to (N, \varphi, \eta, \xi, h)$ be a smooth map between the Kähler manifold M and the Sasaki manifold N. Then f is (J, φ)-pluriharminic morphism if and only if it pulls back local φ-pluriharmonic maps on N to local pluriharmonic maps on M.*

P r o o f. Suppose $f : (M, J, g) \to (N, \varphi, \eta, \xi, h)$ is a (J, φ)-pluriharmonic morphism. Then f is $\pm(J, \varphi)$-holomorphic and pluriharmonic. Let a be a local φ-pluriharmonic map from N to a Riemannian manifold (P, k). We have, for any X and Y vector fields on M:

$$\begin{aligned}&\nabla d(a \circ f)(X,Y) + \nabla d(a \circ f)(JX, JY) = \\ &\quad = (\nabla da)(dfX, dfY) + (\nabla da)(dfJX, dfJY) + \\ &\quad\quad + da((\nabla df)(X,Y) + (\nabla df)(JX, JY)) = \\ &\quad = (\nabla da)(dfX, dfY) + (\nabla da)(\varphi dfX, \varphi dfY) = 0\end{aligned}$$

since f is pluriharmonic and $\pm(J, \varphi)$-holomorphic and a is φ-pluriharmonic. Therefore we get that $a \circ f$ is pluriharmonic.

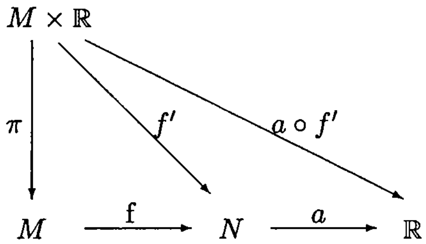
The converse comes directly by the definition of the (J, φ)-pluriharmonic morphism. ■

DEFINITION 2.6. Let $f : (M, \varphi, \eta, \xi, g) \to (N, \varphi', \eta', \xi', h)$ be a smooth map between the Sasaki manifolds M and N. Then f is called φ-pluriharmonic morphism if it pulls back local φ'-pluriharmonic functions on N to local φ-pluriharmonic functions on M.

DEFINITION 2.7. Let $f : (M, \varphi, \eta, \xi, g) \to (N, \varphi', \eta', \xi', h)$ be a smooth map between almost contact metric manifolds M and N. Then f is called φ-holomorphic (or $+\varphi$-holomorphic) if $df \circ \varphi = \varphi' \circ df$ and it is called φ-antiholomorphic (or $-\varphi$-holomorphic) if $df \circ \varphi = -\varphi' \circ df$.

THEOREM 2.3. *Let $f : (M, \varphi, \eta, \xi, g) \to (N, \varphi', \eta', \xi', h)$ be a smooth map between Sasaki manifolds M and N. Then f is a φ-pluriharmonic morphism if and only if it is $\pm\varphi$-holomorphic and φ-pluriharmonic.*

Proof. Consider on $M \times \mathbb{R}$ the Kähler structure given in a canonical way by the Sasaki structure of M, as above. Let $f' = f \circ \pi$, where $\pi : M \times \mathbb{R} \to M$ is the canonical projection.



Suppose f is a φ-pluriharmonic morphism. It means for any local φ'-pluriharmonic function a on N, the composition $a \circ f$ is φ-pluriharmonic. Then $a \circ f \circ \pi = a \circ f'$ is a pluriharmonic map (see Lemma 2.1) , for any local φ'-pluriharmonic function a on N. It follows that f' is a (J', φ')-pluriharmonic morphism. Using Theorem 2.2, it yields that f' is a $\pm(J', \varphi')$-pluriharmonic map which is pluriharmonic. Then [2] f is φ-pluriharmonic and it is easy to see that it is also φ-holomorphic.

The converse comes easy from

$$\nabla d(a \circ f)(X, Y) + \nabla d(a \circ f)(\varphi X, \varphi Y) = (\nabla da)(dfX, dfY)+$$

$$+(\nabla da)(df\varphi X, df\varphi Y) + da((\nabla df)(X, Y) + (\nabla df)(\varphi X, \varphi Y))$$

which is zero since f is φ-pluriharmonic and φ-holomorphic map.

It is easy to check the following:

PROPOSITION 2.2. *A smooth map $f : M \to N$ between the Sasaki manifolds $(M, \varphi, \eta, \xi, g)$ and $(N, \varphi', \eta', \xi', h)$ is a φ-pluriharmonic morphism if and only if it pulls back local φ'-pluriharmonic maps on N to local φ-pluriharmonic maps on M.*

REMARK 2.4. Let f be a $\pm\varphi$-holomorphic map between the almost contact metric manifolds M and N. Then f is a φ-pluriharmonic map [6] if and only if f is an isometric immersion.

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References

[1] D. E. Blair, *Contact Manifolds in Riemannian Geometry*, Lecture Notes, 509, Springer Verlag, (1976).

[2] I. A. Calinov, *Some examples of pluriharmonic maps and φ-pluriharmonic maps*, Ann. Stiint. Univ. Al. I. Cuza Iasi, Mat, 43 (1997), no. 1, 63–72.

[3] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 10 (1978), 1–68.

[4] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. 20 (1988), 385–524.

[5] C. Gherghe, *Harmonic maps on trans-Sasaki manifolds*, Rend. Circ. Mat. Palermo (2), 48 (1999), no. 3, 477–486.

[6] S. Ianus and A. M. Pastore, *Harmonic maps on metric almost contact manifolds*, Ann. Math. Blaise Pascal, vol 2, 2 (1995), 43–53.

[7] E. Loubeau, *Pluriharmonic morphisms*, Math. Scand. 84 (1999), no. 2, 165–178.

[8] Y. Ohnita, *On pluriharmonicity of stable harmonic maps*, J. London Math. Soc. 2, 35 (1987), 563–568.

[9] Y. Ohnita and S. Udagawa, *Stability, complex analiticity and constancy of pluriharmonic maps*, Math. Z. 205 (1990), 629–644.

[10] S. Udagawa, *Pluriharmonic maps and minimal immersions of Kahler manifolds*, J. London Math. Soc, 37 (1988), 375–384.

[11] J. C. Wood, *Harmonic morphisms and Hermitian structures on Einstein manifolds*, Internat. J. Math., vol. 3, 3 (1992), 415–439.

UNIVERSITY OF BUCHAREST
FACULTY OF MATHEMATICS
14 Academiei str.
70109 BUCHAREST, ROMANIA
E-mail: irina_calinov@hotmail.com

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