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CROSSED GROUP RINGS WITH A FINITE SET OF
DEGREES OF INDECOMPOSABLE REPRESENTATIONS
OVER DEDEKIND DOMAINS

Abstract. Let S be a Dedekind domain of characteristic $p > 0$, G be a finite group, $p \mid |G|$ and $S^\lambda G$ be a crossed group ring of the group G and of the ring S with a factor system λ . Under some assumption on G and S , we single out rings $S^\lambda G$ possessing finite set of degrees of indecomposable S -representations.

1. Introduction

Let G be a finite group, $Z(G)$ the center of G , G' the commutant of G , G_p a Sylow p -subgroup of G , G'_p a Sylow p -subgroup of G' , $|g|$ the order of $g \in G$, S a commutative ring with an identity element, $S^p = \{a^p : a \in S\}$, S^* the multiplicative group of the ring S , and $Z^2(G, S^*)$ the group of S -factor systems (2-cocycles) of the group G , where we assume that G acts trivially on S^* (see [23], Chapter 1). Any S -factor system of G is equivalent to some normalized S -factor system of G . From now on we assume that S -factor systems of G are normalized. A crossed group ring $S^\lambda G$ of the group G and of the ring S with a factor system $\lambda \in Z^2(G, S^*)$ is the S -algebra with S -basis $\{u_g : g \in G\}$ satisfying $u_a u_b = \lambda_{a,b} u_{ab}$ for all $a, b \in G$. The S -basis $\{u_g : g \in G\}$ of $S^\lambda G$ will be called natural. Let e be the identity element of G . Then $u_a u_e = u_e u_a = u_a$ for all $a \in G$. We often identify u_e with the identity element of the ring S . That is why, instead of γu_e , we will write γ ($\gamma \in S$). If H is a subgroup of the group G , then the restriction of an S -factor system $\lambda \in Z^2(G, S^*)$ to $H \times H$ will also be denoted by λ . In this case $S^\lambda H$ is a subring of the ring $S^\lambda G$.

Let $\mathbb{M}_n(S)$ be the ring of all $n \times n$ matrices over S . A homomorphism of S -algebras $\Gamma : S^\lambda G \rightarrow \mathbb{M}_n(S)$ is called an matrix S -representation of $S^\lambda G$. The integer n is the degree of Γ . Two matrix S -representations Γ_1 and Γ_2

of $S^\lambda G$ are said to be equivalent if there exist an invertible matrix C over S such that $C^{-1}\Gamma_1(v)C = \Gamma_2(v)$ for all $v \in S^\lambda G$. A matrix S -representation Γ of the ring $S^\lambda G$ is said to be decomposable if it is S -equivalent to a matrix S -representation Γ' of the form

$$\Gamma'(v) = \begin{pmatrix} \Gamma_1(v) & 0 \\ 0 & \Gamma_2(v) \end{pmatrix} \quad (v \in S^\lambda G),$$

where Γ_1 and Γ_2 are matrix S -representations of $S^\lambda G$. A matrix S -representation is indecomposable if it is not decomposable. By an $S^\lambda G$ -module we mean a finitely generated left $S^\lambda G$ -module which is S -free. The study of matrix S -representations of $S^\lambda G$ is essentially equivalent to the study of $S^\lambda G$ -modules (see [7], §10).

Let $D(S^\lambda G)$ be the set of degrees of all indecomposable matrix S -representations of the crossed group ring $S^\lambda G$. If $S^\lambda G = SG$ is the usual group ring, that is λ is trivial, then the problem whether $D(S^\lambda G)$ is finite, is solved in the following cases:

- (a) S is a field of characteristic p [21];
- (b) number ring [3–5], [9], [12], [19], [20], [22], [24], [27];
- (c) ring of residue classes modulo m [15];
- (d) ring of polynomials in t variables with coefficients from a field of characteristic p [17];
- (e) ring of formal power series in t variables with coefficients from a complete discrete valuation ring [16];
- (f) principal ideal domain of characteristic p [13];
- (g) local ring with residue class field of characteristic p [18].

The case when $S^\lambda G$ is not equivalent to the group ring, that is λ is not a coboundary, and S is a ring of P -adic numbers or a field of characteristic p , is studied in [1], [2], [14] and [6].

We note that, in the case S is a field, the set $D(SG)$ is finite if and only if the number of the equivalent classes indecomposable matrix S -representations of SG is finite [21]. If S is a principal ideal domain of characteristic 2 and is not a field, and if G is the group of order 2, then $D(SG) = \{1, 2\}$ [13] and there is an infinite number of pairwise non-equivalent indecomposable matrix S -representations of SG [26].

For a complete discussion of related problems in the modern representation theory of finite groups, algebras, quivers and vector space categories the reader is referred to the monographs [10], [11], [23], [25], [28], [29].

In this paper we study the problem on finiteness of $D(S^\lambda G)$ under the assumption that S is a Dedekind domain of characteristic $p > 0$, and is not a field.

Let us briefly present the main results of the paper. In Section 2, using the articles [13], [18] we prove that if $|G'_p| > 2$, then $D(S^\lambda G)$ is an infinite set for each cocycle $\lambda \in Z^2(G, S^*)$. The case where G_p is abelian and $S = F[x]$ is the ring of polynomials in the variable x over a field F of characteristic p , is studied in Section 3. We establish that if $p \neq 2$ and G_p is a normal subgroup of G , then $D(S^\lambda G)$ is a finite set if and only if $G = G_p \times H$ and $F^\lambda G_p$ is a field. Using a technique developed in [3], [4], we also prove a criterion of finiteness of $D(S^\lambda G)$ for $p = 2$ and $G = G_2$. The criterion asserts that the set $D(S^\lambda G)$ is finite if and only if $\dim_F(F^\lambda G / \text{rad} F^\lambda G) \geq \frac{|G|}{2}$, where $\text{rad} F^\lambda G$ is the radical of the algebra $F^\lambda G$ (see [7], §24). In Section 4, the analogous results are obtained for the case when $S = F[[x]]$ is a ring of formal power series in x over the field F , and the cocycles are taken from $Z^2(G, F^*)$. Throughout this paper we use the terminology and notation introduced in [7], [8] and [23].

2. Necessary conditions for finiteness of $D(S^\lambda G)$

Let K be a quotient field of S , P be a maximal ideal of S , $\pi \in P$ and $\pi \notin P^2$, φ be the P -adic valuation of K , $\hat{K}$ be the φ -adic completion of K , $\hat{S}$ be the valuation ring of $\hat{K}$ and $\hat{P}$ be the maximal ideal of $\hat{S}$. As we know ([8], §4C), $\hat{P} = \pi \hat{S}$ and $S/P \cong \hat{S}/\hat{P}$.

Let $\lambda \in Z^2(G, S^*)$. If $a \in G$ is an element of order n , then the equality

$$u_a^n = \lambda_{a,a} \lambda_{a,a^2} \dots \lambda_{a,a^{n-1}} u_e$$

holds in $S^\lambda G$.

Let $G = (a_1) \times \dots \times (a_m)$ be an abelian group of type $(p^{r_1}, \dots, p^{r_m})$. For each cocycle $\lambda \in Z^2(G, S^*)$, the ring $S^\lambda G$ is commutative. A natural S -basis $\{u_g : g \in G\}$ of the ring $S^\lambda G$ can be chosen in such a way that the following conditions are satisfied:

- (1) if $a = a_1^{i_1} \dots a_m^{i_m} \in G$, where $0 \leq i_q < p^{r_q}$ ($q = 1, \dots, m$), then $u_a = u_{a_1}^{i_1} \dots u_{a_m}^{i_m}$,
- (2) $u_{a_j}^{p^{r_j}} = \delta_j$ ($\delta_j \in S^*$; $j = 1, \dots, m$).

In this connection, we also denote the ring $S^\lambda G$ by $[G, S, \delta_1, \dots, \delta_m]$.

2.1. LEMMA. *Let S be a Dedekind domain of characteristic p . Assume that either $p \neq 2$ and G contains an element a of order p such that*

$$\lambda_{a,a} \lambda_{a,a^2} \dots \lambda_{a,a^{p-1}} = \mu^p, \quad \text{for some } \mu \in S^*,$$

or $p = 2$ and G contains an element a of order 4 such that

$$\lambda_{a,a} \lambda_{a,a^2} \lambda_{a,a^3} = \mu^4, \quad \text{for some } \mu \in S^*.$$

Then $D(S^\lambda G)$ is an infinite set.

Proof. Let

$$\hat{S}^\lambda G = \hat{S} \otimes_S S^\lambda G,$$

$H = (a)$, E_n be the identity matrix of order n , J_n be the Jordan block of order n with the unity on the main diagonal. By virtue of Lemma 2 in [13], the $\hat{S}$ -linear mapping $\hat{\Gamma} : \hat{S}^\lambda H \rightarrow \mathbb{M}_{3n}(\hat{S})$ defined by

$$\hat{\Gamma}(u_a) = \mu \begin{pmatrix} E_n & \pi E_n & J_n \\ 0 & E_n & \pi E_n \\ 0 & 0 & E_n \end{pmatrix},$$

is an indecomposable $\hat{S}$ -representation of the ring $\hat{S}^\lambda H$. Since for $\hat{S}$ -representations of the ring $\hat{S}^\lambda G$, the Krull-Schmidt Theorem [5] is true, we conclude that in virtue of [14], one of the indecomposable summands of the induced representation

$$(2.1) \quad \hat{\Gamma}^{\hat{S}^\lambda G}$$

of the ring $\hat{S}^\lambda G$ has degree greater than or equal to $3n$.

Let Γ be the restriction of $\hat{\Gamma}$ to $S^\lambda H$. Then Γ is an S -representation of the ring $S^\lambda H$ and the representation (2.1) can be obtained from the S -representation

$$(2.2) \quad \Gamma^{S^\lambda G}$$

of the ring $S^\lambda G$ as a result of extension of the ring of scalars S up to the ring $\hat{S}$. Therefore, a decomposition of the representation (2.2) in a direct sum of indecomposable representations has a summand of degree greater than or equal to $3n$. The lemma is proved. ■

2.2. LEMMA. *Let S be a Dedekind domain of characteristic 2. Assume that G contains a subgroup $H = (a) \times (b)$ of type $(2, 2)$ such that $\lambda_{a,a} = \gamma^2$, $\lambda_{b,b} = \delta^2$ for some $\gamma, \delta \in S^*$. Then $D(S^\lambda G)$ is an infinite set.*

Proof. By Lemma 4 [18], the $\hat{S}$ -representation $\hat{\Gamma}$ of the ring $\hat{S}^\lambda H$, defined by the equalities

$$\hat{\Gamma}(u_a) = \gamma \begin{pmatrix} E_n & E_n \\ 0 & E_n \end{pmatrix}, \quad \hat{\Gamma}(u_b) = \delta \begin{pmatrix} E_n & J_n \\ 0 & E_n \end{pmatrix},$$

is indecomposable. Then we use the same reasonings as in the proof of Lemma 2.1. ■

2.3. LEMMA. *Let S be a Dedekind domain of characteristic p , H be a p -subgroup of G' and $\lambda_H : H \times H \rightarrow S^*$ be the restriction of a cocycle $\lambda \in Z^2(G, S^*)$ to $H \times H$. Then λ_H is a coboundary.*

Proof. Since S^* has no elements of order p , the Lemma 2.3 follows by virtue of Corollary 4.10 [23, p. 42]. ■

2.4. THEOREM. Let S be a Dedekind domain of characteristic $p \neq 2$. If the set $D(S^\lambda G)$ is finite for some $\lambda \in Z^2(G, S^*)$, then $G = H \rtimes G_p$ and G_p is abelian. Moreover, if F is a subfield of S and $\lambda \in Z^2(G, F^*)$, then $F^\lambda G_p$ is a field.

Proof. By virtue of Lemmas 2.1 and 2.3, p does not divide $|G'|$. It then follows that $H = \{g \in G : |g| \not\equiv 0 \pmod{p}\}$ is a normal subgroup of G . By Schur's Theorem, $G = H \rtimes G_p$. Let $G_p = (a_1) \times \dots \times (a_m)$ be a group of type $(p^{r_1}, \dots, p^{r_m})$, $S^\lambda G_p = [G_p, S, \delta_1, \dots, \delta_m]$, where $\delta_j \in F^*$ ($j = 1, \dots, m$). Assume that $F[u_{a_1}, \dots, u_{a_j}]$ is a field and $F[u_{a_1}, \dots, u_{a_j}, u_{a_{j+1}}]$ is not a field. It follows that $\delta_{j+1} u_e = v^p$, where $v \in F[u_{a_1}, \dots, u_{a_j}]$. Let

$$\begin{aligned} b_i &= a_i^{p^{r_i-1}}, & u_{b_i} &= u_{a_i}^{p^{r_i-1}} & (i = 1, \dots, m), \\ w &= \delta_{j+1}^{-1} v^{p-1} u_{b_{j+1}}, & B &= (b_1) \times \dots \times (b_m). \end{aligned}$$

We have

$$\begin{aligned} u_{b_i}^p &= \delta_i u_e, & w^p &= u_e, & u_{b_{j+1}} &= vw, \\ v &\in F[u_{b_1}, \dots, u_{b_j}], & F[u_{b_1}, \dots, u_{b_j}, u_{b_{j+1}}] &= F[u_{b_1}, \dots, u_{b_j}, w]. \end{aligned}$$

Therefore, $S^\lambda B = S^\mu N = [N, S, \delta_1, \dots, \delta_j, 1, \delta_{j+2}, \dots, \delta_m]$, where N is an elementary abelian p -group of order $|B|$. Let

$$\Gamma(w) = \begin{pmatrix} E_n & \pi E_n & J_n \\ 0 & E_n & \pi E_n \\ 0 & 0 & E_n \end{pmatrix}.$$

Arguing as in the proof of Lemma 2.1, we obtain that any of the indecomposable components of the induced representation

$$\left(\Gamma^{S^\mu N} \right)^{S^\lambda G}$$

of the ring $S^\lambda G$ has degree greater than or equal to $3n$. This contradicts the assumption that $D(S^\lambda G)$ is a finite set. Therefore, $F^\lambda G_p$ is a field, and the proof is completed. ■

2.5. THEOREM. Let S be a Dedekind domain of characteristic 2. If the set $D(S^\lambda G)$ is finite for some $\lambda \in Z^2(G, S^*)$, then 4 does not divide $|G'|$. Moreover, if $G = G_2 \rtimes H$, then $G = G_2 \times H$.

Proof. The first statement follows immediately from Lemmas 2.1 - 2.3. Let $G = G_2 \rtimes H$ and $[G_2, h] \neq \{e\}$ for some $h \in H$. Assume that G_2 is abelian. If a, b are non-identity elements of G_2 and $h^{-1}ah = ab$, then $b^2 = e$ and $h^{-1}bh = b$. In this case, $h^{-2}ah^2 = a$. But $(h) = (h^2)$, hence,

$h^{-1}ah = a$. The contradiction obtained proves that G_2 is a non-abelian group. Then $G'_2 = (c)$, where $c \in Z(G)$. Let $\bar{G} = (G_2, h)$. Using formula $[xy, z] = [x^y, z^y] \cdot [y, z]$ we get $[\bar{G}, \bar{G}] = [\bar{G}, G_2] = (c)$. In this case, $\bar{G}$ is a nilpotent group. This contradicts to $[G_2, h] \neq \{e\}$, and the proof is finished. ■

3. Crossed group rings with a finite set of degrees of indecomposable representations over a ring of polynomials

In this section, we assume that $S = F[x]$ is a ring of polynomials in the variable x over a field F of characteristic p . As known, if $P = xS$ (we use the notation of Sec. 2), then $\hat{S} = F[[x]]$ is the ring of formal power series in the variable x with coefficients from the field F . Since $S^* = F^*$, sometimes we write $Z^2(G, F^*)$ instead of $Z^2(G, S^*)$. Let us note that in the case under consideration $K = F(x)$.

3.1. THEOREM. *Let F be a field of characteristic $p \neq 2$, $S = F[x]$, $G = G_p \rtimes H$, and $\lambda \in Z^2(G, S^*)$. The set $D(S^\lambda G)$ is finite if and only if $G = G_p \times H$, G_p is an abelian group and $F^\lambda G_p$ is a field.*

Proof. The necessity of the theorem follows from Theorem 2.4. Let us prove the sufficiency. The ring $S^\lambda G_p$ can be viewed as the ring of polynomials in the variable x over the field $L = F^\lambda G_p$. Because of this, $S^\lambda G$ is a crossed group ring $R^\nu H$ of the group H and the ring $R = L[x]$. Let M be a finitely generated S -torsion free $S^\lambda G$ -module. Then M is a finitely generated $R^\nu H$ -module. If $m \in M$, $m \neq 0$ and $f(x)m = 0$ for some $f(x) \in L[x]$, then

$$(f(x))^{p^n} m = 0,$$

where $p^n = \exp G_p$. But

$$(f(x))^{p^n} \in F[x] = S,$$

therefore, $(f(x))^{p^n} = 0$, whence, $f(x) = 0$. This proves that M is an R -torsion free module. The isomorphism and the indecomposability of $S^\lambda G$ -modules are equivalent to those of $R^\nu H$ -modules. Since $p \nmid |H|$, each of the $R^\nu H$ -modules is a completely reducible module. It is known [7] that R -rank of each of the irreducible $R^\nu H$ -modules does not exceed $|H|$. Therefore, $D(S^\lambda G)$ is a finite set, and the proof is finished. ■

Let t be a root of a polynomial of degree $m+1$ with the leading coefficient 1 which is irreducible over F . Let us denote by $\tilde{t}$ the matrix corresponding to the operator of the multiplication by t in the F -basis $1, t, \dots, t^m$ of the field $F[t]$.

3.2. LEMMA. *Let F be a field of characteristic 2, $G = (a)$ a cyclic group of order 2^n ($n \geq 1$), $S = F[x]$, and $S^\lambda G = [G, S, \alpha^{2^m}]$, where $\alpha \notin F^2$. If $m = 0$,*

then up to the equivalence, the ring $S^\lambda G$ has only one indecomposable S -representation $u_a \rightarrow \tilde{\theta}$, where θ is a root of the irreducible polynomial

$$y^{2^n} - \alpha \in F[y].$$

If $m = 1$, then up to equivalence, the indecomposable S -representations of the ring $S^\lambda G$ are the following:

$$\tilde{\rho}, \quad \begin{pmatrix} \tilde{\rho} & \langle f(x, \rho) \rangle \\ 0 & \tilde{\rho} \end{pmatrix},$$

where ρ is a root of the irreducible polynomial

$$y^{2^{n-1}} - \alpha \in F[y]$$

and $\langle f(x, \rho) \rangle$ is the matrix in which all columns but last one being zero, and the last column consisting of the coordinates of $f(x, \rho) \in S[\rho]$ in the S -basis

$$1, \rho, \dots, \rho^{2^{n-1}-1}$$

of the ring $S[\rho]$.

Proof. If $m = 0$, then $S^\lambda G \cong S[\theta] = F[\theta][x]$, where $F[\theta]$ is a field. Each $S^\lambda G$ -module can be considered as a torsion-free module over the principal ideal ring $S[\theta]$, therefore, $M \cong S[\theta] \oplus \dots \oplus S[\theta]$. Hence, up to equivalence, the ring $S^\lambda G$ has only one indecomposable S -representation $u_a \rightarrow \tilde{\theta}$. Let $m = 1$, M be an arbitrary $S^\lambda G$ -module, and

$$N = \{v \in M : (u_a^{2^{n-1}} - \alpha)v = 0\}.$$

One can view the $S^\lambda G$ -module N as a module over the ring $S^\lambda G / (u_a^{2^{n-1}} - \alpha)S^\lambda G \cong S[\rho]$. Since $S[\rho]$ is a principal ideal ring and N is a $S[\rho]$ -torsion free module, we get

$$(3.1) \quad N \cong S[\rho] \oplus \dots \oplus S[\rho].$$

The $S^\lambda G$ -module M/N can also be viewed as a $S[\rho]$ -module. Since M is a free S -module, N is an S -pure submodule of the module M (see [8], p.84). If $\varphi(x, \rho) \in S[\rho]$, then

$$\varphi(x, \rho)^{2^{n-1}} \in S,$$

and, therefore, the equality $\varphi(x, \rho)[v + N] = N$ yields $\varphi(x, \rho) = 0$ or $v \in N$. This means that M/N is a torsion-free module over $S[\rho]$. Therefore,

$$(3.2) \quad M/N \cong S[\rho] \oplus \dots \oplus S[\rho].$$

Using (3.1), (3.2) and the S -purity of the module N , we conclude that any S -representation Γ of the ring $S^\lambda G$ which is realized in the $S^\lambda G$ -module M can be written in the form

$$\Gamma(u_a) = \begin{pmatrix} \tilde{\rho}^{(r)} & * \\ 0 & \tilde{\rho}^{(t)} \end{pmatrix},$$

where $\tilde{\rho}^{(t)} = \text{diag}[\tilde{\rho}_1, \dots, \tilde{\rho}_t]$ and $\rho_1 = \dots = \rho_t = \rho$. Using the technique developed in [3], [4], we conclude that indecomposable S -representations of the ring $S^\lambda G$ for $m = 1$ are the following representations:

$$\tilde{\rho}, \quad \begin{pmatrix} \tilde{\rho} & \langle f(x, \rho) \rangle \\ 0 & \tilde{\rho} \end{pmatrix}.$$

The lemma is proved. ■

3.3. THEOREM. *Let F be a field of characteristic 2, $S = F[x]$, and G be an abelian 2-group. The set $D(S^\lambda G)$ is finite if and only if $\dim(F^\lambda G / \text{rad} F^\lambda G) \geq \frac{|G|}{2}$.*

Proof. If $D(S^\lambda G)$ is finite, then using Lemma 2.2 and reasonings presented in the proof of Theorem 2.4, we conclude that there exists a decomposition $G = H \times (b)$ such that $T = F^\lambda H$ is a field. Let $L = T[x]$, and $B = (b)$. The ring $S^\lambda G$ can be viewed as a crossed group ring

$$L^\mu B = [B, L, \gamma^{2^m}],$$

where either $\gamma = 1$, $m = 0$ or $\gamma \in T$, $\gamma \notin T^2$ and $2^m < |b|$. Since $D(S^\lambda G)$ is finite if and only if $D(L^\mu B)$ is finite, then by Lemma 2.1, we get $m \leq 1$. This means

$$\dim(F^\lambda G / \text{rad} F^\lambda G) \geq \frac{|G|}{2}.$$

The converse statement can be proved by the same reasonings and using Lemma 3.2. ■

3.4. COROLLARY. *Let F be a field of characteristic 2, $S = F[x]$, G be an abelian 2-group, and $H = \{g \in G : g^4 = e\}$. The set $D(S^\lambda G)$ is finite if and only if the set $D(S^\lambda H)$ is finite.*

3.5. REMARK. Let F be a field of characteristic 2, $S = F[x]$, G be an abelian 2-group, and N be the socle of G . If $D(S^\lambda G)$ is a finite set, then the set $D(S^\lambda N)$ is also finite. The converse does not hold.

4. Crossed group rings with a finite set of degrees of indecomposable representations over a ring of formal power series

We assume that F is a field of characteristic $p > 0$, $S = F[[x]]$, G is a finite group, G_p is a Sylow p -subgroup of the group G . If Ω is an algebraic closure of the field F , then we assume $F[[x]] \subset \Omega[[x]]$. Therefore, for each $\theta \in \Omega$ we have $F[[x]][\theta] = F[\theta][[x]]$. Theorem 3.3 is also valid for rings $F[[x]]^\lambda G$ under the same assumption $\lambda \in Z^2(G, F^*)$.

4.1. PROPOSITION. *Let F be a field of characteristic $p \neq 2$, $S = F[[x]]$, $G = H \rtimes G_p$ and G_p be an abelian group. If $\lambda \in Z^2(G, F^*)$, then $D(S^\lambda G)$ is finite if and only if $F^\lambda G_p$ is a field.*

Proof. Apply Theorem 2.4 and Higman's theory of relative projectivity [21] that was developed in [1,14] for crossed group rings. ■

We remark that if $G = H \rtimes G_p$, then for each $\mu \in Z^2(G_p, F^*)$ there exists a $\lambda \in Z^2(G, F^*)$ such that $\lambda_{G_p \times G_p} = \mu$.

4.2. PROPOSITION. *Let F be a field of characteristic 2, $S = F[[x]]$, G be a finite group, $|G'_2| \neq 2$, and $\lambda \in Z^2(G, F^*)$. The set $D(S^\lambda G)$ is finite if and only if $G = H \rtimes G_2$, G_2 is abelian and $\dim(F^\lambda G_2 / \text{rad } F^\lambda G_2) \geq \frac{|G_2|}{2}$.*

Proof. Apply Theorem 2.5 and an analogue of Theorem 3.3 for $F[[x]]$. ■

4.3. PROPOSITION. *Let F be a perfect field of characteristic p , $S = F[[x]]$, G a finite group, and $\lambda \in Z^2(G, F^*)$. The set $D(S^\lambda G)$ is finite if and only if $p \nmid |G|$ for $p \neq 2$ and $4 \nmid |G|$ for $p = 2$.*

Proof. Since $S^\lambda G_p$ is equivalent to the group ring SG_p [23, p.68], the Proposition 4.3 follows by virtue of Theorem 1 [13]. ■

4.4. PROPOSITION. *Let F be a perfect field of characteristic $p \neq 2$, $S = F[[x]]$, G a finite group, $\lambda \in Z^2(G, S^*)$. Suppose that the group G contains an element a of order p such that*

$$(4.1) \quad \lambda_{a,a} \lambda_{a,a^2} \dots \lambda_{a,a^{p-1}} = \gamma + x\sigma,$$

where $\gamma \in F$, $\sigma \in S$ and $\sigma \not\equiv 0 \pmod{x}$. Then the set $D(S^\lambda G)$ is finite if and only if G_p is a cyclic group.

Proof. Since F is a perfect field, then going (if necessary) to the equivalent factor set, we can assume $\lambda_{g_1, g_2} \equiv 1 \pmod{x}$ for arbitrary $g_1, g_2 \in G_p$. In particular, in (4.1) we have $\gamma = 1$.

Assume that $D(S^\lambda G)$ is a finite set. Then by virtue of Theorem 2.4, G_p is an abelian group. Let H be the socle of G_p and $S^\lambda H = [H, S, \delta_1, \dots, \delta_m]$. Without loss of generality, we can assume $\delta_1 \equiv 1 \pmod{x}$ and $\delta_1 \not\equiv 1 \pmod{x^2}$. Let $\bar{x} = \delta_1 - 1$. Then $xS = \bar{x}S$. Hence, each element $\mu \in S$ can be presented as a formal series $\mu = \gamma_0 + \gamma_1 \bar{x} + \gamma_2 \bar{x}^2 + \dots$, where $\gamma_j \in F$ ($j = 0, 1, 2, \dots$). Let θ be a root of the polynomial $y^p - \delta_1$ and $\pi = \theta - 1$. Then $\pi^p = \bar{x}$, and hence

$$\mu = (\rho_0 + \rho_1 \pi + \rho_2 \pi^2 + \dots)^p,$$

where $\rho_j^p = \gamma_j$ and $\rho_j \in F$ for each $j = 0, 1, 2, \dots$. It follows that $\delta_i = \omega_i^p$, where $\omega_i \in S[\theta]$ for each $i = 2, \dots, m$. By the same reasonings as in the proof of Theorem 2.4 we obtain that if $m \neq 1$, then $D(S^\lambda G)$ is an infinite set. Therefore, G_p is cyclic.

Conversely, assume that $G_p = \langle b \rangle$, where $|b| = p^n$. On the basis of (4.1) we get

$$u_b^{p^n} = 1 + \bar{x},$$

where $\bar{x} \in S$ and $\bar{x}S = xS$. It follows that $S[u_b]$ is the ring of all S -integral elements of the field $K[u_b]$, where K is the quotient field of the ring S . Since $S[u_b]$ is a principal ideal ring, each finitely generated S -torsion free $S[u_b]$ -module is isomorphic to $S[u_b] \oplus \dots \oplus S[u_b]$. Then the finiteness of $D(S^\lambda G)$ follows from the finiteness of $D(S^\lambda G_p)$. The proposition is proved. ■

4.5. PROPOSITION. *Let F be a perfect field of characteristic 2, $S = F[[x]]$, G a finite group, G_2 a Sylow 2-subgroup of G , and $\lambda \in Z^2(G, S^*)$. Suppose that G_2 is abelian and G_2 contains an element a of order 4 such that $\lambda_{a,a}\lambda_{a,a^2}\lambda_{a,a^3} = \gamma + x\rho$, where $\gamma \in F$, $\rho \in S$ and $\rho \not\equiv 0 \pmod{x}$. Then $D(S^\lambda G)$ is a finite set if and only if G_2 is a cyclic group or group of type $(2^n, 2)$.*

Proof. We can assume $G_2 = (a_1) \times \dots \times (a_m)$, $|a_i| = 2^{n_i}$ ($i = 1, \dots, m$), $n_1 \geq 2$ and

$$u_{a_1}^{2^{n_1}} = 1 + x.$$

The ring $S^\lambda G_2$ can be viewed as a crossed group ring of the group $(a_2) \times \dots \times (a_m)$ and of the ring $S[u_{a_1}]$. Let θ be a root of the polynomial

$$y^{2^{n_1}} - x.$$

Then $S[u_{a_1}] \cong S[\theta]$ and each element $\mu \in S$ can be expressed as $\mu = \rho^{2^{n_1}}$, where $\rho \in S[\theta]$. Hence, by Lemmas 2.1 and 2.2, we conclude that if $D(S^\lambda G)$ is a finite set, then $m \leq 2$, and, if $m = 2$, then $n_2 = 1$. This proves the necessity.

Let us prove the sufficiency. If G_2 is cyclic, then $S^\lambda G_2 \cong S[\theta]$. Then we use the same reasonings as in the proof of Proposition 4.4. If G_2 is of type $(2^n, 2)$, then $S^\lambda G_2$ can be viewed as a group ring of the group of order 2 and of the ring $S[\theta]$. The set of degrees of indecomposable $S(\theta)$ -representations of such a group ring is $\{1, 2\}$ [13]. Therefore, $D(S^\lambda G_2) = \{2^n, 2^{n+1}\}$. The finiteness of $D(S^\lambda G)$ follows from the finiteness of $D(S^\lambda G_2)$. The proposition is proved. ■

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