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ON THE APPLICATION OF SOME MEASURES
OF NONCOMPACTNESS TO EXISTENCE THEOREMS
FOR AN m TH ORDER DIFFERENTIAL EQUATION

Assume that $I = [0, a]$, E is a Banach space, $B = \{x \in E : \|x\| \leq b\}$ and $f : I \times B \rightarrow E$ is a bounded uniformly continuous function.

In this paper we shall give sufficient conditions for the existence of solutions of the Cauchy problem

$$(1) \quad x^{(m)} = f(t, x)$$

$$(2) \quad x(0) = 0, x'(0) = \eta_1, \dots, x^{(m-1)}(0) = \eta_{m-1},$$

where $\eta_1, \dots, \eta_{m-1} \in E$.

These conditions are expressed in terms of axiomatic measures of noncompactness introduced by Banaś and Goebel [2]. Our considerations are connected with the paper [5]. In contrast to [5], we shall use in our proof some ideas from papers [3] and [1].

Analogously, as in [2] and [1], let M_E denote the family of all nonempty and bounded subsets of E and N_E the family of all nonempty and relatively compact sets in E .

Recall that a function $\mu : M_E \rightarrow [0, \infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

1° the family $\ker \mu = \{X \in M_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \in N_E$,

2° if $X \subset Y$, then $\mu(X) \leq \mu(Y)$,

3° $\mu(\bar{X}) = \mu(X)$,

4° $\mu(\text{conv} X) = \mu(X)$,

5° $\mu(\lambda \cdot X + (1 - \lambda)Y) = \lambda \cdot \mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$,

6° if $X_n \in M_E$, $X_n = \bar{X}_n$ and $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and

$\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

Let μ be a measure of noncompactness in the above sense. Moreover, we assume

$$7^\circ \mu(\{a\}) = 0 \text{ for all } a \in E,$$

$$8^\circ \mu(a + X) = \mu(X) \text{ for } a \in E \text{ and } X \in M_E.$$

Then, by 5° we have

$$9^\circ \mu(\lambda X) \leq \lambda \mu(X) \text{ for } 0 \leq \lambda \leq 1.$$

Set $E_\mu = \{x \in E : \{x\} \in \ker \mu\}$.

Denote by $B(0, r)$ the ball with center 0 and radius r .

The main result of the paper is the following

THEOREM. *Suppose that there is a continuous function $u : [0, a] \mapsto \mathbb{R}_+$ such that $u(t) > 0$ for $t > 0$, $u(0) = \dots u^{(m-1)}(0) = 0$ and $u^{(m)}(t) > 0$ is Lebesgue integrable. If*

$$(3) \quad \mu(f(t, X)) \leq \frac{u^{(m)}(t)}{u(t)} \mu(X) \text{ for } X \subset B \text{ and } t \in (0, a)$$

and

$$(4) \quad \lim_{\substack{t \rightarrow 0^+ \\ r \rightarrow 0^+}} \frac{\mu(f(t, B(0, r)))}{u^{(m)}(t)} = 0,$$

then there exists an interval $J = [0, d]$ such that the problem (1)–(2) has at least one solution defined on J .

Proof. Put $M = \sup\{\|f(t, x)\| : t \in I, x \in B\}$. Choose a positive number d such that $d \leq a$ and

$$(5) \quad \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^j}{j!} + M \frac{d^m}{m!} \leq b.$$

Let $C = C(J, E)$ be the Banach space of continuous functions $J \mapsto E$ and let $\tilde{B} \subset C$ be the subset of those functions with values in B .

We define a mapping F by

$$F(x)(t) = p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, x(s)) ds \quad (t \in J, x \in \tilde{B}),$$

where $p(t) = \sum_{j=1}^{m-1} \eta_j \frac{t^j}{j!}$. It is known (cf. [4]) that F is a continuous mapping

$\tilde{B} \mapsto \tilde{B}$ and a function $x \in C$ is a solution of (1)–(2) iff $x = F(x)$. Moreover, by the Lagrange theorem, we have

$$\|F(x)(t) - F(x)(\tau)\| \leq K|t - \tau| \text{ for } x \in \tilde{B}, t, \tau \in J$$

where $K = \sum_{j=1}^{m-1} \|\eta_j\| \frac{d^{j-1}}{(j-1)!} + M \frac{d^{m-1}}{(m-1)!}$.

Denote by X_o the set of all functions $x \in C(J, E)$ such that $x(0) = 0$ and $\|x(t) - x(\tau)\| \leq K|t - \tau|$ for $t, \tau \in J$. The set X_o is nonempty, bounded, closed, convex and equicontinuous.

Set $X_{n+1} = \overline{\text{conv}} F(X_n)$ ($n = 0, 1, \dots$). The mapping F is continuous and maps X_o into itself. Since the set X_o is convex and closed,

$$X_1 = \overline{\text{conv}} F(X_o) \subset \overline{\text{conv}} X_o = X_o.$$

Similarly $X_{n+1} \subset X_n$. Let $X_n(t) = \{x(t) : x \in X_n\}$. As $X_{n+1} \subset X_n$, we have $X_{n+1}(t) \subset X_n(t)$. Hence $\mu(X_{n+1}(t)) \leq \mu(X_n(t))$.

Putting $v_n(t) = \mu(X_n(t))$, we get $0 \leq v_{n+1}(t) \leq v_n(t)$ ($n = 0, 1, \dots$) and all these functions are equicontinuous. Therefore the sequence $v_n(t)$ converges uniformly to a function $v_\infty(t) = \lim_{n \rightarrow \infty} v_n(t)$.

First we assume that $d \leq 1$.

By (4) for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(6) \quad \mu(f(t, B(0, Kt))) \leq \varepsilon u^{(m)}(t) \quad \text{for } 0 \leq t \leq \delta.$$

Since f is uniformly continuous and X_n is equicontinuous, it follows that for given $t > 0$ the set of functions

$$s \mapsto (t-s)^{m-1} f(s, x(s))', \quad \text{for } x \in X_n$$

is equicontinuous on $[0, t]$. For $x \in X_o$ we have $\|x(t)\| \leq Kt$. Fix $t \in (0, \delta]$.

Applying 7°, 8° and Lemma 7.2.4 of [1] we obtain

$$\begin{aligned} v_{n+1}(t) &= \mu(X_{n+1}(t)) = \mu(\overline{\text{conv}} F(X_n)(t)) = \mu(F(X_n)(t)) \\ &= \mu \left(\left\{ p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, x(s)) ds : x \in X_n \right\} \right) \\ &= \mu \left(\left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, x(s)) ds : x \in X_n \right\} \right) \\ &\leq \int_0^t \mu \left(\left\{ \frac{1}{(m-1)!} (t-s)^{m-1} f(s, x(s)) : x \in X_n \right\} \right) ds \\ &\leq \int_0^t \frac{1}{(m-1)!} (t-s)^{m-1} \mu(\{f(s, x(s)) : x \in X_n\}) ds. \end{aligned}$$

The last inequality follows from 9°, because

$$\frac{(t-s)^{m-1}}{(m-1)!} \leq 1.$$

As $X_n \subset X_o$, we have $\|x(t)\| \leq Kt$ for $x \in X_n$. In view of (6) this implies that

$$v_{n+1}(t) \leq \int_0^t \frac{1}{(m-1)!} (t-s)^{m-1} \varepsilon u^{(m)}(s) ds = \varepsilon u(t),$$

so that

$$(7) \quad v_{n+1}(t) = \mu(X_{n+1}(t)) \leq \varepsilon u(t) \quad \text{for } t \in [0, \delta) \quad \text{and } n = 0, 1, \dots$$

Thus $v_\infty(t) = \mu(X_\infty(t)) \leq \varepsilon u(t)$ for $t \in (0, \delta]$. Fix $t \in (0, d)$.

Similar arguments show that

$$\begin{aligned} v_{n+1}(t) &= \mu(X_{n+1}(t)) = \mu(\overline{\text{conv}} F(X_n)(t)) = \mu(F(X_n)(t)) \\ &= \mu \left(\left\{ p(t) + \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, x(s)) ds : x \in X_n \right\} \right) \\ &= \mu \left(\left\{ \frac{1}{(m-1)!} \int_0^t (t-s)^{m-1} f(s, x(s)) ds : x \in X_n \right\} \right) \\ &\leq \int_0^t \mu \left(\left\{ \frac{1}{(m-1)!} (t-s)^{m-1} f(s, x(s)) : x \in X_n \right\} \right) ds \\ &\leq \int_0^t \frac{1}{(m-1)!} (t-s)^{m-1} \mu(\{f(s, x(s)) : x \in X_n\}) ds \\ &\leq \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} \frac{u^{(m)}(s)}{u(s)} \mu(X_n(s)) ds. \end{aligned}$$

From (7) it is clear that there exists the integral

$$\int_0^t \frac{(t-s)^{m-1}}{(m-1)!} \frac{u^{(m)}(s)}{u(s)} \mu(X_n(s)) ds.$$

Thus

$$v_{n+1}(t) \leq \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} \frac{u^{(m)}(s)}{u(s)} v_n(s) ds.$$

By the Lebesgue dominated convergence theorem, we get

$$(8) \quad v_\infty(t) \leq \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} \frac{u^{(m)}(s)}{u(s)} v_\infty(s) ds \quad \text{dla } t \in (0, d).$$

Put $z(t) = \frac{v_\infty(t)}{u(t)}$ for $t \in (0, d)$ and $z(0) = 0$. By (7) the function z is continuous. Suppose that z is not identically zero on $[0, d)$. Let τ be the

smallest real number such that $z(\tau) = \max_{t \in J} z(t)$. Then $0 \leq z(s) < z(\tau)$ for $s \in [0, \tau)$. By (8) we have

$$\begin{aligned} z(\tau) &= \frac{v_\infty(\tau)}{u(\tau)} \leq \frac{1}{u(\tau)(m-1)!} \int_0^\tau (\tau-s)^{m-1} \frac{u^{(m)}(s)}{u(s)} v_\infty(s) ds \\ &= \frac{1}{u(\tau)(m-1)!} \int_0^\tau (\tau-s)^{m-1} u^{(m)}(s) z(s) ds \\ &< \frac{z(\tau)}{u(\tau)} \frac{1}{(m-1)!} \int_0^\tau (\tau-s)^{m-1} u^{(m)}(s) ds = \frac{z(\tau)}{u(\tau)} \cdot u(\tau) = z(\tau), \end{aligned}$$

a contradiction. Thus $z(t) \equiv 0$ on $[0, d]$, i.e. $v_\infty(t) = 0$ on $[0, d]$.

Set $\mu_C(X) = \max\{\mu(X(t)) : t \in J\}$ for X equicontinuous. Then μ_C is a measure of noncompactness (cf. [2], Theorem 11.2, p.59). Since

$$\lim_{n \rightarrow \infty} \max\{v_n(t) : t \in J\} = 0,$$

$\mu_C(X_n) \rightarrow 0$ as $n \rightarrow \infty$. Hence the set $X_\infty = \bigcap_{n=1}^\infty X_n$ is nonempty, convex, closed and $X_\infty \subset E_\mu$.

Now we can apply the Schauder fixed point theorem to the mapping $F|X_\infty$, which yields the existence of $x \in X_\infty$ such that $x = F(x)$, i.e. x is a solution of (1) – (2).

Suppose now $d > 1$. For $t \in [1, d]$ the function $f(t, x)$ satisfies the μ -Lipschitz condition with the integrable function $\lambda(t) = \frac{u^{(m)}(t)}{u(t)}$, i.e.

$$\mu(f(t, X)) \leq \lambda(t)\mu(X) \quad \text{for } t \in [1, d].$$

The equation $x^{(m)} = f(t, x)$ is equivalent to the first order system. Applying now Theorem 7.3.1 of [1], we see that the solution can be continued to the interval $[0, h]$, $1 < h \leq d$. If $h < d$, then

$$\|x(t)\| \leq \sum_{j=1}^{m-1} \|\eta_j\| \frac{h^j}{j!} + M \frac{h^m}{m!} < b \quad \text{for } t \in [0, h],$$

so that $x(h) \in \text{Int}B$. Thus the solution $x(t)$ can be continued to the interval $J = [0, d]$.

REMARK. Assume that $A : (0, d] \mapsto \mathbb{R}$ is a function such that $A'(t) = a(t)$ for almost every $t \in (0, d]$, $\lim_{t \rightarrow 0+} A(t) = -\infty$, where $a(t) \geq 0$ for $(0, d]$. If we take $m = 1$ and $u(t) = \exp A(t)$ in our theorem, we obtain Theorem 7.4.2 from [1].

The following functions will be useful as some examples

- 1) $a(t) = \frac{c}{t}$ ($c > 1$), $A(t) = c \ln t$, $t \in (0, 1]$,
- 2) $a(t) = \frac{1}{t^2}$, $A(t) = -\frac{1}{t}$, $t \in (0, 1]$.

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