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COMMON FIXED POINTS FOR EXPANSION MAPPINGS IN GENERALIZED METRIC SPACES

Abstract. The purpose of this paper is to obtain some common fixed point theorems for expansion mappings in recently developed space D-metric space. Our results generalize several corresponding results in metric spaces.

1. Introduction

Rhoades [5] summarized contractive maps of some types and discussed on fixed points. Wang-Li-Gao and Is'eki [8] proved some fixed point theorems for expansion maps, which correspond to some contractive maps in [5]. Khan-Khan and Sessa [3] generalized the results of [8]. Rhoades [6] and Taniguchi [7] have further extended, generalized the results of [8] for a pair of maps in metric spaces. Most of these results do not ensure the uniqueness of the fixed point. Popa [4] established a supplementary condition, in which the fixed point is unique, and improved some results for expansion mappings in metric spaces.

Motivated by the measures of nearness between two or more objects with respect to a specific property, called the parameter of the nearness, Dhage [2] introduced a new structure of a generalized metric spaces (or D-metric spaces) and proved some fixed point theorems. Generalized metric D is a slight variant of 2-metric.

In this paper we prove some common fixed point theorems for expansion maps, in which the fixed point is unique, in the setting of generalized metric spaces. Our results generalize several results established in [3], [4], [6]–[8].

2. Results

Throughout the discussion (X, D) stands for D-metric space. For terminologies and basic properties of D-metric spaces, the reader is referred to [2].

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Let R^+ be the set of all non-negative real numbers, following Khan-Khan and Sessa [3], let Φ be the collection of all functions $\phi : (R^+)^3 \to R^+$ satisfying:

(a) ϕ is lower semi continuous in each co-ordinate variable.

(b) Let $u \geq \phi(u, v, v)$ or $u \geq \phi(v, u, v)$. Then $u \geq kv$, for $u, v \in R^+$, where $\phi(1, 1, 1) = k > 1$.

(c) $\phi(u, u, u) > u$, for $u \in R^+ - \{0\}$.

THEOREM 2.1. *Let S and T be surjective, self mappings of a complete and bounded D-metric space X satisfying:*

$$D(Sx, Ty, z) \geq \phi\{D(x, Sx, z), D(y, Ty, z), D(x, y, z)\}, \quad \text{for all } x, y, z \text{ in } X, \text{ where } \phi \in \Phi. \tag{2.1.1}$$

Then S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Since S and T are surjective, we choose x_1 in X such that $Sx_1 = x_0$ and for this x_1 there exists x_2 in X such that $Tx_2 = x_1$. Inductively, we can construct a sequence $\{x_n\}$ in X such that

$$Sx_{2n+1} = x_{2n} \quad \text{and} \quad Tx_{2n+2} = x_{2n+1}; \qquad n = 0, 1, 2, 3, \ldots \tag{2.1.2}$$

Using (2.1.1) for any $m \geq 2$, we have

$$\begin{aligned} D(x_0, x_1, x_m) &= D(Sx_1, Tx_2, x_m) \\ &\geq \phi\{D(x_1, Sx_1, x_m), D(x_2, Tx_2, x_m), D(x_1, x_2, x_m)\} \\ &= kD(x_1, x_2, x_m) \quad \text{[by (b)]}. \end{aligned}$$

Similarly for any $m \geq 3$, we have

$$\begin{aligned} D(x_1, x_2, x_m) &= D(Sx_3, Tx_2, x_m) \\ &\geq \phi\{D(x_3, Sx_3, x_m), D(x_2, Tx_2, x_m), D(x_3, x_2, x_m)\} \\ &= kD(x_2, x_3, x_m). \end{aligned}$$

Therefore

$$D(x_2, x_3, x_m) \leq (1/k)D(x_1, x_2, x_m) \leq (1/k)^2 D(x_0, x_1, x_m).$$

Inductively for any $m \geq n + 1$, we have

$$D(x_n, x_{n+1}, x_m) \leq (1/k)^n D(x_0, x_1, x_m) \leq (1/k)^n M.$$

(Since X is bounded, there exists $M > 0$ such that $D(x, y, z) \leq M$ $\forall x, y, z \in X$).

Now for $p, t \in N$, we have

$$
\begin{aligned}
&D(x_n, x_{n+p}, x_{n+p+t}) \\
&\quad \le D(x_n, x_{n+1}, x_{n+p+t}) + D(x_n, x_{n+p}, x_{n+1}) + D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\
&\quad \le 2(1/k)^n M + D(x_{n+1}, x_{n+p}, x_{n+p+t}) \\
&\quad \le 2[(1/k)^n + (1/k)^{n+1}]M + D(x_{n+2}, x_{n+p}, x_{n+p+t}) \le \cdots \\
&\quad \le 2\sum_{i=n}^{n+p+t} (1/k)^i M < \{(1/k)^n/(1-1/k)\}2M \to 0, \qquad \text{as } n \to \infty.
\end{aligned}
$$

This shows that $\{x_n\}$ is a Cauchy sequence in X. By completeness of X, $\{x_n\}$ and also its subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to some z in X.

Since S and T are surjective, there exists u and v in X such that $z = Su$ and $z = Tv$. Using (2.1.1), we have

$$
\begin{aligned}
D(x_{2n}, z, z) &= D(Sx_{2n+1}, Tv, z) \\
&\ge \phi\{D(x_{2n+1}, Sx_{2n+1}, z), D(v, Tv, z), D(x_{2n+1}, v, z)\}.
\end{aligned}
$$

Letting $n \to \infty$ and using (b), we get $0 \ge kD(z, v, z)$, yields $z = v$. Similarly $z = u$. Hence z is the common fixed point of S and T in X.

Now for uniqueness of z, let z' be another common fixed point of S and T then from (2.1.1) and (c), we have $D(z, z, z') > D(z, z, z')$, yields $z = z'$. This completes the proof.

The following corollary of Theorem 2.1 of this paper, extends Theorem 1 of Taniguchi [7] to D-metric space.

COROLLARY 2.2. *Let S and T be surjective, self mappings of a complete and bounded D-metric space X satisfying:*

(2.2.1) $$D(Sx, Ty, z) \ge aD(x, Sx, z) + bD(y, Ty, z) + cD(x, y, z), \quad \text{for all } x, y, z \text{ in } X,$$

where a, b, c are non-negative numbers with $a + b + c > 1$ and $a < 1$, $b < 1$. Then S and T have a unique common fixed point in X.

Proof. If we set $k = a+b+c$ and $\phi(r, s, t) = ar+bs+ct$ for all $r, s, t \in R^+$. If $u \ge \phi(u, v, v) = au+bv+cv$ then $u \ge \{(b+c)/(1-a)\}v = (a+b+c)v = kv$, similarly if $u \ge \phi(v, u, v)$ then $u \ge kv$ for some $u, v \in R^+$. Also $\phi(u, u, u) = (a + b + c)u > u$. Therefore $\phi \in \Phi$. Hence from Theorem 2.1, the proof of the corollary is complete.

Now suppose that Φ' be the family of all functions $\phi' : (R^+)^3 \to R^+$ satisfying (a), (c) and

(d) Let $u \geq \phi'(u,v,v)$ or $u \geq \phi'(v,u,v)$ then $u \geq kv$, for $u, v \in R^+ - \{0\}$, where $\phi'(1,1,1) = k > 1$. Here we observe that $\Phi \subset \Phi'$ and if $\phi' : (R^+)^3 \to R^+$ is defined as $\phi'(r,s,t) = k \min\{r,s,t\}$ for all $r,s,t \in R^+$, $k > 1$. If $u \geq \phi'(u,v,v) = k \min\{u,v,v\}$ then $u \geq kv$, similarly $u \geq \phi'(v,u,v)$ then $u \geq kv$ for some $u, v \in R^+ - \{0\}$. Also $\phi(u,u,u) = k \min\{u,u,u\} = ku > u$. Thus $\phi' \in \Phi'$ but $\phi' \notin \Phi$. Hence Φ' is strictly larger than Φ.

THEOREM 2.3. *Let S and T be continuous, surjective, self mappings of a complete and bounded D-metric space X satisfying:*

(2.3.1) $$D(Sx,Ty,z) \geq \phi'\{D(x,Sx,z), D(y,Ty,z), D(x,y,z)\},$$ *for all $x, y, z \in X$,*

where $\phi' \in \Phi'$. Then S and T have a unique common fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X defind by (2.1.2). As in the proof of Theorem 2.1, $\{x_n\}$ is a Cauchy sequence in X and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = z$$

for some z in X. Now by continuity of S and T, we have $Sx_{2n+1} = x_{2n} \to S_z = z$ and $Tx_{2n+1} = x_{2n+1} = Tz = z$, as $n \to \infty$. Hence as in the proof of Theorem 2.1, z is the unique common fixed point of S and T. This completes the proof.

The following corollary of Theorem 2.3 of this paper, extends Theorem 2 of Taniguchi [7] to D-metric space.

COROLLARY 2.4. *Let S and T be continuous, surjective, self mappings of a complete and bounded D-metric space X satisfying:*

(2.4.1) $$D(Sx,Ty,z) \geq k \min\{D(x,Sx,z), D(y,Ty,z), D(x,y,z)\},$$ *for all x, y, z in X,*

where $k > 1$. Then S and T have a unique common fixed point in X.

Proof. If we set $\phi'(r,s,t) = k \min\{r,s,t\}$, for all $r,s,t \in R^+$, $k > 1$. If $u \geq \phi'(u,v,v) = k \min\{u,v,v\}$ then $u \geq kv$, similarly if $u \geq \phi'(v,u,v)$ then $u \geq kv$ for some $u, v \in R^+ - \{0\}$. Also $\phi'(u,u,u) = k \min\{u,u,u\} = ku > u$. Therefore $\phi \in \Phi'$. Hence from Theorem 2.3, the proof of the corollary is complete.

Following Boyd and Wong [1], let Ψ be the collection of all functions $\psi : R^+ \to R^+$ such that

(e) ψ is upper semi continuous.
(f) ψ is non decreasing.
(g) $\psi(t) < t$ for all $t > 0$.
(h) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$.

THEOREM 2.5. *Let S and T be continuous, surjective, self mappings of a complete and bounded D-metric space X satisfying:*

$$(2.5.1) \quad \psi(D(Sx,Ty,z)) \geq \min\{D(x,Sx,z), D(y,Ty,z), D(x,y,z)\}, \quad \text{for all } x,\ y,\ z \text{ in } X,$$

where $\psi \in \Psi$. Then S and T have a unique common fixed point in X.

Proof. Let $\{x_n\}$ be a sequence in X defined by (2.1.2). Using (2.5.1) for any $m \geq 2$, we have

$$\begin{aligned} \psi(D(x_0,x_1,x_m)) &= \psi(D(Sx_1,Tx_2,x_m)) \\ &\geq \min\{D(x_1,Sx_1,x_m), D(x_2,Tx_2,x_m), D(x_1,x_2,x_m)\} \\ &= D(x_1,x_2,x_m). \end{aligned}$$

Similarly for any $m \geq 3$, we have

$$\psi(D(x_1,x_2,x_m)) \geq D(x_2,x_3,x_m).$$

Therefore $D(x_2,x_3,x_m) \leq \psi(D(x_1,x_2,x_m)) \leq \psi^2(D(x_0,x_1,x_m))$. Inductively for any $m \geq n+1$, we have $D(x_n,x_{n+1},x_m) \leq \psi^n(D(x_0,x_1,x_m)) \leq \psi^n(M)$. (Since X is bounded).

Now for $p,t \in N$, we have

$$\begin{aligned} &D(x_n,x_{n+p},x_{n+p+t}) \\ &\quad \leq D(x_n,x_{n+1},x_{n+p+t}) + D(x_n,x_{n+p},x_{n+1}) + D(x_{n+1},x_{n+p},x_{n+p+t}) \\ &\quad \leq 2\psi^n(M) + D(x_{n+1},x_{n+2},x_{n+p+t}) + D(x_{n+1},x_{n+p},x_{n+2}) \\ &\qquad + D(x_{n+2},x_{n+p},x_{n+2}) \\ &\qquad \leq 2[\psi^n(M) + \psi^{n+1}(M)] + D(x_{n+2},x_{n+p},x_{n+p+t}) \leq \cdots \\ &\qquad \leq 2\sum_{i=n}^{n+p+t} \psi^i(M). \end{aligned}$$

From (h), it follows that $\{x_n\}$ is a Cauchy sequence in X. By completeness of X,$\{x_n\}$ and also its subsequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ converge to some z in X. Hence as in the proof of Theorem 2.3, z is the common fixed point of S and T.

Now for uniqueness of z, let z' be another common fixed point of S and T then from (2.5.1), we have $\psi(D(z,z,z')) = \psi(D(Sz,Tz,z')) \geq D(z,z,z')$, yields z=z'. This completes the proof.

REMARK 2.1. For $S = T$, the results of this paper generalize some results of Khan-Khan and Sessa [3], Wang-Li-Gao and Is'eki [8] to D-metric spaces.

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