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FREE ACTIONS OF SEMIPRIME RINGS WITH INVOLUTION INDUCED BY A DERIVATION

Abstract. Let R be an associative ring. An element $a \in R$ is said to be dependent of a mapping $F : R \rightarrow R$ in case $F(x)a = ax$ holds for all $x \in R$. A mapping $F : R \rightarrow R$ is called a free action in case zero is the only dependent element of F . In this paper free actions of semiprime $*$ -rings induced by a derivation are considered. We prove, for example, that in case we have a derivation $D : R \rightarrow R$, where R is a semiprime $*$ -ring, then the mapping F defined by $F(x) = D(x^*) + D(x)^*$, $x \in R$, is a free action. It is also proved that any Jordan $*$ -derivation on a 2-torsion free semiprime $*$ -ring is a free action.

This research has been motivated by the work of Laradji and Thaheem [8] and is a continuation of [20]. Throughout, R will represent an associative ring with center $Z(R)$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that a ring R is prime if $aRb = (0)$ implies that $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $x \rightarrow x^*$ on a ring R is called involution in case $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$ -ring. An additive mapping $D : R \rightarrow R$ is called a derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. A derivation D is inner in case there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. Following [8], an element $a \in R$ is said to be dependent element of a mapping $F : R \rightarrow R$ if $F(x)a = ax$ holds for all $x \in R$. A mapping $F : R \rightarrow R$ is called a free action in case zero is the only dependent element of F . It is easy to see that in semiprime rings there are no nonzero nilpotent dependent elements (see [8]). Depen-

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dent elements were implicitly used by Kallman [7] to extend the notion of free action of automorphisms of abelian von Neumann algebras of Murray and von Neumann [9, 10]. They were later on introduced by Choda, Kasahara and Nakamoto [3]. Several other authors have been studying dependent elements in operator algebras (see [4] and [5]). A brief account of dependent elements in W^* -algebras has been also appeared in the book of Stratila [11]. Some results concerning dependent elements of some mappings related to derivations and automorphisms on prime and semiprime rings can be found in [19]. It is our aim in this paper to investigate elements which are dependent of some additive mappings on semiprime $*$ -ring.

In [19] Vukman and Kosi-Ulbl have proved that the mapping $x \mapsto x^* + x$ is a free action on a semiprime $*$ -ring. The question arises if the same is true in case we have the mapping $x \mapsto x^* - x$ on a semiprime $*$ -ring. The mapping $x \mapsto x^* - x$ is a special case of so-called Jordan $*$ -derivations. The concept of Jordan $*$ -derivation appears naturally in the theory of representability of quadratic form by bilinear form (Halperin problem). For results concerning representability of quadratic form by bilinear form we refer to [12, 13, 15–18]. An additive mapping $D : R \rightarrow R$, where R is an arbitrary ring is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$. Obviously, any derivation is a Jordan derivation. The converse is in general not true. It is well known that any Jordan derivation on a 2-torsion free semiprime ring is a derivation (see [1] and [6]). An additive mapping $D : R \rightarrow R$, where R is an arbitrary $*$ -ring is called a $*$ -derivation in case $D(xy) = D(x)y^* + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan $*$ -derivation if $D(x^2) = D(x)x^* + xD(x)$ is fulfilled for all $x \in R$. For some results concerning Jordan $*$ -derivations we refer to [2] and [14]. One might expect that any Jordan $*$ -derivation on a 2-torsion free semiprime $*$ -ring is a $*$ -derivation, but this is not the case. It is easy to prove that there are no nonzero $*$ -derivations on a 2-torsion free non-commutative prime $*$ -ring (see [2] for the details). It is trivial to see that in any $*$ -ring R the mappings $x \mapsto x^* - x$ and $x \mapsto ax^* - xa$, where $a \in R$ is a fixed element, are Jordan $*$ -derivations.

We are ready for our first result.

THEOREM 1. *Let R be a 2-torsion free semiprime $*$ -ring and let $D : R \rightarrow R$ be a Jordan $*$ -derivation. In this case D is a free action.*

Proof. Assume that $a \in R$ is an element such that

$$(1) \quad D(x)a = ax,$$

for all $x \in R$. We prove that D is a free action by showing that $a = 0$.

Putting $xy + yx$ for x in the above relation we obtain

$$(D(x)y^* + D(y)x^* + xD(y) + yD(x))a = a(xy + yx), \quad x, y \in R.$$

According to (1) one can replace in the above relation $D(y)a$ and $D(x)a$ by ay and ax , respectively, which gives

$$(2) \quad D(x)y^*a + D(y)x^*a + [x, a]y + [y, a]x = 0, \quad x, y \in R.$$

Putting $x = y = a^*$ in the above relation we obtain

$$2D(a^*)a^2 + 2[a^*, a]a^* = 0.$$

Replacing $D(a^*)a$ by aa^* in the above relation and using the fact that R is 2-torsion free, we arrive at

$$(3) \quad aa^*a + a^*aa^* - aa^{*2} = 0.$$

Putting $x = y = a$ in (2) we obtain

$$(4) \quad D(a)a^*a = 0.$$

Left multiplication of the relation (3) by $D(a)$ gives because of (4)

$$D(a)aa^*a - D(a)aa^{*2} = 0.$$

In the above relation one can replace $D(a)a$ by a^2 which gives

$$a^2(a^*a - a^{*2}) = 0.$$

Left multiplication of the above relation by $D(x)$ gives because of (1)

$$axa(a^*a - a^{*2}) = 0, \quad x \in R.$$

Putting $(a^*a - a^{*2})x$ for x in the above relation we arrive at

$$a(a^*a - a^{*2})xa(a^*a - a^{*2}) = 0, \quad x \in R,$$

which gives

$$(5) \quad a(a^*a - a^{*2}) = 0.$$

According to (5) the relation (3) reduces to

$$a^*aa^* = 0.$$

Now we have $0 = (a^*aa^*)^* = aa^*a$. We have therefore

$$(6) \quad aa^*a = 0.$$

Because of (6) the relation (5) reduces to $aa^{*2} = 0$. Now we have $0 = (aa^{*2})^* = a^2a^*$. Thus we have

$$a^2a^* = 0.$$

Left multiplication of the above relation by $D(x)$ gives $axaa^* = 0$, $x \in R$, whence it follows $aa^*xaa^* = 0$, $x \in R$, which gives

$$(7) \quad aa^* = 0.$$

Now right multiplication of (1) by a^* gives because of (7)

$$axa^* = 0, \quad x \in R.$$

Left multiplication of the above relation by a^* and right multiplication by a gives $a^*axa^*a = 0$, $x \in R$, whence it follows

$$(8) \quad a^*a = 0.$$

Putting $x = a$ and $y = a^*$ in the relation (2) we obtain because of (7) and (8) $D(a)a^2 = 0$ which gives $a^3 = 0$, which means that also $a = 0$. The proof of the theorem is complete. ■

COROLLARY 1. *Let R be a 2-torsion free semiprime * -ring. In this case the mappings $x \mapsto x^* - x$ and $x \mapsto ax^* - xa$, where $a \in R$ is a fixed element, are free actions.*

Proof. Both mappings are Jordan * -derivations.

For the proof of Theorem 2 we need the following lemma.

LEMMA 1. *Let R be a semiprime ring and let $a, b, c \in R$ be fixed elements. If $axa + bxc = 0$ holds for all $x \in R$ then $axb = bxa$ is fulfilled for all $x \in R$.*

Proof. Putting in the relation

$$(9) \quad axa + bxc = 0, \quad x \in R$$

ybx for x we obtain

$$(10) \quad aybxa + bybxc = 0, \quad x, y \in R.$$

Left multiplication of the relation (9) by by gives

$$(11) \quad byaxa + bybxc = 0, \quad x, y \in R.$$

Subtracting (11) from (10) we arrive at

$$(12) \quad (ayb - bya)xa = 0, \quad x, y \in R.$$

Putting first in the above relation xy for x then multiplying the relation (12) from the right side by yb and subtracting the relations so obtained, we obtain $(ayb - bya)x(ayb - bya) = 0$, $x, y \in R$, whence it follows $ayb = bya$, $y \in R$. The proof of the lemma is complete. ■

THEOREM 2. *If R is a semiprime * -ring and $G : R \rightarrow R$ an additive mapping satisfying the relation $G(xy) = G(y)x^* + y^*G(x)$, for all pairs $x, y \in R$, then G is a free action on R .*

Proof. Assume that $a \in R$ is such that

$$(13) \quad G(x)a = ax$$

holds for all $x \in R$. It is our aim to prove that $a = 0$. Putting xy for x in the above relation we obtain

$$G(y)x^*a + y^*G(x)a = axy, \quad x, y \in R.$$

According to (13) one can replace in the above relation $G(x)a$ with ax , which gives

$$G(y)x^*a + y^*ax = axy, \quad x, y \in R.$$

The substitution $x = a^*$ in the above relation gives because of (13)

$$(14) \quad axa + x^*aa^* = aa^*x, \quad x \in R.$$

Let x be xa in the above relation. We have

$$(15) \quad axa^2 + a^*x^*aa^* = aa^*xa, \quad x \in R.$$

Right multiplication of (14) by a gives

$$(16) \quad axa^2 + x^*aa^*a = aa^*xa, \quad x \in R.$$

Subtracting (16) from (15) we arrive at

$$a^*xaa^* = xaa^*a, \quad x \in R.$$

Left multiplication of the above relation by a gives

$$(17) \quad aa^*xaa^* = axaa^*a, \quad x \in R.$$

From the above relation and Lemma 1 one obtains

$$(18) \quad aa^*xa = axaa^*, \quad x \in R.$$

According to (18) one can replace in the right side of the relation (17) $axaa^*$ by aa^*xa , which gives

$$(19) \quad aa^*x(aa^* - a^2) = 0, \quad x \in R.$$

Similarly one can replace on the right side of the relation (16) aa^*xa by $axaa^*$, which gives

$$ax(a^2 - aa^*) + x^*aa^*a = 0, \quad x \in R.$$

The substitution a^*x for x in the above relation gives $aa^*x(a^2 - aa^*) + x^*a^2a^*a = 0$, $x \in R$, which reduces because of (19) to $xa^2a^*a = 0$, $x \in R$, whence it follows

$$a^2a^*a = 0.$$

Left multiplication of the above relation by $G(x)$ gives $axaa^*a = 0$, $x \in R$, whence it follows

$$aa^*axaa^*a = 0, \quad x \in R,$$

and finally

$$aa^*a = 0.$$

Repeating again the same approach one obtains

$$(20) \quad a^*a = 0.$$

Because of (20) the relation (17) reduces to $aa^*xaa^* = 0$, $x \in R$, whence it follows

$$(21) \quad aa^* = 0.$$

Because of (21) the relation (14) reduces to $axa = 0$, $x \in R$, which gives finally $a = 0$. The proof of the theorem is complete. ■

COROLLARY 2. *Let R be a semiprime $*$ -ring and let $D : R \rightarrow R$ be a derivation. We associate with D four mappings $F_D^{(1)}, F_D^{(2)}, F_D^{(3)}, F_D^{(4)} : R \rightarrow R$ defined by $x \mapsto D(x^*)$, $x \mapsto D(x)^*$, $x \mapsto D(x^*) + D(x)^*$ and $x \mapsto D(x^*) - D(x)^*$, respectively. Then $F_D^{(1)}, F_D^{(2)}, F_D^{(3)}$, and $F_D^{(4)}$ are free actions on R .*

Proof. Denote by $G : R \rightarrow R$ any of the mappings $F_D^{(1)}, F_D^{(2)}, F_D^{(3)}$, or $F_D^{(4)}$. It is easy to see that G satisfies the relation $G(xy) = G(y)x^* + y^*G(x)$, for all pairs $x, y \in R$ which means that all assumptions of Theorem 2 are fulfilled. ■

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