Urszula Bednarz

STABILITY OF THE HADAMARD PRODUCT OF k -UNIFORMLY CONVEX AND k -STARLIKE FUNCTIONS IN CERTAIN NEIGHBOURHOOD

Abstract. For a constant $k \in [0, \infty)$ a normalized function f , analytic in the unit disk, is said to be k -uniformly convex if $\operatorname{Re}(1 + zf''(z)/f'(z)) > k|zf''(z)/f'(z)|$ at any point in the unit disk. The class of k -uniformly convex functions is denoted $k\text{-}\mathcal{UCV}$ (cf. [3]). The function g is said to be k -starlike if $g(z) = zf'(z)$ and $f \in k\text{-}\mathcal{UCV}$.

For analytic functions f, g , where $f(z) = z + a_2z^2 + \dots$ and $g(z) = z + b_2z^2 + \dots$, the convolution is defined as follows:

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Generalized neighbourhood is defined as:

$$TN_{\delta}(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{H} : \sum_{n=2}^{\infty} T_n |a_n - b_n| \leq \delta \right\}.$$

In this note some properties of the convolution of k -uniformly convex and k -starlike functions are investigated.

1. Introduction and definitions

Let $\mathcal{H}$ denote the class of functions f analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

and let $\mathcal{S}$ denote the subclass of functions in $\mathcal{H}$ which are also *univalent* in $\mathcal{U}$. Further, for $k \in [0, \infty)$, let $k\text{-}\mathcal{UCV}$ denote the subclass of $\mathcal{S}$, consisting

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of the k -uniformly convex functions in $\mathcal{U}$ define as follows:

$$(1.2) \quad k\text{-}\mathcal{UCV} := \left\{ f \in \mathcal{S} : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > k \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in \mathcal{U} \right\},$$

and k - $\mathcal{ST}$ be the subclasses of $\mathcal{S}$ consisting of the k -starlike functions in $\mathcal{U}$, and defined as:

$$(1.3) \quad k\text{-}\mathcal{ST} := \left\{ f \in \mathcal{S} : \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \mathcal{U} \right\}.$$

Observe that the classes $k\text{-}\mathcal{UCV}$ and $k\text{-}\mathcal{ST}$ are related by the classical Alexander theorem, that is known also as an equivalence between usual class of convex functions $\mathcal{CV}$ and the class of starlike functions $\mathcal{ST}$. Also note, that the class $k\text{-}\mathcal{UCV}$ was defined pure geometrically as a subclass of univalent functions, that map each circular arc contained in the unit disc $\mathcal{U}$ with a center ξ , $|\xi| \leq k$ ($0 \leq k < \infty$), onto a convex arc. Classes $k\text{-}\mathcal{UCV}$ and $k\text{-}\mathcal{ST}$ were introduced by Kanas and Wiśniowska ([3], [5]) and studied in the series of papers ([1],[3],[4],[5],[6]). Some of properties of $k\text{-}\mathcal{UCV}$ and $k\text{-}\mathcal{ST}$, in particular, concerning the stability of the Hadamard product, integral convolution and integral transformation were studied by Bednarz and Kanas in [1],[2].

The *convolution*, or *Hadamard product*, of two power series $f(z) = z + a_2z^2 + \dots$, and $g(z) = z + b_2z^2 + \dots$, convergent in $\mathcal{U}$ is the function $h = f * g$ with the power series

$$(1.4) \quad h(z) = (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Convolution has the algebraic properties of ordinary multiplication, the geometric series $K(z) = z + z^2 + \dots = \frac{z}{1-z}$ acts as the identity element under convolution: $(f * K)(z) = f(z)$ for any $f \in \mathcal{H}$, $z \in \mathcal{U}$.

Note that, if I denotes $I(z) \equiv z$ then

$$(f * I)(z) = I(z).$$

Various properties of Hadamard product were studied by several authors, e.g. Robertson ([13]), Ruscheweyh and Sheil-Small ([12]).

In accordance to Ruscheweyh ([10]), let $\mathcal{V}^*$ denote the *dual set* of $\mathcal{V} \subset \mathcal{H}$ (cf. [6]). Then

$$(1.5) \quad \mathcal{V}^* = \{g \in \mathcal{H} : (f * g)(z) \neq 0, \quad \forall f \in \mathcal{V}, \quad \forall z \in \mathcal{U}_0\},$$

where $\mathcal{U}_0 = \{z : 0 < |z| < 1\}$, and $\mathcal{V}^{**} = (\mathcal{V}^*)^*$ denote the *second dual*, which is the smallest of all dual classes containing $\mathcal{V}$. The *duality principle* states, that for compact and complete $\mathcal{V}$ the closed convex hull of $\mathcal{V}$ is the same as the closed convex hull of $\mathcal{V}^{**}$. This means, that under weak conditions

on $\mathcal{V}$, many extremal problems in $\mathcal{V}$ are solved in $\mathcal{V}^{**}$ and can be obtained by unified approach.

Dual sets for the classes $k\text{-}\mathcal{ST}$ and $k\text{-}\mathcal{UCV}$ were found by Kanas, Wiśniowska (cf. [3], [6]). Let us denote the dual set for $k\text{-}\mathcal{ST}$ by $\mathcal{B}$ and for $k\text{-}\mathcal{UCV}$ by $\mathcal{G}$. Then

$$(1.6) \quad f \in k\text{-}\mathcal{ST} \iff \frac{(f * g)(z)}{z} \neq 0, \quad \forall h \in \mathcal{B}, \quad \forall z \in \mathcal{U},$$

and

$$(1.7) \quad f \in k\text{-}\mathcal{UCV} \iff \frac{(f * g)(z)}{z} \neq 0, \quad \forall h \in \mathcal{G}, \quad \forall z \in \mathcal{U},$$

respectively (cf [6]). For $h(z) = z + c_2 z^2 + \dots \in \mathcal{B}$ we have the following estimates (cf [6]),

$$(1.8) \quad |c_n| \leq n + (n-1)k, \quad n \geq 2,$$

and if $h \in \mathcal{G}$, then

$$(1.9) \quad |c_n| \leq n[n + (n-1)k], \quad n \geq 2.$$

Further, sufficient conditions to be in $k\text{-}\mathcal{ST}$ and $k\text{-}\mathcal{UCV}$ are (cf.[6])

$$(1.10) \quad \text{if } \sum_{n=2}^{\infty} [n + (n-1)k] |a_n| \leq 1 \text{ then } f \in k\text{-}\mathcal{ST},$$

$$(1.11) \quad \text{if } \sum_{n=2}^{\infty} n[n + (n-1)k] |a_n| \leq 1 \text{ then } f \in k\text{-}\mathcal{UCV}$$

respectively. For $\delta \geq 0$ Ruscheweyh ([11]) defined N_δ neighbourhood of a function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ by

$$(1.12) \quad N_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{H} : \sum_{n=2}^{\infty} n |a_n - b_n| \leq \delta \right\}.$$

By $N_\delta(A)$, $A \subset \mathcal{H}$, we denote the union of all neighbourhoods $N_\delta(f)$ with f ranging over the class A . The quantity $\sum_{n=2}^{\infty} n |a_n - b_n|$ can be regarded as the distance between two functions f and g in a some subclass of $\mathcal{H}$, equipped with the pre-norm of functions $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ defined as $\|F\| = \sum_{n=2}^{\infty} n |A_n|$. Ruscheweyh proved certain inclusions for mentioned above neighbourhoods, in particular that $N_{1/4}(f) \subset \mathcal{ST}$ holds for all $f \in \mathcal{CV}$. Problem of neighbourhoods in various classes of functions was also studied in papers [1], [6], [9], [15], [16].

Sheil-Small and Silvia (cf.[14]) have introduced a more general concept of a T_δ -neighbourhood:

$$(1.13) \quad TN_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{H} : \sum_{n=2}^{\infty} T_n |a_n - b_n| \leq \delta \right\},$$

where $T = \{T_n\}_{n=2}^\infty$ is a sequence of positive numbers. Hence for $T_n = n$, $n \geq 2$ we obtain $TN_\delta(f) = N_\delta(f)$. The problem of TN_δ neighbourhood was considered by Bednarz and Kanas in [1] mostly for $t_n = n^2$. They proved, that $TN_{1/2}(f) \subset ST$ for $f \in \mathcal{CV}$. Observe, that $1/2$ -the radius of TN_δ neighbourhood is greater than the radius $1/4$, obtained by Ruscheweyh for N_δ neighbourhood. All depends on the choice of T_n .

DEFINITION. Assume that A, B, C are subclasses of the class $\mathcal{H}$ with

$$A * B = \{f * g : f \in A \text{ and } g \in B\}$$

(cf. eg. [12], [15]). Let $A * B \subset C$, then the convolution (1.4) is called *T-C-stable on the pair of classes (A, B)* if there exists $\delta > 0$, such that $TN_\delta(A) * TN_\delta(B) \subset C$ (cf. [7], [8]).

The constant δ , which characterizes the stability of Hadamard product is defined as :

$$(1.14) \quad \delta(A * B, C) = \sup\{\delta : TN_\delta(A) * TN_\delta(B) \subset C\}.$$

If the value (1.14) is positive, then there exists a neighbourhood of the class A and B mapped by the convolution into C . The stability of the Hadamard product can be regarded as a problem of preserving a product of the topology of neighbourhoods in A and B onto the product $A * B$.

Problem of the stability of the inclusion for the Hadamard product in the classes of univalent, starlike and convex functions in N_δ and also TN_δ neighbourhoods was considered by Nezhmetdinov ([7], [8]). Numerous results concerning the stability of the Hadamard product were obtained by Kanas and Bednarz ([1], [2], [6]). In particular in [6] Kanas proved, that

$$(1.15) \quad N_{1/4(k+1)}(f) \subset k-ST \quad \text{for } f \in k-UCV,$$

that is a k -uniform version of the Ruscheweyh result.

In the present article we study a problem of the stability of convolution in TN_δ neighbourhood for the class $k-UCV$ and $k-ST$. We give some conditions, in the form of restrictions onto a sequence T , for convolution to be $T - k-UCV$ or $T - k-ST$ stable. We also find estimates for constants δ connected with stability. Although we do not know, if they are sharp.

2. Stability of the Hadamard product

In the sequel the following notation will be used : $f(z) = z + \sum_{n=2}^\infty a_n z^n$, $f_0(z) = z + \sum_{n=2}^\infty a_{0n} z^n$, $g(z) = z + \sum_{n=2}^\infty b_n z^n$, $g_0(z) = z + \sum_{n=2}^\infty b_{0n} z^n$, and $A_n = O(B_n)$, which means, that the expression A_n/B_n is limited as $n \rightarrow \infty$.

THEOREM. If $\frac{1}{T_n} = O(\frac{1}{n})$ as $n \rightarrow \infty$, then the convolution (1.4) is:

1. $T - k\mathcal{UCV}$ - stable on $(\{I\}, \{I\})$, besides

$$(2.1) \quad \delta(\{I\} * \{I\}, k\mathcal{UCV}) \geq \inf \left\{ T_n \frac{1}{n\sqrt{k+1}} : n \geq 2 \right\},$$

2. $T - k\mathcal{ST}$ - stable on $(\{I\}, \{I\})$, moreover

$$(2.2) \quad \delta(\{I\} * \{I\}, k\mathcal{ST}) \geq \inf \left\{ T_n \frac{\sqrt{2}}{n\sqrt{k+1}} : n \geq 2 \right\},$$

3. $T - k\mathcal{ST}$ - stable on $(k\mathcal{UCV}, \{I\})$, besides for $\delta = \delta_T(k\mathcal{UCV} * \{I\}, k\mathcal{ST})$ the following estimate is valid

$$(2.3) \quad \delta \geq \inf \left\{ \frac{T_n^2}{n(k+1)(T_n + \delta)} : n \geq 2 \right\},$$

4. $T - k\mathcal{ST}$ - stable on $(k\mathcal{UCV}, \mathcal{CV})$, in addition for $\delta = \delta_T(k\mathcal{UCV} * \mathcal{CV}, k\mathcal{ST})$ the following estimate is valid

$$(2.4) \quad \delta \geq \inf \left\{ \frac{T_n^2}{8n(k+1)^2(\delta + 2T_n)} : n \geq 2 \right\}.$$

Proof. For any $f, g, f_0, g_0 \in \mathcal{H}$ and $h \in \mathcal{B}$ (or $h \in \mathcal{G}$), $z \in \mathcal{U}$ we have

$$(2.5) \quad \left| \frac{(f * g * h)(z)}{z} \right| \geq \left| \frac{(f_0 * g_0 * h)(z)}{z} \right| - \left| \frac{(f_0 * (g - g_0) * h)(z)}{z} \right| \\ - \left| \frac{((f - f_0) * g_0 * h)(z)}{z} \right| - \left| \frac{((f - f_0) * (g - g_0) * h)(z)}{z} \right|.$$

Assume, that $\frac{1}{T_n} = O(\frac{1}{n})$ with $n \rightarrow \infty$. This means, that for some $P > 0$ the estimate

$$(2.6) \quad \frac{1}{T_n} \leq \frac{P}{n}$$

is valid, where $n \geq 2$.

CASE (2.1). There also exists, according to (2.6), $M > 0$ such, that $\frac{1}{T_n} \leq \frac{M}{n\sqrt{k+1}}$ for $n \geq 2$.

Hence $\delta_0 = \inf \left\{ T_n \frac{1}{n\sqrt{k+1}} : n \geq 2 \right\} > 0$. Assume, that $0 < \delta \leq \delta_0$.

Let $f_0 = g_0 = I$ and let $f, g \in TN_\delta(I)$. We shall show, that $(f * g * h)(z)/z \neq 0$ for $h \in \mathcal{G}$, what is equivalent to the fact $f * g \in k\mathcal{UCV}$. Observe that, by properties of the Hadamard product we have $(f_0 * (g - g_0) * h)(z) = 0$, $((f - f_0) * g_0 * h)(z) = 0$ and $(f_0 * g_0 * h)(z) = I(z)$.

Moreover, by the fact that $f, g \in TN_\delta(I)$ we obtain $\sum_{n=2}^{\infty} T_n |a_n| \leq \delta$ and $T_n |b_n| \leq \delta$, therefore $|b_n| \leq \frac{\delta}{T_n}$ for $n \geq 2$. Since $\delta \leq \delta_0$ we have $n\sqrt{k+1} \leq \frac{T_n}{\delta}$, and in this case, in view of (1.9), the inequality (2.5) becomes:

$$\left| \frac{(f * g * h)(z)}{z} \right| > 1 - \sum_{n=2}^{\infty} |a_n| |b_n| |c_n| \geq 1 - \sum_{n=2}^{\infty} n[n + (n-1)k] |a_n| |b_n|$$

$$\begin{aligned}
&\geq 1 - (k+1) \sum_{n=2}^{\infty} n^2 |a_n| |b_n| \\
&\geq 1 - (k+1) \sum_{n=2}^{\infty} \frac{\delta}{T_n} \frac{T_n^2}{(k+1)\delta^2} |a_n| \\
&\geq 1 - \frac{1}{\delta} \sum_{n=2}^{\infty} T_n |a_n|,
\end{aligned}$$

that is nonnegative. From this we conclude, that $f * g \in k\text{-UCV}$ and condition $\frac{1}{T_n} = O(\frac{1}{n})$ is sufficient for the convolution to be $T - k\text{-UCV} - \text{stable}$ on $(\{I\}, \{I\})$, moreover $\delta(\{I\} * \{I\}, k\text{-UCV}) \geq \inf \left\{ T_n \frac{1}{n\sqrt{k+1}} : n \geq 2 \right\}$.

CASE (2.2). Because of (2.6) for some $M > 0$ holds $\frac{M\sqrt{2}}{n\sqrt{k+1}}$ and $n \geq 2$. Hence $\delta_0 = \inf \left\{ T_n \frac{\sqrt{2}}{n\sqrt{k+1}} : n \geq 2 \right\} > 0$. Let $0 < \delta \leq \delta_0$, so we obtain $\delta \leq T_n \frac{\sqrt{2}}{n\sqrt{k+1}}$.

Similarly, like in the previous case, we assume that $f_0 = g_0 = I$ and let $f, g \in TN_\delta(I)$, and $h \in \mathcal{B}$. From this we have $\sum_{n=2}^{\infty} T_n |a_n| \leq \delta$ and $|b_n| \leq \frac{\delta}{T_n} \leq \frac{\sqrt{2}}{2\sqrt{k+1}}$ for $n \geq 2$. Then applying (2.5) we have:

$$\begin{aligned}
\left| \frac{(f * g * h)(z)}{z} \right| &> 1 - \sum_{n=2}^{\infty} |a_n| |b_n| |c_n| \geq 1 - \sum_{n=2}^{\infty} [n + (n-1)k] |a_n| |b_n| \\
&\geq 1 - (k+1) \sum_{n=2}^{\infty} n |a_n| |b_n| \\
&\geq 1 - (k+1) \sum_{n=2}^{\infty} \frac{T_n \sqrt{2}}{\delta \sqrt{k+1}} |a_n| |b_n| \\
&\geq 1 - (k+1) \frac{\sqrt{2}}{\delta \sqrt{k+1}} \frac{\sqrt{2}}{2\sqrt{k+1}} \sum_{n=2}^{\infty} T_n |a_n|.
\end{aligned}$$

From this we conclude, that $(f * g * h)(z)/z \neq 0$, which means that $f * g \in k\text{-ST}$ for $\delta \geq \inf \left\{ T_n \frac{\sqrt{2}}{n\sqrt{k+1}} \right\}$.

CASE (2.3) According to our assertion (2.6) there exists $M > 0$ such, that $\frac{1}{T_n} \leq \frac{M}{n(k+1)}$ for all $n \geq 2$. Note, that also $\frac{1}{T_n} \leq M$, $n \geq 2$ is valid.

Let $f_0 \in k\text{-UCV}$ and $g_0 = I$. Also, let $f \in TN_\delta(f_0)$, $g \in TN_\delta(g_0)$, $h \in \mathcal{B}$. The inequality (2.5) takes the form

$$\begin{aligned}
(2.7) \quad \left| \frac{(f * g * h)(z)}{z} \right| &\geq 1 - \left| \frac{(f_0 * (g - g_0) * h)(z)}{z} \right| - \left| \frac{((f - f_0) * (g - g_0) * h)(z)}{z} \right| \\
&> 1 - \sum_{n=2}^{\infty} |a_{0n}| |b_n| |c_n| - \sum_{n=2}^{\infty} |a_n - a_{0n}| |b_n| |c_n|.
\end{aligned}$$

Since $f_0 \in k\text{-}\mathcal{UCV} \subset \mathcal{CV}$, then $|a_{0n}| \leq 1$. By the fact that $g \in TN_\delta(g_0)$ we have $\sum_{n=2}^{\infty} T_n |b_n| \leq \delta$. Similarly, since $f \in TN_\delta(f_0)$ we have $|a_n - a_{0n}| \leq \frac{\delta}{T_n}$.

Now, applying above inequalities to (2.7), we obtain for all $z \in \mathcal{U}$:

$$\begin{aligned} \left| \frac{(f * g * h)(z)}{z} \right| &> 1 - (k+1) \sum_{n=2}^{\infty} n |b_n| (1 + |a_n - a_{0n}|) \\ &\geq 1 - (k+1) \sum_{n=2}^{\infty} n |b_n| \left(1 + \frac{\delta}{T_n}\right) \\ &\geq 1 - \delta M (1 + \delta M). \end{aligned}$$

We see, that for $\delta < \frac{1}{2M}$ the expression $\left| \frac{(f * g * h)(z)}{z} \right|$ does not vanish in the $\mathcal{U}$ for $h \in \mathcal{B}$. Hence the convolution (1.4) is $T - k\text{-}ST$ - stable.

Now suppose, that $0 < \delta \leq \delta_0$, where δ_0 is the unique non-negative root of the equation $\delta = inf \left\{ \frac{T_n^2}{n(k+1)(T_n + \delta)} : n \geq 2 \right\}$.

Then $\delta \leq \frac{T_n^2}{n(k+1)(T_n + \delta)}$ for all $n \geq 2$. Therefore

$$(k+1)n \left(1 + \frac{\delta}{T_n}\right) \leq \frac{T_n}{\delta}, \quad n \geq 2$$

and, as well as above, we have, that $\left| \frac{(f * g * h)(z)}{z} \right| \neq 0$ provided that $h \in \mathcal{B}$ and $z \in \mathcal{U}$. Consequently, $f * g \in k\text{-}ST$ and the convolution (1.4) is $T - k\text{-}ST$ - stable.

CASE (2.4). If the estimate (2.6) holds, that is also true, that for some $M > 0$ we have $\frac{1}{T_n} \leq \frac{M}{n(k+1)}$.

Let $f_0 \in k\text{-}\mathcal{UCV}$, $g_0 \in \mathcal{CV}$ and $f \in TN_\delta(f_0)$, $g \in TN_\delta(g_0)$ and $h \in \mathcal{B}$. Since $f_0 \in k\text{-}\mathcal{UCV}$, $g_0 \in \mathcal{CV}$ we have $f_0 * g_0 \in k\text{-}\mathcal{UCV}$ (cf. [4]). Thus, in view of (1.15) $|(f_0 * g_0 * h)(z)/z| > 1/(4(k+1))$. By the above relation the inequality (2.5) becomes

$$\begin{aligned} (2.8) \quad \left| \frac{(f * g * h)(z)}{z} \right| &> \frac{1}{4(k+1)} - \sum_{n=2}^{\infty} |a_{0n}| |b_n - b_{0n}| |c_n| \\ &\quad - \sum_{n=2}^{\infty} |b_{0n}| |a_n - a_{0n}| |c_n| - \sum_{n=2}^{\infty} |a_n - a_{0n}| |b_n - b_{0n}| |c_n|. \end{aligned}$$

The coefficients of $f_0 \in k\text{-}\mathcal{UCV} \subset \mathcal{CV}$ satisfy inequality $|a_{0n}| \leq 1$ for $n \geq 2$ then, by (1.8) we have:

$$(2.9) \quad \sum_{n=2}^{\infty} |a_{0n}| |b_n - b_{0n}| |c_n| \leq (k+1) \sum_{n=2}^{\infty} n |b_n - b_{0n}|.$$

Similarly $g_0 \in \mathcal{CV}$, then $|b_{0n}| \leq 1$, and

$$(2.10) \quad \sum_{n=2}^{\infty} |b_{0n}| |a_n - a_{0n}| |c_n| \leq (k+1) \sum_{n=2}^{\infty} n |a_n - a_{0n}|.$$

Finally

$$(2.11) \quad \begin{aligned} \sum_{n=2}^{\infty} |a_n - a_{0n}| |b_n - b_{0n}| |c_n| &\leq (k+1) \sum_{n=2}^{\infty} n |a_n - a_{0n}| |b_n - b_{0n}| \\ &\leq \frac{1}{2} (k+1) \sum_{n=2}^{\infty} n (|a_n - a_{0n}|^2 + |b_n - b_{0n}|^2). \end{aligned}$$

The function $f \in TN_{\delta}(f_0)$, then $\sum_{n=2}^{\infty} T_n |a_n - a_{0n}| \leq \delta$ and therefore $|a_n - a_{0n}| \leq \frac{\delta}{T_n}$ for $n \geq 2$. Similarly, since $g \in TN_{\delta}(g_0)$ we have $\sum_{n=2}^{\infty} T_n |b_n - b_{0n}| \leq \delta$ and therefore $|b_n - b_{0n}| \leq \frac{\delta}{T_n}$ for $n \geq 2$.

By virtue of (2.9), (2.10), and (2.11), and also the above conditions the inequality (2.8) gives

$$\begin{aligned} \left| \frac{(f * g * h)(z)}{z} \right| &> \frac{1}{4(k+1)} - (k+1) \sum_{n=2}^{\infty} n [|a_n - a_{0n}| + |b_n - b_{0n}|] \\ &\quad + \frac{\delta}{2T_n} (|a_n - a_{0n}| + |b_n - b_{0n}|) \\ &\geq \frac{1}{4(k+1)} - (k+1) \sum_{n=2}^{\infty} n \left(1 + \frac{\delta}{2T_n} \right) (|a_n - a_{0n}| + |b_n - b_{0n}|) \\ &\geq \frac{1}{4(k+1)} - \delta \sup \left\{ 2(k+1)n \left(T_n + \frac{\delta}{2} \right) \frac{1}{T_n^2} : n \geq 2 \right\}. \end{aligned}$$

From virtue of the above assumptions, we have

$$2(k+1)n \left(T_n + \frac{\delta}{2} \right) \frac{1}{T_n^2} \leq M \left(2 + \frac{1}{n(k+1)} \right), \quad n \geq 2,$$

and thus $\delta > 0$ can be found such, that $\frac{(f * g * h)(z)}{z} \neq 0$ for all $f \in k\text{-UCV}$, $g \in \mathcal{CV}$, $h \in \mathcal{B}$ and $z \in \mathcal{U}$.

It implies the T - k - ST -stability of the convolution (1.4) on $(k\text{-UCV}, \mathcal{CV})$, provided that the condition $\frac{1}{T_n} = O(\frac{1}{n})$ as $n \rightarrow \infty$ holds.

REMARK. By setting $T_n = n$, for $n \geq 2$ in the Theorem we have all the results for constants δ (1.14) obtained in [6], and for $T_n = n^2$ some of the results from paper [1].

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DEPARTMENT OF MATHEMATICS
 TECHNICAL UNIVERSITY OF RZESZÓW
 ul. W. Pola 2
 35-959 RZESZÓW, POLAND
 E-mail: ubednarz@prz.edu.pl

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