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ON A GENERALIZATION OF CLOSE-TO-CONVEXITY FOR COMPLEX HOLOMORPHIC FUNCTIONS IN $\mathbb{C}^n$

Abstract. Sveral authors (I. I. Bawrin [1], K. Dobrowolska, I. Dziubiński, P. Liczberski, R. Sitarski [3], [4], [5], [13], S. Gong, S. S. Miller [6], Z. J. Jakubowski and J. Kamiński [8], J. Janiec [10] and others) studied various families of complex holomorphic functions in $\mathbb{C}^n$ and in Banach space, corresponding with famous subclasses of univalent functions. In this paper we study a class of holomorphic functions of n complex variables analogous to the class of close-to-convex functions of one variable considered by M. Biernacki, W. Kaplan and Z. Lewandowski (see [2], [11], [12]).

1. Introduction

Let Ω be a bounded complete circular and convex domain in $\mathbb{C}^n$. If we put

$$\|\omega\| = \inf\{c > 0 : \frac{1}{c}\omega \in \Omega\}, \quad \omega \in \mathbb{C}^n,$$

then $(\mathbb{C}^n, \|\cdot\|)$ is a Banach space, Ω is a unit ball in this space and $\partial\Omega$ is a unit sphere (see e.g. [14]). In the paper we use this norm.

Let $I : \mathbb{C}^n \rightarrow \mathbb{C}$ be the following linear functional

$$I(z) = z_1 + z_2 + \dots + z_n, \quad z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$$

and let

$$\chi(I) = \{z \in \mathbb{C}^n : I(z) \neq 0\}, \quad \gamma(I) = \mathbb{C}^n \setminus \chi(I).$$

Clearly, $\gamma(I)$ is a hyperplane, $\chi(I)$ is dense in $\mathbb{C}^n$ and $\chi(I) \cap \partial\Omega$ is dense in $\partial\Omega$.

Let $H(\Omega)$ denote the class of all holomorphic functions $f : \Omega \rightarrow \mathbb{C}$ of the form

$$(1) \quad f(z) = I(z) + \sum_{m=2}^{\infty} P_{m,f}(z), \quad z \in \Omega,$$

where $P_{m,f}(z) = \frac{1}{m!} D^m f(0)(z, \dots, z)$, and $D^m f(z)$ is the n -th Frechet derivative of the function f at the point z , while $D^m f(0)(w^1, \dots, w^m)$ is the value of $D^m f(z)$ on the vectors $w^1, \dots, w^m \in \mathbb{C}^n$.

There were examined (see [10]) the following families of functions

$$S^*(\Omega) = \{f \in H(\Omega) : f(z) \neq 0, \operatorname{Re} \frac{Df(z)(z)}{f(z)} > 0, z \in \chi(I) \cap \Omega\},$$

$$S^c(\Omega) = \{f \in H(\Omega) : Df(z)(z) \neq 0, \\ \operatorname{Re} \left(1 + \frac{D^2 f(z)(z, z)}{Df(z)(z)} \right) > 0, z \in \chi(I) \cap \Omega\}.$$

In the geometric theory of functions on the complex plane well known is the class S of functions $F(\zeta) = \zeta + a_2 \zeta^2 + \dots$, holomorphic and univalent in the unit disc $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and its subclasses S^* (starlike functions), S^c (convex functions) and S^{cc} (close-to-convex functions).

The definitions of the families $S^*(\Omega)$ and $S^c(\Omega)$ are based on the analytical conditions: $\operatorname{Re} \frac{\zeta F'(\zeta)}{F(\zeta)} > 0$ for $F \in S^*$ and $\operatorname{Re} \left(1 + \frac{\zeta F''(\zeta)}{F'(\zeta)} \right) > 0$ for $F \in S^c$ (see [10]).

In this paper we shall consider the family $S^{cc}(\Omega)$ which is connected with the class S^{cc} in a similar way. We say that $f \in H(\Omega)$ belongs to the class $S^{cc}(\Omega)$ if there exists a function $g \in S^*(\Omega)$ such that

$$(2) \quad \operatorname{Re} \frac{Df(z)(z)}{g(z)} > 0, \quad z \in \chi(I) \cap \Omega.$$

We say that functions $f \in S^{cc}(\Omega)$ and $g \in S^*(\Omega)$ are associated.

Let $h \in H(\Omega)$ and $g(z) = Dh(z)(z)$ for $z \in \Omega$. By generalization of Alexander's Theorem (see [10]), we know that $h \in S^c(\Omega)$ if and only if the function g belongs to $S^*(\Omega)$, so from (2), f belongs to $S^{cc}(\Omega)$ if and only if, there exists a function $h \in S^c(\Omega)$ such that

$$(3) \quad \operatorname{Re} \frac{Df(z)(z)}{Dh(z)(z)} > 0, \quad z \in \chi(I) \cap \Omega.$$

Analogously as below, we say that functions $f \in S^{cc}(\Omega)$ and $h \in S^c(\Omega)$ are associated.

Now we prove the announced relation between the classes $S^{cc}(\Omega)$ and S^{cc} .

PROPOSITION 1. *If $f \in S^{cc}(\Omega)$, then for any $w \in \chi(I) \cap \partial\Omega$ the function F_w ,*

$$(4) \quad F_w(\zeta) = \frac{f(\zeta w)}{I(w)}, \quad \zeta \in \Delta,$$

is close-to-convex ($F_w \in S^{cc}$).

Proof. Let us fix $f \in S^{cc}(\Omega)$ and let $g \in S^*(\Omega)$ be a function associated with f . Then the function $G_w(\zeta) = \frac{g(\zeta w)}{I(w)}$ belongs to S^* (see [10]) and

from (2) we obtain

$$0 < \operatorname{Re} \frac{Df(\zeta w)(\zeta w)}{g(\zeta w)} = \operatorname{Re} \frac{\zeta [Df(\zeta w)(w)]}{g(\zeta w)} = \operatorname{Re} \frac{\frac{\zeta Df(\zeta w)(w)}{I(w)}}{\frac{g(\zeta w)}{I(w)}}$$

for $w \in \chi(I) \cap \partial\Omega$ and $\zeta \in \Delta$. In view of (4) we have for $w \in \chi(I) \cap \partial\Omega$

$$F'_w(\zeta) = \frac{Df(\zeta w)(w)}{I(w)}, \quad \zeta \in \Delta$$

thus

$$\operatorname{Re} \frac{\zeta F'_w(\zeta)}{G_w(\zeta)} > 0, \quad \zeta \in \Delta,$$

and we conclude that $F_w \in S^{cc}$ (see [12], [11]). ■

PROPOSITION 2. $S^{cc}(\Omega)$ is a proper superclass of $S^*(\Omega)$.

Proof. The inclusion $S^*(\Omega) \subset S^{cc}(\Omega)$ follows from the definitions of $S^*(\Omega)$ and $S^{cc}(\Omega)$. Indeed, if $f \in S^*(\Omega)$, then for $z \in \chi(I) \cap \partial\Omega$ we have that (2) is fulfilled with $g = f$. To prove that $S^*(\Omega) \neq S^{cc}(\Omega)$ we consider the function $f_\alpha(z) : \Omega \rightarrow \mathbb{C}$ of the form

$$f_\alpha(z) = \frac{I(z)(1 - AZ_w^*(z))}{(1 - BZ_w^*(z))^2},$$

where $Z_w^*(z) = \frac{I(z)}{I(w)}$, $A = e^{2i\alpha} \cos \alpha$, $B = e^{i\alpha}$, $\alpha \in (0, \pi)$ and $w \in \chi(I) \cap \partial\Omega$ is such that $|I(w)| = \max_{z \in \bar{\Omega}} |I(z)|$. Let us notice that Z_w^* is a functional over $\mathbb{C}^n$ ($Z_w^* \in \mathbb{C}^{n*}$) such that

$$(5) \quad Z_w^*(w) = 1 \quad \text{and} \quad \|Z_w^*\| = 1.$$

We will prove that $f_\alpha \in S^{cc}(\Omega)$ but for $\alpha \neq \pi/2$, $f_\alpha \notin S^*(\Omega)$. Let F_w be defined by (4) for $f = f_\alpha$. To show that $f_\alpha \notin S^*(\Omega)$ it suffices to prove that $F_w \notin S^*$ (see [10]).

From the definitions of f_α and F_w we have

$$F_w(\zeta) = \frac{\zeta(1 - AZ_w^*(\zeta w))}{(1 - BZ_w^*(\zeta w))^2} = \frac{\zeta(1 - A\zeta)}{(1 - B\zeta)^2}, \quad \zeta \in \Delta.$$

Thus F_w maps Δ onto the complex plane minus a vertical slit that starts at

$$s_0 = \frac{-ie^{2i\alpha}}{4 \sin \alpha}$$

and extends upward to $-(\cos \alpha)/2 + i\infty$ (see [6], vol I, p. 196). For $\alpha \neq \pi/2$, we have $\operatorname{Re} s_0 \neq 0$, and $F_w(\Delta)$ is not starlike. Then $F_w \notin S^*$ and consequently $f_\alpha \notin S^*(\Omega)$ (see [10]).

To show that $f_\alpha \in S^{cc}(\Omega)$ let us consider the associated function $g_\alpha \in S^*(\Omega)$ of the form

$$g_\alpha(z) = \frac{I(z)}{(1 - BZ_w^*(z))^2}, \quad z \in \Omega.$$

For $\alpha \neq \pi/2$ we have

$$(6) \quad \operatorname{Re} \frac{Df_\alpha(z)(z)}{g_\alpha(z)} > 0, \quad z \in \chi(I) \cap \Omega.$$

Indeed, from definition of f_α we obtain

$$Df_\alpha(z)(z) = \frac{1 + B \frac{I(z)}{I(w)} - 2A \frac{I(z)}{I(w)}}{\left(1 - B \frac{I(z)}{I(w)}\right)^3} I(z),$$

so for $\alpha \neq \pi/2$

$$\frac{Df_\alpha(z)(z)}{g_\alpha(z)} = 1 - \frac{2}{i \frac{I(w)}{I(z)} + 1} = 1 - \frac{2 \left(1 + i \frac{\overline{I(w)}}{I(z)}\right)}{\left|1 + i \frac{I(w)}{I(z)}\right|^2}.$$

Thus (6) is true if and only if

$$1 - \frac{2 \left(1 - \operatorname{Im} \frac{I(w)}{I(z)}\right)}{\left|1 + i \frac{I(w)}{I(z)}\right|^2} > 0, \quad z \in \chi(I) \cap \Omega,$$

equivalently

$$\left| \frac{I(w)}{I(z)} \right|^2 > 1, \quad z \in \chi(I) \cap \Omega.$$

The last inequality is obvious by the Maximum Principle and by the choice of the point w . In consequence we obtain that $f \in S^{cc}(\Omega)$. ■

THEOREM 1. *If $f \in S^{cc}(\Omega)$ and $w \in \partial\Omega$, then for $m \in \mathbb{N}$*

$$(7) \quad |P_{m,f}(w)| \leq m |I(w)|.$$

The above bound is sharp. The equality is attained by the function $\tilde{f} \in S^{cc}(\Omega)$ of the form

$$(8) \quad \tilde{f}(z) = \frac{I(z)}{(1 - Z_w^*(z))^2},$$

where $Z_w^ \in (\mathbb{C}^n)^*$ satisfies the conditions (5).*

Proof. Let first $w \in \chi(I) \cap \partial\Omega$. Then from (1) for function F_w defined by (4) we have

$$F_w(\zeta) = \frac{I(\zeta w) + \sum_{m=2}^{\infty} P_{m,f}(\zeta w)}{I(w)} = \zeta + \sum_{m=2}^{\infty} \frac{P_{m,f}(w)}{I(w)} \zeta^m, \quad \zeta \in \Delta.$$

From Proposition 1 it follows that $F_w \in S^{cc}$, so from the coefficient estimates in the class S^{cc} (see [6]) we have

$$\left| \frac{P_{m,f}(w)}{I(w)} \right| \leq m, \quad m \in \mathbb{N} \setminus \{1\}.$$

Hence, we have (7) for $w \in \chi(I) \cap \partial\Omega$.

Let us assume now that $w \in \gamma(I) \cap \partial\Omega$. In view of density $\chi(I) \cap \partial\Omega$ in $\partial\Omega$, there exists $w = \lim_{k \rightarrow \infty} w^k$, where $w^k \in \chi(I) \cap \partial\Omega$, $k \in \mathbb{N}$. Accordingly to the first case we get

$$|P_{m,f}(w^k)| \leq m |I(w^k)|.$$

Therefore, by the continuity of $P_{m,f}$, I and $|\cdot|$, letting $k \rightarrow \infty$ we obtain (7).

Now, we will prove the sharpness of (7). Let us observe, that the function $\tilde{f}$ defined by (8) has the following expansion

$$\tilde{f}(z) = I(z) + \sum_{m=2}^{\infty} m I(z) (Z_w^*(z))^{m-1}, \quad z \in \Omega.$$

By the fact that $\tilde{f} \in S^{cc}(\Omega)$ and by (1), in view of the identity theorem for series of homogeneous polynomials, we have

$$P_{m,\tilde{f}}(z) = m I(z) (Z_w^*(z))^{m-1} \quad \text{for } m \in \mathbb{N} \setminus \{1\}.$$

Since $rw \in \Omega$ for $r \in (0, 1)$, then putting $z = rw$ we get

$$P_{m,\tilde{f}}(rw) = m I(rw) (Z_w^*(rw))^{m-1} \quad \text{for } m \in \mathbb{N} \setminus \{1\}.$$

Then, in view of m -homogeneity $P_{m,\tilde{f}}$ and (5), we have

$$P_{m,\tilde{f}}(w) = m I(w), \quad m \in \mathbb{N} \setminus \{1\}$$

for $w \in \gamma(I) \cap \partial\Omega$. ■

From the above theorem we obtain

COROLLARY 1. *If $f \in S^{cc}(\Omega)$, then $f(z) = 0$ for $z \in \gamma(I) \cap \Omega$.*

COROLLARY 2. *If $f \in S^{cc}(\Omega)$, then $\|P_{m,f}\| \leq m \|I\|$, where $\|P_{m,f}\| = \max_{\|x\|=1} |P_{m,f}(x)|$. This estimate is sharp and attained by the function defined by (8).*

THEOREM 2. *Let $f \in S^{cc}(\Omega)$ and $z \in \Omega$. Then*

$$(9) \quad \frac{|I(z)|}{(1 + \|z\|)^2} \leq |f(z)| \leq \frac{|I(z)|}{(1 - \|z\|)^2}.$$

The bounds (9) are sharp.

Proof. For $z \in \gamma(I) \cap \Omega$ the inequality (9) holds by Corollary 1. So let us assume that $z \in \chi(I) \cap \Omega$ and F_w is defined by (4). Using the growth theorem for the class S^{cc} (see [16], [6]), we get

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq |F_w(\zeta)| \leq \frac{|\zeta|}{(1-|\zeta|)^2}, \quad |\zeta| < 1.$$

If we set $w = \frac{z}{\|z\|}$, then $w \in \chi(I) \cap \partial\Omega$ and by (4) we have

$$\frac{\frac{1}{\|z\|} |\zeta| |I(z)|}{(1+|\zeta|)^2} \leq \left| f\left(\zeta \frac{z}{\|z\|}\right) \right| \leq \frac{\frac{1}{\|z\|} |\zeta| |I(z)|}{(1-|\zeta|)^2},$$

so for $\zeta = \|z\|$ we obtain (9).

Now we shall prove the sharpness of bounds (9). Let z be an arbitrarily fixed point from Ω and $w = \frac{z}{\|z\|}$. For $\varepsilon = \pm 1$ we consider the function

$$(10) \quad f_\varepsilon(x) = \frac{I(x)}{(1 - \varepsilon Z_w^*(x))^2}, \quad x \in \Omega,$$

where $Z_w^* \in (C^n)^*$ fulfils (5). Putting $\varepsilon = 1$ we obtain

$$|f_1(z)| = \frac{|I(z)|}{|1 - Z_w^*(z)|^2} = \frac{|I(z)|}{|1 - \|z\| Z_{\frac{z}{\|z\|}}^*\left(\frac{z}{\|z\|}\right)|^2} = \frac{|I(z)|}{(1 - \|z\|)^2},$$

and the upper bound is achieved.

The proof that equality in the lower bound is attained for the function f_{-1} is similar.

THEOREM 3. *If $f \in S^{cc}(\Omega)$, $z \in \Omega$, then*

$$(11) \quad \frac{|I(z)| (1 - \|z\|)}{(1 + \|z\|)^3} \leq |Df(z)(z)| \leq \frac{|I(z)| (1 + \|z\|)}{(1 - \|z\|)^3}.$$

The above estimate is sharp.

Proof. Let $z \in \gamma(I) \cap \Omega$. Then $I(z) = 0$. Simultaneously $Df(z)(z) = 0$, because $Df(z)(z) = I(z) + \sum_{m=2}^{\infty} m P_{m,f}(z)$ and by (7) $P_{m,f}(z) = 0$. Thus we obtain equality in (11).

Let now $z \in \chi(I) \cap \Omega$ be arbitrarily fixed. Then for a function F_w defined by (4) we get

$$F'_w(\zeta) = \frac{1}{I(w)} (Df(\zeta w)(w)).$$

From the distortion theorem for the class S^{cc} (see [16], [6]) we have

$$\frac{1 - |\zeta|}{(1 + |\zeta|)^3} \leq |F'_w(\zeta)| \leq \frac{1 + |\zeta|}{(1 - |\zeta|)^3}, \quad |\zeta| < 1.$$

Hence, for $w = \frac{z}{\|z\|}$ we obtain

$$\frac{1-|\zeta|}{(1+|\zeta|)^3} \left| I\left(\frac{z}{\|z\|}\right) \right| \leq \left| Df\left(\zeta \frac{z}{\|z\|}\right) \left(\frac{z}{\|z\|}\right) \right| \leq \frac{1+|\zeta|}{(1-|\zeta|)^3} \left| I\left(\frac{z}{\|z\|}\right) \right|.$$

Putting $\zeta = \|z\|$ we obtain (11).

To show the sharpness in (11) let us consider the function f_ε , $\varepsilon = \pm 1$ defined in (10). For $\varepsilon = 1$ we obtain

$$Df_1(x)(x) = \frac{I(x)(1 + Z_w^*(x))}{(1 - Z_w^*(x))^3}.$$

Putting $x = z$ we have

$$Df_1(z)(z) = \frac{I(z)(1 + \|z\| Z_{\frac{z}{\|z\|}}^*(\frac{z}{\|z\|}))}{(1 - \|z\| Z_{\frac{z}{\|z\|}}^*(\frac{z}{\|z\|}))^3} = \frac{I(z)(1 + \|z\|)}{(1 - \|z\|)^3}.$$

Analogously for $\varepsilon = -1$ we prove that for the function f_{-1} we obtain equality in the lower bound. ■

The following theorem is a generalization of the above result.

THEOREM 4. Let $n \in \mathbf{N}$ and $0 \neq z \in \Omega$. If $f \in S^{\text{cc}}(\Omega)$, then

$$\frac{|I(z)|(n - \|z\|)}{(1 + \|z\|)^{(n+2)}} \leq \frac{|D^{(n)}f(z)(z, \dots, z)|}{n! \|z\|^{n-1}} \leq \frac{|I(z)|(n + \|z\|)}{(1 - \|z\|)^{(n+2)}}.$$

The above bounds are sharp and being attained by functions (10).

THEOREM 5. If $f \in S^{\text{cc}}(\Omega)$ is associated with $h \in S^{\text{c}}(\Omega)$ by (3), then

$$\operatorname{Re} \frac{f(z)}{h(z)} > 0, \quad z \in \chi(I) \cap \Omega.$$

The proof is based on the following n -dimensional version of Jack Lemma (compare [4], [7]).

LEMMA 1. Let $f: \Omega \rightarrow \mathbf{C}$ be a holomorphic function such that $f(0) = 0$ and $f \not\equiv 0$. If for a point $0 \neq \hat{z} \in \Omega$ we have

$$|f(\hat{z})| = \max_{\|z\| \leq \|\hat{z}\|} |f(z)|,$$

then there exists $s \geq 1$ such that

$$\frac{Df(\hat{z})(\hat{z})}{f(\hat{z})} = s.$$

Proof of the lemma. We will show that the function

$$F(\zeta) = f\left(\zeta \frac{\hat{z}}{\|\hat{z}\|}\right), \quad \zeta \in \Delta,$$

satisfies the assumptions of classical Jack Lemma ([8]). Indeed, this function is holomorphic in Δ , $F(0) = 0$ and $F \not\equiv 0$. Moreover $|F(\zeta_0)| =$

$\max_{|\zeta| \leq |\zeta_0|} |F(\zeta)|$ for $\zeta_0 = \|\dot{z}\|$. Indeed, from the assumptions of the lemma and the definition of F we have

$$\begin{aligned} |F(\zeta_0)| &\leq \max_{|\zeta| \leq |\zeta_0|} |F(\zeta)| = \max_{|\zeta| \leq |\zeta_0|} \left| f\left(\zeta \frac{\dot{z}}{\|\dot{z}\|}\right) \right| \leq \\ &\leq \max_{\|z\| \leq \|\dot{z}\|} |f(z)| = |f(\dot{z})| = |F(\|\dot{z}\|)| = |F(\zeta_0)|. \end{aligned}$$

In view of Jack Lemma there exists $s \geq 1$ such that

$$\zeta_0 \frac{F'(\zeta_0)}{F(\zeta_0)} = s.$$

Then, by the computation of the derivative of F we obtain

$$s = \frac{\zeta_0 Df(\zeta_0 \frac{\dot{z}}{\|\dot{z}\|})(\frac{\dot{z}}{\|\dot{z}\|})}{f(\zeta_0 \frac{\dot{z}}{\|\dot{z}\|})}.$$

This completes the proof, because $\zeta_0 = \|\dot{z}\|$. ■

Proof of Theorem 5. It is sufficient to consider $f \in S^{cc}(\Omega)$ such that $f \neq h$. Since $h \in S^c(\Omega)$, we have $h(z) \neq 0$ and $Dh(z)(z) \neq 0$ for $z \in \chi(I) \cap \Omega$ (see [10]). Denoting

$$(12) \quad \frac{f(z)}{h(z)} = \frac{1+q(z)}{1-q(z)} \quad \text{for } q(z) \neq 1, z \in \Omega,$$

we can deduce that the function q is holomorphic in Ω , $q(0) = 0$ and $q \neq 0$. It is sufficient to prove that $|q(z)| < 1$ for $z \in \chi(I) \cap \Omega$. In the opposite case, there exists $\dot{z} \in \chi(I) \cap \Omega$ such that

$$1 = |q(\dot{z})| = \max_{\|z\| \leq \|\dot{z}\|} |q(z)|.$$

In view of Lemma 1, we conclude that there exists a real number s such that

$$(13) \quad \frac{Dq(\dot{z})(\dot{z})}{q(\dot{z})} = s \geq 1.$$

From (12) we obtain

$$Df(z)(z) = \frac{1+q(z)}{1-q(z)} Dh(z)(z) + h(z) \frac{2Dq(z)(z)}{(1-q(z))^2}, \quad z \in \chi(I) \cap \Omega.$$

Hence

$$(14) \quad \frac{Df(\dot{z})(\dot{z})}{Dh(\dot{z})(\dot{z})} = \frac{1+q(\dot{z})}{1-q(\dot{z})} \left[1 + \frac{Dq(\dot{z})(\dot{z})}{q(\dot{z})} \frac{2q(\dot{z})}{1-q^2(\dot{z})} \frac{h(\dot{z})}{Dh(\dot{z})(\dot{z})} \right].$$

Since $|q(\dot{z})| = 1$, so $\operatorname{Re} \frac{1+q(\dot{z})}{1-q(\dot{z})} = 0$. Putting $\frac{1+q(\dot{z})}{1-q(\dot{z})} = iy$, where y is a real

number, we obtain $q(\dot{z}) = \frac{iy-1}{iy+1}$ and

$$\frac{2q(\dot{z})}{1-q^2(\dot{z})} = \frac{i(y^2+1)}{2y}.$$

From the above considerations and in view of (13), we can rewrite (14) in the following form

$$\operatorname{Re} \frac{Df(\dot{z})(\dot{z})}{Dh(\dot{z})(\dot{z})} = -\frac{s(y^2+1)}{2} \operatorname{Re} \frac{h(\dot{z})}{Dh(\dot{z})(\dot{z})}.$$

Since $h \in S^c(\Omega) \subset S^*(\Omega)$, we have $\operatorname{Re} \frac{Dh(\dot{z})(\dot{z})}{h(\dot{z})} > 0$. Therefore

$$\operatorname{Re} \frac{Df(\dot{z})(\dot{z})}{Dh(\dot{z})(\dot{z})} < 0,$$

which (by (3)) contradicts the fact that $f \in S^{\text{cc}}(\Omega)$. ■

Theorem 5 we can prove also using a modified version of the Sakaguchi Lemma (see [15]).

References

- [1] I. I. Bawrin, *Klassy golomorfnykh funkcyj mnogich kompleksnykh peremennnykh i ekstremalnyje voprosy dla etich klassow funkcyj*, "Nauka" Moskwa, (1976).
- [2] M. Biernacki, *Sur la representation conforme des domaines lineairement accessibles*, Prace Mat.-Fiz., **44** (1936), 293–314.
- [3] K. Dobrowolska, I. Dziubiński, *Of starlike and convex functions of many variables*, Demonstratio Math. **11**(2), (1978), 545–555.
- [4] K. Dobrowolska, P. Liczberski, *On some differential inequalities for holomorphic functions of many variables*, Demonstratio Math. **14**(2), (1981), 383–398.
- [5] I. Dziubiński, R. Sitarski, *On classes of holomorphic functions of many variables starlike and convex on some hipersurfaces*, Demonstratio Math. **13**(3), (1980), 619–632.
- [6] A. W. Goodman, *Univalent functions*, Mariner Publishing Company 1983.
- [7] S. Gong, S. S. Miller, *Partial differential subordinations and inequalities defined on complete circular domains*, Comm. Partial Differential Equations, **11** (1986), No. 11, 1243–1255.
- [8] I. S. Jack, *Function starlike and convex of order alpha*, J. London Math. Soc. **3** (1971), 469–474.
- [9] Z. J. Jakubowski, J. Kamiński, *On some classes of Mocanu- Bazilevic functions*, Acta Univ. Lodz., Folia Math., **5** (1992), 39–61.
- [10] E. Janiec, *On starlike and convex maps of a Banach space into the complex plane*, Demonstratio Math. **27** (1994), 511–516.
- [11] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. **1** (1952) 169–185.
- [12] Z. Lewandowski, *Sur l'identite de certaines classes de fonctions univalentes*, part I, Ann. Univ. M. Curie-Sklodowska, **12**, (1958), 131–146; part II, ibid. **14** (1960), 19–46.

- [13] P. Liczberski, *On Baurin families of holomorphic maps of Banach spaces into the complex plane*, Zesz. Nauk. Polit. Rzesz, **48** (1988), 77–85.
- [14] W. Rudin, *Function Theory in the Unit Ball*, Springer-Verlag, New York Inc., 1980.
- [15] K. Sakaguchi, *On certain univalent mapping*, J. Math. Soc. Japan **11** (1959), 72–75.
- [16] G. Schober, *Univalent Functions—Selected Topics*, Springer-Verlag, New York Inc., 1975.

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Received February 25, 2004; revised version April 5, 2005.