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ABOUT OSCILLATORY BEHAVIOR OF SOLUTIONS WHEN
PASSING FROM DIFFERENCE TO DIFFERENTIAL
EQUATIONS

Abstract. We study a second order difference equations. We obtain conditions of preservation of solutions when passing from difference to differential equations.

A study of properties of differential equations poses, a question of whether an oscillatory behavior of solutions is preserved when passing from difference equations to the corresponding differential equations.

The converse problem, i.e. the problem of whether solutions of difference equations will have an oscillatory behavior if solutions of the corresponding differential equations do was studied in [3].

So, consider a second order finite-difference equation with variable coefficients and step h which is written in the form

$$(1) \quad y_{n+2} + p_n y_{n+1} + q_n y_n = 0,$$

where

$$\begin{aligned} t_n &= t_0 + nh, & y_n &= y(t_n), \\ p_n &= p(t_n) & q_n &= q(t_n) \neq 0 \end{aligned}$$

are uniquely defined for any t_n .

We will say that y_n changes sign at $t = t_n$ if one of the following conditions is satisfied:

- 1) $y_n y_{n+1} < 0$
or
2) $y_n = 0$,
and
 $y_{n-1} y_{n+1} < 0$.

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If y_n has at least two changes of sign on the interval (a, b) , then y_n is called oscillating on this interval. Let $y_n^{(1)}$ and $y_n^{(2)}$ be two linearly independent solutions of equation (1). The determinant constructed from these solutions,

$$(2) \quad D_n = \begin{vmatrix} y_n^{(1)} & y_{n+1}^{(1)} \\ y_n^{(2)} & y_{n+1}^{(2)} \end{vmatrix},$$

can be calculated by the formula [1]

$$D_n = D_0 \prod_{k=0}^n q_k.$$

For the sake of definiteness, we will assume in the sequel that $D_0 > 0$. Let us consider in detail the case where the free term in equation (1) is positive definite ($q_k > 0$ for all k). It is clear that, in such a case, $D_n > 0$, i.e.

$$y_n^{(1)} y_{n+1}^{(2)} - y_{n+1}^{(1)} y_n^{(2)} > 0 \quad \text{for any } n.$$

This inequality shows that there is the following relation between changes of signs of two linearly independent solutions:

- a) if $y_n^{(1)} > 0$ and $y_{n+1}^{(1)} > 0$,
then $y_n^{(2)} > 0$, and $y_{n+1}^{(2)} < 0$ is impossible;
- b) if $y_n^{(1)} > 0$ and $y_{n+1}^{(1)} < 0$,
then $y_n^{(2)} < 0$, and $y_{n+1}^{(2)} < 0$ is impossible;
- c) if $y_n^{(1)} < 0$ and $y_{n+1}^{(1)} > 0$,
then $y_n^{(2)} > 0$, and $y_{n+1}^{(2)} > 0$ is impossible;
- d) if $y_n^{(1)} < 0$ and $y_{n+1}^{(1)} < 0$,
then $y_n^{(2)} < 0$, and $y_{n+1}^{(2)} > 0$ is impossible.

From these observations one can draw the following conclusion:

- 1) If one of the solutions has a change of sign at a given point, then the second solution can not preserve the sign of $y_{n+1}^{(1)}$ at this point.
- 2) If one of the solutions preserves the sign at a given point, then the second solution can not change the sign to the opposite.

If we know the behavior of one solution, these observations allow to make a "prediction" about the behavior of the other. Let us give an example.

Let

$$y_n^{(1)} < 0, \quad y_{n+1}^{(1)} > 0, \quad y_{n+2}^{(1)} < 0.$$

For $y_n^{(2)}$, the following cases can occur:

if $y_n^{(2)} > 0$, then $y_{n+1}^{(2)} < 0$, $y_{n+2}^{(2)} > 0$;

if $y_n^{(2)} < 0$, then $y_{n+2}^{(2)} > 0$ in the case, where $y_{n+1}^{(2)} < 0$,

$$y_{n+2}^{(2)} \geq 0 \quad \text{if} \quad y_{n+1}^{(2)} > 0.$$

From this, it becomes clear that changes of sign of one solution, generally speaking, do not make turns with changes of sign of the other solution. (The same thing can be observed solving equations with constant coefficients.) A certain connection between behavior of two solutions is established in the following theorem.

THEOREM 1. *If $y_n^{(1)}$ has two zeros, then $y_n^{(2)}$ has either a zero or a change of sign between them.*

Proof. Let $y_m^{(1)} = y_{m+s}^{(1)} = 0$ for any n ,

$$m \leq n \leq m + s.$$

By using (2), one can write

$$y_n^{(1)} y_{n+1}^{(2)} - y_{n+1}^{(1)} y_n^{(2)} = D_n.$$

If $y_n^{(2)}$ has at least one zero for the considered n , the theorem is proved. If

$$y_n^{(2)} \neq 0, \quad m \leq n \leq m + s,$$

then one can write

$$\frac{y_n^{(1)} y_{n+1}^{(2)} - y_{n+1}^{(1)} y_n^{(2)}}{y_n^{(2)} y_{n+1}^{(2)}} = \frac{D_n}{y_n^{(2)} y_{n+1}^{(2)}}$$

or

$$-\Delta \left(\frac{y_n^{(1)}}{y_n^{(2)}} \right) = \frac{D_n}{y_n^{(2)} y_{n+1}^{(2)}}$$

for any n , $m \leq n < m + s$.

If we sum both sides of the equalities, we get

$$-\sum_{n=m}^{m+s-1} \Delta \left(\frac{y_n^{(1)}}{y_n^{(2)}} \right) = \sum_{n=m}^{m+s-1} \frac{D_n}{y_n^{(2)} y_{n+1}^{(2)}},$$

but

$$\sum_{n=m}^{m+s-1} \Delta \left(\frac{y_n^{(1)}}{y_n^{(2)}} \right) = \frac{y_{m+s}^{(1)}}{y_{m+s}^{(2)}} - \frac{y_m^{(1)}}{y_m^{(2)}} = 0$$

because

$$y_m^{(1)} = y_{m+s}^{(1)} = 0.$$

Hence

$$\sum_{n=m}^{m+s-1} \frac{D_n}{y_n^{(2)} y_{n+1}^{(2)}} = 0,$$

$D_n > 0$ as it was remarked before.

If $y_n^{(2)} y_{n+1}^{(2)} > 0$ for all n , then the inequality can not hold. Consequently, there exists such n , $m \leq n < m + s$, for which $y_n^{(2)}$ has a change of sign which was to be proved. If the free term of the equation has negative sign, then this reasoning will be somewhat modified.

Indeed, if $q_k < 0$ for all n , then

$$D_n = D_0 \prod_{k=0}^n q_k = D_0 (-1)^{n+1} \prod_{k=0}^n |q_k|,$$

i.e.

$$D_{2k} < 0, \quad D_{2k+1} > 0.$$

For solutions at even times ($n = 2k$) the above given properties a), b), c), d) will hold, at odd times — the same will be true but with opposite signs. The above theorem remains to be valid if $s = 2l$. If in the case where q_k , we have a sign changing coefficient

$$q_k = (-1)^k |q_k|,$$

then

$$D_{4k} > 0, \quad D_{4k+1} > 0,$$

and

$$D_{4k+2} < 0 \quad D_{4k+3} < 0,$$

and all the conclusions with the corresponding changes remain valid.

Consider the corresponding finite-difference second order differential equation written in the form

$$(3) \quad y'' + p_1(t) y' + q_1(t) y = 0,$$

where p_1 and q_1 are functions continuous on (a, b) , and $p(t_n)$, $q(t_n)$ are related to p_1 and q_1 via the formulas

$$p(t_n) = h p_1(t_n) - 2,$$

$$q(t_n) = 1 + h^2 q_1(t_n) - h p_1(t_n).$$

Let us study conditions that would imply that solutions of equation (3) have oscillatory behavior on the interval (a, b) if such are solutions of (1). Recall that a solution of (3) has an oscillatory behavior on (a, b) if it has on (a, b) at least two zeros.

Equations (1) and (3) are equivalent to systems of two first order equations

$$(4) \quad \begin{cases} \Delta y_n^h = h z_n^h, \\ \Delta z_n^h = -h(q_{1n} y_n^h + p_{1n} z_n^h) \end{cases}$$

and

$$(5) \quad \begin{cases} y' = z, \\ z' = -q_1 y - p_1 z. \end{cases}$$

We will need the following lemma on behavior of solutions of a system of linear equations on a finite interval. So, let the system of differential equations be given:

$$(6) \quad \frac{dx}{dt} = A(t)x,$$

where $x \in \mathbb{R}^n$, $A(t)$ is a continuous on (a, b) matrix.

System (6) is equivalent to the system of integral equations:

$$(7) \quad x(t) = x_0 + \int_{t_0}^t A(s)x(s) ds.$$

It follows from (7) that

$$(8) \quad \|x(t)\| \leq \|x_0\| + \int_{t_0}^t \|A(s)\| \cdot \|x(s)\| ds.$$

Whence, from the Gronwall-Bellman lemma, we get the following estimate for $\|x(t)\|$:

$$(9) \quad \|x(t)\| \leq \|x_0\| \exp \left(\int_{t_0}^t \|A(s)\| ds \right).$$

We have the following lemma.

LEMMA. *Solutions of system (6) on the interval (a, b) satisfy estimate (6) with t_0 , $t \in (a, b)$.*

Let us prove a theorem about preservation of the oscillatory behavior.

THEOREM 2. *If, for a sufficiently small $h = h_0$, equation (1) has a solution that has oscillatory behavior on an interval (a, b) , then equation (3) also has a solution that has oscillatory behavior on the interval (a, b) .*

Proof. Suppose that $\tilde{y}_n^{h_0}$ is a solution of equation (1) that has an oscillatory behavior on (a, b) . This means that $\tilde{y}_n^{h_0}$, on (a, b) , has at least two changes of sign. Then, in virtue of the definition, there are four patterns of behavior of $y_n^{h_0}$ on (a, b) .

First consider the case, where the first change of sign enjoys property 1) and the second one — 2). So, let n_0 and m_0 be such that

$$\tilde{y}_{n_0}^{h_0} \cdot \tilde{y}_{n_0+1}^{h_0} < 0,$$

and

$$y_{m_0}^{h_0} = 0 \quad \tilde{y}_{m_0-1}^{h_0} \cdot \tilde{y}_{m_0+1}^{h_0} < 0.$$

Denote

$$\delta = \min\{|\tilde{y}_{n_0}^{h_0}|, |\tilde{y}_{n_0+1}^{h_0}|, |\tilde{y}_{m_0-1}^{h_0}|, |\tilde{y}_{m_0+1}^{h_0}|\}.$$

Consider such a solution of (3) that

$$t = t_0 + n_0 h_0$$

takes the value $\tilde{y}_{n_0}^{h_0}$, and its derivative equals zero. The interval

$$[t_0 + n_0 h_0, t_0 + (m_0 + 1)h_0]$$

obviously entirely lies inside (a, b) , and hence the coefficients $p_1(t)$ and $q_1(t)$ will be continuous and bounded on this interval. Matrix $A(t)$, written for a system of linear equations which is equivalent to (3), clearly has the form

$$(10) \quad A(t) = \begin{pmatrix} 0 & 1 \\ -q_1(t) & -p_1(t) \end{pmatrix}.$$

By the remark above, on the interval

$$[t_0 + n_0 h_0, t_0 + (m_0 + 1)h_0],$$

it is bounded with respect to the norm, i.e.

$$\|A(t)\| \leq (1 + \max |q_1(t)| + \max |p_1(t)|) = k,$$

where the maximum is taken over the interval

$$[t_0 + n_0 h_0, t_0 + (m_0 + 1)h_0].$$

Hence, by Lemma 1 and using estimate (9), one can assert that the solution of (3) satisfying the above initial conditions is subject to the inequality

$$(11) \quad |\tilde{y}(t)| \leq |\tilde{y}_{n_0}^{h_0}| e^{k(b-a)} = N$$

on the considered interval.

For further considerations, we will need a lemma from [3].

LEMMA 2. For corresponding solutions of systems (4) and (5),

$$y_0 = y_0^h, \quad z_0 = z_0^h,$$

the following estimates hold:

$$(12) \quad \begin{aligned} |y_n - y_n^h| &\leq \frac{hN}{2}(e^{b-a} - 1), \\ |z_n - z_n^h| &\leq \frac{hN}{2}(e^{b-a} - 1), \end{aligned}$$

where

$$y_n = y(t_0 + nh), \quad z_n = z(t_0 + nh).$$

Let us choose h_0 so small that

$$|\tilde{y}_n - \tilde{y}_n^{h_0}| \leq \frac{\delta}{2}.$$

Then, by this lemma, we have at the points

$$\begin{aligned} t_0 + n_0 \tilde{h}_0, \\ t_0 + (n_0 + 1) \tilde{h}_0, \\ t_0 + (m_0 - 1) \tilde{h}_0, \\ t_0 + (m_0 + 1) \tilde{h}_0, \end{aligned}$$

that solution $\tilde{y}_n$ of equation (3) preserves the sign of solution $\tilde{y}_n^{h_0}$ of difference equation (1). And hence, being continuous, it equals zero on the interval (a, b) at least twice, and this means that it has oscillatory behavior.

Other cases of behavior of the solution $\tilde{y}_n^{h_0}$ can be treated similarly. This finishes the proof of the theorem.

REMARK. It follows from the proof of the theorem that it is possible to find an estimate for the smallness of h_0 . Namely, it will be sufficient for h_0 to be such that

$$(13) \quad hN(e^{b-a} - 1) \leq \delta,$$

i.e.

$$(14) \quad h \leq \frac{\delta}{N(e^{b-a} - 1)},$$

where expressions for δ and N can be written in terms of the coefficients $p(t)$ and $q(t)$ by using the oscillating solution $\tilde{y}_n^{h_0}$ of finite-difference equation (1). Using this reasoning and the proof of the theorem, it is possible to obtain sufficient conditions such that

$$(15) \quad h_0 \leq \frac{\min\{|\tilde{y}_{n_0}^{h_0}|, |\tilde{y}_{n_0+1}^{h_0}|, |\tilde{y}_{m_0-1}^{h_0}|, |\tilde{y}_{m_0+1}^{h_0}|\}}{|\tilde{y}_{n_0}^{h_0}|e^{k(b-a)}(e^{b-a} - 1)},$$

where

$$k = (1 + \max |q_1(t)| + \max |p_1(t)|)$$

and the maximum is taken over the interval

$$[t_0 + n_0 h_0, t_0 + (m_0 + 1) h_0].$$

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