Bożena Piątek

ON THE SINCOV FUNCTIONAL EQUATION

Abstract. The Riemann integral, the Hukuhara differences and derivatives will be applied to obtain solutions of the Sincov functional equation.

1. Definition of Riemann integral and its properties

Let $(C, +)$ be an Abelian semigroup with the identity element and let the operation $\cdot : [0, \infty) \times C \rightarrow C$ of multiplication of elements of C by nonnegative numbers $[0, \infty) \times C \ni (\lambda, s) \rightarrow \lambda s \in C$ be given. A triple $(C, +, \cdot)$ is said to be an *abstract convex cone* if the operations $+$ and $\cdot$ satisfy the following properties

$$\lambda(s + t) = \lambda s + \lambda t, \quad (\lambda + \mu)s = \lambda s + \mu s, \quad \lambda(\mu s) = (\lambda\mu)s, \quad 1 \cdot s = s$$

for every $s, t \in C$ and $\lambda \geq 0, \mu \geq 0$.

If $s, t, t' \in C$, $s + t = s + t'$ implies that $t = t'$, then C is said to satisfy the *cancellation law*. Suppose further that a metric ρ is given in C and (C, ρ) is a complete metric space. Moreover, let ρ be *translation invariant*, i.e.,

$$\rho(s + t, s + t') = \rho(t, t') \quad \text{for all } s, t, t' \in C$$

and *positively homogeneous*, i.e.,

$$\rho(\lambda s, \lambda t) = \lambda \rho(s, t) \quad \text{for all } s, t \in C \text{ and } \lambda \geq 0.$$

A metric abstract convex cone with all above properties is said to be a *complete metric cone*.

Let $f : [a, b] \rightarrow C$ be a function defined on interval $[a, b] \subset \mathbb{R}$ with values in a complete metric cone C . A set $\Delta = \{x_0, x_1, \dots, x_n\}$, where

$$a = x_0 < x_1 < \dots < x_n = b,$$

is said to be a *partition* of $[a, b]$. For given partition Δ we set $\delta(\Delta) = \max \{x_i - x_{i-1} : i \in \{1, \dots, n\}\}$. For a partition Δ and for a system $\tau =$

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$(\tau_1, \dots, \tau_n)$ of intermediate points $\tau_i \in [x_{i-1}, x_i]$ we form the Riemann sum

$$S(\Delta, \tau) = \sum_{i=1}^n (x_i - x_{i-1}) f(\tau_i).$$

Let I be an element of C . If for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\rho(I, S(\Delta, \tau)) < \varepsilon$$

for every partition $\Delta = \{x_0, x_1, \dots, x_n\}$ such that $\delta(\Delta) < \delta$ and for all intermediate points $\tau_i \in [x_{i-1}, x_i]$, $i \in \{1, \dots, n\}$, then I is said to be the *Riemann integral* of f over $[a, b]$ and it is denoted by $\int_a^b f(x) dx := I$. In this case f is said to be *Riemann integrable*.

An ordinary difference is not defined in a cone. Thus proofs of some properties of the Riemann integral are not always the same as for real functions. Nevertheless it is able to prove that bounded function $f : [a, b] \rightarrow C$ which is continuous expect a finite number of points of $[a, b]$ is Riemann integrable. Also we can prove that the integrability of f on $[a, b]$ implies the integrability of this function on each subinterval $[c, d] \subset [a, b]$ and the equality

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

holds for all $c \in (a, b)$.

2. Definition of derivative

In an Abelian semigroup we are able to define a difference of two elements of C . For given $s, t \in C$, an element $u \in C$ is said to be the *difference* $t - s$ if $t = s + u$ (see [5, p. 210]). If an abstract convex cone satisfies the cancellation law and the difference exists, then it is unique. The definition of the derivative given below is due to M. Hukuhara (cf. [5, pp. 210–211]). Let $f : [a, b] \rightarrow C$ and let $x_0 \in (a, b)$. f is *differentiable* at x_0 if there exist the limits

$$\lim_{x \rightarrow x_0^+} \frac{f(x) - f(x_0)}{x - x_0} =: f'_+(x_0)$$

and

$$\lim_{x \rightarrow x_0^-} \frac{f(x_0) - f(x)}{x_0 - x} =: f'_-(x_0)$$

and they are equal. Then the derivative of f at x_0 is defined by $f'(x_0) := f'_+(x_0) = f'_-(x_0)$. In this definition it is assumed that the differences $f(x) - f(x_0)$ and $f(x_0) - f(x)$ exist for all $x > x_0$ and $x < x_0$, respectively, sufficiently close to x_0 . We say that f is *differentiable* at a if there exists $f'_+(a)$ and f is differentiable at b if $f'_-(b)$ exists. In these cases we set $f'(a) := f'_+(a)$ and $f'(b) := f'_-(b)$.

It can be shown that if $f : [a, b] \rightarrow C$ is continuous and C is a complete metric cone, then $g : [a, b] \rightarrow C$ defined by the formula

$$g(x) = \int_a^x f(y) dy$$

is differentiable on $[a, b]$ and $g' = f$.

We are also able to prove that if $f, g : [a, b] \rightarrow C$ are continuous and $g'_+(x) = f(x)$, $x \in [a, b)$, then the difference $g(b) - g(a)$ exists and the formula $\int_a^b f(x) dx = g(b) - g(a)$ holds. Moreover, g is differentiable on $[a, b]$.

3. Sincov functional equation

The functional equation

$$(1) \quad f(x, y) + f(y, z) = f(x, z)$$

was studied by D. M. Sincov in 1903. Assume that M is a nonempty set and let $(X, +)$ be an Abelian group. It is known (see e.g. [1, p. 223]) that the general solution of (1) is of the form

$$(2) \quad f(x, y) = g(y) - g(x),$$

where $g : M \rightarrow X$ is an arbitrary function.

Now let $(C, +)$ be a semigroup satisfying the cancellation law. Assume that D is a nonempty subset of $M \times M$ for which the following condition holds

$$(3) \quad (x, y) \in D \wedge (y, z) \in D \Rightarrow (x, z) \in D.$$

Suppose that $f : D \rightarrow C$ is a solution of (1). Let us take $(x, y) \in D$, $(a, x) \in D$. Then $(a, y) \in D$ and there exists the difference $f(a, y) - f(a, x)$. Thus f is of the form (2) for all $(x, y) \in D$, where $g(x) := f(a, x)$.

Conversely, if $g : M \rightarrow C$ is an arbitrary function such that there exist all differences $g(y) - g(x)$ for every $(x, y) \in D$, then $f(x, y) := g(y) - g(x)$ is a solution of (1) in D . Thus we have the following theorem.

THEOREM 1. *Let $(C, +)$ be a semigroup satisfying the cancellation law. Assume that M is a set and D is a nonempty subset of $M \times M$ such that condition (3) holds. Then a function $f : D \rightarrow C$ is a solution of (1) if and only if there exists a function $g : M \rightarrow C$ such that all differences $g(y) - g(x)$ exist and $f(x, y) = g(y) - g(x)$ for $(x, y) \in D$.*

In the sequel we denote by T the set

$$\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y < \infty\}.$$

THEOREM 2. *Assume that C is a complete metric cone and $g : [0, \infty) \rightarrow C$ is a Riemann integrable function on every interval $[a, b] \subset [0, \infty)$. Then*

the function

$$f(x, y) := \int_x^y g(t) dt$$

defined on T is a solution of (1).

Proof. Let $0 \leq x \leq y \leq z < \infty$. We have by the additivity of an integral

$$f(x, y) + f(y, z) = \int_x^y g(t) dt + \int_y^z g(t) dt = \int_x^z g(t) dt = f(x, z). \quad \blacksquare$$

THEOREM 3. Let C be a complete metric cone. If $f : T \rightarrow C$ is a solution of (1) and if $\frac{\partial}{\partial y} f(0, \cdot)$ exists and is continuous then

$$f(x, y) = \int_x^y \frac{\partial}{\partial y} f(0, t) dt$$

holds true for all $(x, y) \in T$.

Proof. Define $h : T \rightarrow C$ by

$$h(x, y) := \int_x^y \frac{\partial}{\partial y} f(0, t) dt, \quad (x, y) \in T.$$

Since $f(0, \cdot)$ is an antiderivative of $\frac{\partial}{\partial y} f(0, \cdot)$ and f is a solution of (1), we obtain

$$h(x, y) = f(0, y) - f(0, x) = f(x, y)$$

for all $(x, y) \in T$. $\blacksquare$

4. Riemann integral of multifunction

Let X be a real Banach space. Denote by $clb(X)$ the set of all nonempty convex closed bounded subsets of X . For given $A, B \in clb(X)$, we set $A + B = \{a + b : a \in A, b \in B\}$, $\lambda A = \{\lambda a : a \in A\}$ for $\lambda \geq 0$ and $A \overset{*}{+} B = cl(A + B) = cl(clA + clB)$, where clA means the closure of A in X . It is easy to see that $(clb(X), \overset{*}{+}, \cdot)$ is an abstract convex cone with the identity element $\{0\}$. If $A, B, C \in clb(X)$, then the equality $A \overset{*}{+} C = B \overset{*}{+} C$ implies $A = B$, thus $clb(X)$ with addition $\overset{*}{+}$ satisfies the cancellation law (see [6] and [10]).

The abstract convex cone $clb(X)$ is a metric space with the Hausdorff metric d defined by the relation

$$d(A, B) = \max \{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} (\inf_{b \in B} \|a - b\|)$. The metric space $(clb(X), d)$ is complete (see e.g. [2, Theorem II-3, p. 40]). Moreover, the Hausdorff metric

d is translation invariant since

$$d(A \overset{*}{+} C, B \overset{*}{+} C) = d(A + C, B + C) = d(A, B)$$

(cf. [3, Lemma 2.2] and [6, Lemma 3]) and positively homogeneous

$$d(\lambda A, \lambda B) = \lambda d(A, B)$$

for all $\lambda \geq 0$ and $A, B, C \in clb(X)$ (cf. [3, Lemma 2.2]). Consequently $clb(X)$ is a complete metric cone.

Let F be a multifunction on an interval $[a, b]$ with convex closed bounded values in X . We can define the Riemann integral of the multifunction on $[a, b]$ like in part 1. In this case the Riemann sum $S(\Delta, \tau)$ has the form

$$S(\Delta, \tau) = (x_1 - x_0)F(\tau_1) \overset{*}{+} \dots \overset{*}{+} (x_n - x_{n-1})F(\tau_n).$$

EXAMPLE. Let $[c_1, d_1], [c_2, d_2]$ be arbitrary intervals on the line. It is known that

$$d([c_1, d_1], [c_2, d_2]) = \max\{|c_1 - c_2|, |d_1 - d_2|\}.$$

Let $f, g : [a, b] \rightarrow \mathbb{R}$ be real functions such that $f \leq g$. It is not difficult to verify that multifunction $F : [a, b] \rightarrow clb(\mathbb{R})$, $F(x) = [f(x), g(x)]$, $x \in [a, b]$ is Riemann integrable on $[a, b]$ if and only if the both functions f and g are Riemann integrable on this interval and the equality

$$\int_a^b F(x)dx = \left[\int_a^b f(x)dx, \int_a^b g(x)dx \right]$$

holds.

The Riemann integral for multifunctions with nonempty compact convex values in $\mathbb{R}^n$ was introduced by Alexander Dinghas [4] in 1956 and by Masuo Hukuhara [5] in 1967. Its some applications can be found in papers of Dmitry B. Silin (see e.g. [7], [8]). The paper [9] of D. B. Silin contains some version of the Riemann integral. For continuous multifunctions with values as above the integral coincides with the Riemann one defined by Dinghas and Hukuhara (cf. [9, p. 133]). Further papers that deal with the problems of set-valued differentiation and integration are quoted in [9].

Let $A, B, C \in clb(X)$. A set C is said to be the *Hukuhara difference* of A and B , i.e., $C = A \overset{*}{-} B$ if $A = B \overset{*}{+} C$ (see [5, p. 210] for convex compact sets A, B, C). By Rådström's theorem (see [6] and [10]) it follows that if the difference $A \overset{*}{-} B$ exists, then it is exactly one.

The differentiability of multifunction $F : [a, b] \rightarrow clb(X)$ is understood as in part 2.

5. Multivalued version of Sincov functional equation

Let X be a Banach space. Consider equation (1)

$$(4) \quad F(x, y) \overset{*}{+} F(y, z) = F(x, z)$$

for $0 \leq x \leq y \leq z < \infty$, where $F : T \rightarrow clb(X)$ and

$$T := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y < \infty\}.$$

Theorems 1–3 can be reformulated in the following way.

THEOREM 1'. Assume that X is a Banach space. If $P : [0, \infty) \rightarrow clb(X)$ is a multifunction such that the Hukuhara difference $F(x, y) := P(y) \overset{*}{-} P(x)$ exists for all $(x, y) \in T$, then the multifunction $F : T \rightarrow clb(X)$ satisfies (4) in T . Conversely, if $F : T \rightarrow clb(X)$ is a solution of (4) in T , then there exists the Hukuhara difference $F(x, z) \overset{*}{-} F(x, y)$ for every $0 \leq x \leq y \leq z < \infty$ and

$$F(y, z) = F(x, z) \overset{*}{-} F(x, y)$$

for all $0 \leq x \leq y \leq z < \infty$.

REMARK 1. If $F : T \rightarrow clb(X)$ is a solution of (4), then

$$F(x, z) \overset{*}{-} F(x, y) = F(0, z) \overset{*}{-} F(0, y)$$

for all $0 \leq x \leq y \leq z < \infty$.

THEOREM 2'. Assume that X is a Banach space and $G : [0, \infty) \rightarrow clb(X)$ is a Riemann integrable multifunction on every interval $[a, b] \subset [0, \infty)$. Then the multifunction

$$F(x, y) := \int_x^y G(t) dt$$

defined on T is a solution of (4).

THEOREM 3'. Assume that X is a Banach space. If $G : [0, \infty) \rightarrow clb(X)$ is continuous then the multifunction

$$F(x, y) := \int_x^y G(t) dt$$

satisfies (4) and the derivative $\frac{\partial F}{\partial y}(0, y)$ with respect to the second variable of $F(0, \cdot)$ exists and $\frac{\partial F}{\partial y}(0, y) = G(y)$, $y \geq 0$. Conversely, if $F : T \rightarrow clb(X)$ is a solution of (4) and if the derivative $\frac{\partial F}{\partial y}(0, \cdot)$ exists and it is continuous

on $[0, \infty)$, then the formula

$$F(x, y) = \int_x^y \frac{\partial F}{\partial y}(0, t) dt$$

holds true for all $(x, y) \in T$.

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INSTITUTE OF MATHEMATICS
SILESIA UNIVERSITY OF TECHNOLOGY
Kaszubska 23
44-100 GLIWICE, POLAND
E-mail: Bozena.Piatek@polsl.pl

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