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THE STRICT TOPOLOGY ON TOPOLOGICAL ALGEBRAS

Abstract. In this paper, we define strict, uniform and compact-open topologies on a topological algebra and investigate their properties. These results unify and include several results of other authors.

1. Introduction

Let X be a Banach algebra and A a closed two-sided ideal in X having a two-sided approximate identity. In the case of $X = C_b(S)$ with S a locally compact Hausdorff space and $A = C_0(S)$, the strict topology β on X was first introduced by Buck [2] in 1958 as the locally convex topology generated by the seminorms $f \rightarrow \|\varphi f\|$ for $\varphi \in A$. Wang [23] and Busby [3] defined in an analogous way the strict topology on the multiplier (or centralizer) algebras $M(A)$ and $M_d(A)$ of a Banach algebra A . Busby in [3] also defined the general strict topology on X in the case of X a B^* -algebra as the one given by the seminorms $x \rightarrow \max\{\|ax\|, \|xa\|\}$ for $a \in A$. Independently, this topology was studied in greater detail and in a more general setting by Sentilles and Taylor in an interesting paper [19] where X is a Banach left A -module. As pointed out in [19], the essential ingredients of arguments used by Buck [2] and later authors is the presence of an approximate identity in $C_0(S)$ or A .

The aim of the present paper is to define the strict, uniform, and compact-open topologies on a class of non-locally convex topological algebras and investigate their properties. In this setting, our results unify and include several results of other authors (e.g., Busby [3], Sentilles-Taylor [19], Taylor [21, 22], Shapiro [20], Collins [4], Ruess [16], Davenport [5], Phillips [12, 13], Khan-Mohammad-Thaheem [8], etc.). Our main results are given in

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section 3 while section 2 is devoted to some technical preliminaries required for the development of our results. In section 4, we present some examples and applications. The authors had initially established these results for locally m -convex algebra but later concluded that “local convexity” can be dispensed with. We mention that the strict and related topologies have also been considered earlier on the space $C_b(S, E)$ for E a topological vector space, not necessarily locally convex [7]. In view of the increasing applications of (non-normed) topological algebras in other fields such as quantum mechanics and quantum statistics (see e.g. Lassner [9, 10]), these results would provide a framework for further applications.

2. Preliminaries

An algebra X (over $\mathbb{R}$ or $\mathbb{C}$) with a topology τ is called a *topological algebra* (resp. *semitopological algebra*) if it is a topological vector space in which multiplication is jointly (resp. separately) continuous. If (X, τ) is a topological algebra, there exists a base $\mathcal{W}$ of τ -neighborhoods of 0 consisting of closed, balanced, and absorbing sets such that for each $W \in \mathcal{W}$, there exists a $U \in \mathcal{W}$ with $U^2 \subseteq W$. (Here $U^2 = UU = \{ab : a, b \in U\}$). A subset U of X is called *idempotent* if $U^2 \subseteq U$. Following Żelazko ([24, p. 31], [25, p. 88]), A semitopological algebra X is said to be *locally idempotent* if it has a base of neighborhoods of 0 consisting of idempotent sets. X is called *locally bounded* if there is a bounded neighborhood of 0. Every normed algebra and, more generally, every locally m -convex algebra is locally idempotent; every Hausdorff locally bounded algebra is metrizable. Every complete metrizable semitopological algebra is a topological algebra; every locally idempotent algebra is also a topological algebra. A complete metrizable topological algebra is called an *F-algebra*. A net $\{e_\lambda : \lambda \in I\}$ in a topological algebra X is called a *two-sided approximate identity* if, for all $x \in X$, $\lim_\lambda e_\lambda x = \lim_\lambda x e_\lambda = x$. A topological vector space (TVS) E is called *ultrabornological* if every linear map from E into any TVS which takes bounded sets into bounded sets is continuous. Every metrizable TVS is ultrabornological. For the general theory of TVS, see e.g. [15, 17] and of topological algebras, see [11, 25].

Let (X, τ) be a topological algebra, A any closed two-sided ideal in X , and $\mathcal{W}$ a base of τ -neighborhoods of 0 in X . For any bounded set $D \subseteq A$ and $W \in \mathcal{W}$, we set

$$N(D, W) = \{x \in X : Dx \subseteq W, xD \subseteq W\}.$$

If $\{a\} \subseteq A$ is a singleton, we write $N(a, W)$ instead of $N(\{a\}, W)$. The *uniform topology* $u = u_A$ (resp. *strict topology* $\beta = \beta_A$) on X is defined as the linear topology which has a base of neighborhoods of 0 consisting of all

sets of the form $N(D, W)$, where D is a bounded (resp. finite) subset of A and $W \in \mathcal{W}$. If A has a two-sided approximate identity $\{e_\lambda : \lambda \in I\}$, the compact-open topology $k = k_A$ on X is defined just as above with D being a finite subset of $\{e_\lambda : \lambda \in I\}$ (cf. [19], §.3).

3. Main Results

Throughout this section we shall assume, unless stated otherwise, that (X, τ) is a locally idempotent F -algebra, A a closed two-sided ideal in X having a bounded two-sided approximate identity $\{e_\lambda : \lambda \in I\}$, and $\mathcal{W}$ a base of τ -neighborhoods of 0 in X consisting of balanced idempotent sets. We shall also assume that A is *faithful* in X in the sense that, for each $x \neq 0$ in X , there exist $a, b \in A$ with $ax \neq 0$ and $bx \neq 0$; that is, if $\{x \in X : Ax = \{0\} = xA\} = \{0\}$ (cf [6, 19, 23]). The assumption of an F -algebra is required essentially to apply the uniform boundedness principle, the open mapping theorem and the closed graph theorem [15].

THEOREM 3.1. (a) $k \leq \beta \leq u \leq \tau$.

(b) k, β , and u are Hausdorff.

(c) (X, k) , (X, β) , and (X, u) are semi-topological algebras.

(d) For each $x \in X$, $e_\lambda x \rightarrow^\beta x$ and $xe_\lambda \rightarrow^\beta x$.

(e) $\bigcup_{\lambda \in I} e_\lambda X$ and $\bigcup_{\lambda \in I} X e_\lambda$ are β -dense in X ; in particular, A is β -dense in X .

Proof. (a) It is clear that $k \leq \beta \leq u$. Let D be a bounded set in A and $W \in \mathcal{W}$. Choose $U \in \mathcal{W}$ with $U^2 \subseteq W$ and $r > 0$ with $D \subseteq rU$. Then $V = \frac{1}{r}U \in \mathcal{W}$ and it is easy to see that $V \subseteq N(D, W)$. This proves that $u \leq \tau$.

(b) We need only to show that k is Hausdorff. Let

$$x \in \bigcap_{\lambda \in I} \bigcap_{W \in \mathcal{W}} N(e_\lambda, W).$$

Since τ is Hausdorff, $\bigcap_{W \in \mathcal{W}} W = \{0\}$ and so $e_\lambda x = 0 = xe_\lambda$ for all $\lambda \in I$. Then, for any $a \in A$, $ax = \lim_\lambda (ae_\lambda)x = 0$ and also $xa = 0$. Since A is faithful in X , we have $x = 0$.

(c) We prove the result only for (X, u) as the proofs for (X, β) and (X, k) are similar to it. Fix $z \neq 0$ in X , and let D be a bounded set in A and $W \in \mathcal{W}$. Choose a balanced idempotent $U \in \mathcal{W}$ with $U \subseteq W$ and $r > 1$ with $z \in rU$. Put $D_1 = D \cup Dz \cup zD$ and $U_1 = \frac{1}{r}U$. Then it is easy to verify that $zN(D_1, U_1) \subseteq N(D, W)$ and $N(D_1, U_1)z \subseteq N(D, W)$. This implies that the multiplication in (X, u) is separately continuous.

(d) Let $x \in X$, and let D be a finite set in A and $W \in \mathcal{W}$. Choose a balanced idempotent $U \in \mathcal{W}$ with $U \subseteq W$ and $r > 1$ with $x \in rU$. There exists

$\lambda_0 \in I$ such that $ae_\lambda - a \in \frac{1}{r}U$ and $e_\lambda xa - xa \in \frac{1}{r}U$ for all $a \in D$ and $\lambda \geq \lambda_0$. Then, for any $a \in D$ and $\lambda \geq \lambda_0$, we have $a(e_\lambda x - x) = (ae_\lambda - a)x \in W$ and $(e_\lambda x - x)a = e_\lambda xa - xa \in W$. Hence $e_\lambda x \rightarrow^\beta x$. In a similar way, we have $xe_\lambda \rightarrow^\beta x$.

(e) This follows immediately from (d). ■

The following result is an extension of ([5], Lemma 4).

LEMMA 3.2. *Let S be a β -bounded set in X . Then, for any bounded set $D \subseteq A$, $DS \cup SD$ is τ -bounded in X .*

Proof. For any $x \in X$, define $L_x, R_x : A \rightarrow A$ by $L_x(a) = xa$ and $R_x(a) = ax, a \in A$. Since S is β -bounded, for each $a \in A$, $aS \cup Sa$ is τ -bounded in X . Hence $B = \{L_x, R_x : x \in S\}$ is a pointwise bounded subset of $CL(A)$, the space of all continuous linear maps from A into A . Since A is an F -algebra, by the uniform boundedness principle ([15], Theorem 2.6), B is equicontinuous and hence uniformly bounded in $CL(A)$. This implies that, for any bounded set $D \subseteq A$, $DS \cup SD$ is τ -bounded. ■

THEOREM 3.3.

- (a) β and u have the same bounded sets.
- (b) If (X, β) is ultrabornological, then $\beta = u$ on X .

Proof. (a) This follows immediately from Lemma 3.2.

(b) By (a), the identity map $i : (X, \beta) \rightarrow (X, u)$ takes bounded sets into bounded sets. Hence, by hypothesis, i is continuous; that is, $u \leq \beta$. ■

THEOREM 3.4.

- (a) $k = \beta$ on τ -bounded sets in X .
- (b) If a sequence $\{x_n\} \subseteq X$ is β -convergent, it is u -bounded and k -convergent; conversely, if $\{x_n\}$ is τ -bounded and k -convergent, it is β -convergent.

Proof. (a) Let S be a τ -bounded set in X , and let $\{x_\alpha : \alpha \in J\} \subseteq S$ with $x_\alpha \rightarrow^k x \in S$. Let D be a finite set in A and $W \in \mathcal{W}$. Choose a balanced idempotent $U \in \mathcal{W}$ with $U + U + U \subseteq W$. Choose $r > 1$ with $D \cup S \subseteq rU$. There exists $\lambda_0 \in I$ such that

$$ae_{\lambda_0} - a \in \frac{1}{r}U, \quad e_{\lambda_0}a - a \in \frac{1}{r}U$$

for all $a \in D$. Choose $\alpha_0 \in J$ such that

$$e_{\lambda_0}(x_\alpha - x) \in \frac{1}{r}U, \quad (x_\alpha - x)e_{\lambda_0} \in \frac{1}{r}U$$

for all $\alpha \geq \alpha_0$. Now, let $a \in D$ and $\alpha \geq \alpha_0$. Then

$$\begin{aligned} a(x_\alpha - x) &= (a - ae_{\lambda_0})x_\alpha + ae_{\lambda_0}(x_\alpha - x) + (ae_{\lambda_0} - a)x \\ &\in U + U^2 + U^2 \subseteq U + U + U \subseteq W. \end{aligned}$$

Similarly, we have $(x_\alpha - x)a \in W$. So $x_\alpha \rightarrow^\beta x$ and hence $\beta \leq k$.

(b) Let $\{x_n\} \subseteq X$ with $x_n \rightarrow^\beta x \in X$. Then it is β -bounded and hence u -bounded by Theorem 3.3(a). Further, since $k \leq \beta$, $x_n \rightarrow^k x$. Conversely, suppose $\{x_n\}$ is τ -bounded and $x_n \rightarrow^k x$. By (a), $k = \beta$ on the τ -bounded set $\{x, x_n : n \geq 1\}$; hence $x_n \rightarrow^\beta x$. ■

If, in addition, X is locally bounded, we obtain a stronger version of Theorem 3.4 which generalizes the corresponding results of Buck [2] and Santilles and Taylor [19].

THEOREM 3.5. *Suppose (X, τ) is locally bounded. Then*

(a) $k = \beta$ on u -bounded sets.

(b) A sequence $\{x_n\} \subseteq X$ is β -convergent $\Leftrightarrow$ it is u -bounded and k -convergent.

Proof. (a) Let S be a u -bounded set in X , and let $\{x_\alpha : \alpha \in J\} \subseteq S$ with $x_\alpha \rightarrow^k x \in S$. Let D be a finite set in A and $W \in \mathcal{W}$. Choose a balanced idempotent $U \in \mathcal{W}$ with $U + U + U \subseteq W$. Let V be a closed bounded τ -neighborhood of 0 in X . Choose $r > 1$ with $V \subseteq rU$. Since S is u -bounded, it follows that $(V \cap A)S$ is τ -bounded. Choose $t > 1$ with $\{x\} \cup D \cup (V \cap A)S \subseteq tV$. There exists $\lambda_0 \in I$ such that

$$ae_{\lambda_0} - a \in \frac{1}{r^2t}V, \quad e_{\lambda_0}a - a \in \frac{1}{r^2t}V$$

for all $a \in D$. Since $x_\alpha \rightarrow^k x$, choose $\alpha_0 \in J$ such that

$$e_{\lambda_0}(x_\alpha - x) \in \frac{1}{r^2t}V, \quad (x_\alpha - x)e_{\lambda_0} \in \frac{1}{r^2t}V$$

for all $\alpha \geq \alpha_0$. Now, let $a \in D$ and $\alpha \geq \alpha_0$. Then

$$a(x_\alpha - x) = (a - ae_{\lambda_0})x_\alpha + ae_{\lambda_0}(x_\alpha - x) + (ae_{\lambda_0} - a)x.$$

If $p = p_V$ is the Minkowski functional of V , then

$$\frac{(a - ae_{\lambda_0})x_\alpha}{p(a - ae_{\lambda_0})} \in (V \cap A)S \subseteq tV \text{ and so } (a - ae_{\lambda_0})x_\alpha \in \frac{1}{r^2}V \subseteq U.$$

Hence $a(x_\alpha - x) \in U + U^2 + U^2 \subseteq W$. Similarly, we have $(x_\alpha - x)a \in W$. So $x_\alpha \rightarrow^\beta x$ and hence $\beta \leq k$.

(b) The proof is almost the same as that of Theorem 3.4(b). ■

The next result characterizes the equivalence of bounded sets in the topologies β , u , and τ on X .

THEOREM 3.6. *Suppose (X, τ) is locally bounded. Then the following are equivalent:*

- (a) u and τ have the same bounded sets.
- (b) β and τ have the same bounded sets.
- (c) The β -closure of a τ -bounded set is τ -bounded.
- (d) $u = \tau$.

Proof. (a) $\Leftrightarrow$ (b) since β and u have the same bounded sets by Theorem 3.3(a).

(b) $\Rightarrow$ (c). Let S be a τ -bounded set in X . Since $\{e_\lambda : \lambda \in I\}$ is τ -bounded, $P = \bigcup_{\lambda \in I} e_\lambda S$ is also τ -bounded. Since $\beta \leq \tau$, P and hence $\overline{P}^\beta$ is β -bounded. By (b), $\overline{P}^\beta$ is τ -bounded. By Theorem 3.1(d), for any $x \in S$, $e_\lambda x \rightarrow^\beta x$ and so $S \subseteq \overline{P}^\beta$. Hence $\overline{S}^\beta$ is τ -bounded.

(c) $\Rightarrow$ (a). Let S be a u -bounded set in X . Then $P = \bigcup_{\lambda \in I} e_\lambda S$ is τ -bounded. Indeed, if $W \in \mathcal{W}$, then taking $D = \{e_\lambda : \lambda \in I\}$ as a bounded set in A , there exists $r > 0$ such that $S \subseteq rN(D, W)$ and so $P \subseteq rW$. By (c), $\overline{P}^\beta$ is τ -bounded. But, as before $S \subseteq \overline{P}^\beta$; hence S is τ -bounded.

(a) $\Rightarrow$ (d). Let $W \in \mathcal{W}$, and let V be a bounded τ -neighborhood of 0 in X . Then $D = V \cap A$ is a bounded τ -neighborhood of 0 in A , and so $N(D, V)$ is a bounded u -neighborhood of 0 in X . Further $N(D, V)$ is u -bounded in X . Indeed, if B is a bounded set in A and $U \in \mathcal{W}$, choose $t > 0$ with $B \subseteq tD$ and $V \subseteq tU$; consequently, it follows that $N(D, V) \subseteq t^2N(B, U)$. Now, by (a), $N(D, V)$ is τ -bounded. There exists $r > 0$ such that $N(D, V) \subseteq rW$ or equivalently $N(D, \frac{1}{r}V) \subseteq W$. This shows that $\tau \leq u$.

(d) $\Rightarrow$ (a). This is obvious. ■

We now consider a characterization of the equivalence of the topologies β , u , and τ on X .

THEOREM 3.7. *Consider the following conditions:*

- (a) *There exists $a \in A$ such that the map $x \rightarrow ax$ is an isomorphism.*
- (b) $\beta = \tau$ on X .
- (c) (X, β) is metrizable.
- (d) (X, β) is ultrabornological.
- (e) $\beta = u$ on X .
- (f) *For each $x \in X$, $e_\lambda x \rightarrow^u x$ and $xe_\lambda \rightarrow^u x$.*

Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f). If (X, τ) is a Banach algebra, then (f) $\Rightarrow$ (a); that is, (a)–(f) are equivalent.

Proof. (a) $\Rightarrow$ (b). Suppose (a) holds, and let $x_\alpha \rightarrow^\beta x$ in X . Now $\phi : x \rightarrow ax$ is a continuous, linear and one-one map from (X, τ) onto (X, τ) . As a consequence of the open mapping theorem [15], the inverse map $\phi^{-1} : ax \rightarrow x$

is continuous. Since $ax_\alpha \rightarrow^\tau ax$, we have $\phi^{-1}(ax_\alpha) \rightarrow^\tau \phi^{-1}(ax)$; that is, $x_\alpha \rightarrow^\tau x$. So $\tau \leq \beta$.

(b) $\Rightarrow$ (c). This is obvious since (X, τ) is metrizable.

(c) $\Rightarrow$ (d) since every metrizable TVS is ultrabornological by ([15], Theorem 1.32).

(d) $\Rightarrow$ (e). This follows from Theorem 3.3(b).

(e) $\Rightarrow$ (f). This follows from Theorem 3.1(d).

Now suppose that (X, τ) is a Banach algebra. Then (f) $\Rightarrow$ (a) follows from ([19], Theorem 2.4). ■

Following Rieffel [14], the *essential part* X_e of X is defined as

$$X_e = \{x \in X : e_\lambda x \rightarrow^u x, xe_\lambda \rightarrow^u x\}.$$

X is called *essential* if $X = X_e$. Let $[\overline{AX \cup XA^u}]$ denote the u -closed linear subalgebra of X spanned by $AX \cup XA$.

THEOREM 3.8. $X_e = [\overline{AX \cup XA^u}]$.

Proof. If $x \in X_e$, then $e_\lambda x \rightarrow^u x$ and $xe_\lambda \rightarrow^u x$, and so $x \in [\overline{AX \cup XA^u}]$. Now let $x \in [\overline{AX \cup XA^u}]$, and consider any bounded set $D \subseteq A$ and $W \in \mathcal{W}$. Choose a balanced idempotent $U \in \mathcal{W}$ with $U + U + U \subseteq W$. Put $D_1 = \{a, e_\lambda, ae_\lambda, e_\lambda a : a \in D, \lambda \in I\}$, and choose $r > 1$ with $D_1 \subseteq rU$. We may assume that $x \in \overline{AX}^u$ and so there exist $a_1, \dots, a_n \in A$ and $x_1, \dots, x_n \in X$ such that

$$\sum_{i=1}^n a_i x_i - x \in N\left(D_1, \frac{1}{r}U\right).$$

Choose $\lambda_0 \in I$ such that

$$e_\lambda a_i x_i - a_i x_i \in \frac{1}{nr}U, \quad a_i x_i e_\lambda - a_i x_i \in \frac{1}{nr}U$$

for all $\lambda \in \lambda_0$ and $1 \leq i \leq n$. Then, for any $a \in D$ and $\lambda \in \lambda_0$,

$$\begin{aligned} a(e_\lambda x - x) &= ae_\lambda \left(x - \sum_{i=1}^n a_i x_i \right) + \\ &\quad + a \sum_{i=1}^n (e_\lambda a_i x_i - a_i x_i) + a \left(\sum_{i=1}^n a_i x_i - x \right) \\ &\in U + U^2 + U \subseteq W, \end{aligned}$$

and similarly $(e_\lambda x - x)a \in W$. Hence $e_\lambda x \rightarrow^u x$. In a similar way we can show that $xe_\lambda \rightarrow^u x$. ■

Under some additional hypotheses and using a factorization theorem we shall next show that $X_e = AX = XA$. But first we recall some terminology. Following Ansari-Piri [1], a TVS E is called *fundamental* if there exists $r > 1$

such that, for any sequence $\{x_n\}$ in E , the convergence of $r^n(x_{n+1} - x_n)$ to 0 in E implies that $\{x_n\}$ is a Cauchy sequence. Every locally convex space and every locally bounded space is fundamental. An approximate identity $\{b_\alpha : \alpha \in J\}$ in a topological algebra B is said to be *uniformly bounded* if there exists $r > 0$ such that $\{(\frac{b_\alpha}{r})^n : \alpha \in J, n = 1, 2, \dots\}$ is a bounded set in B . If E is a left B -module, then E is called a *topological left B -module* if the multiplication map $(b, x) \rightarrow bx$ of $B \times E$ into E is jointly continuous. A topological left B -module E is called *essential* if B has a uniformly bounded left approximate identity $\{b_\alpha : \alpha \in J\}$ such that $b_\alpha x \rightarrow x$ for all $x \in E$.

We shall require the following generalization of the famous Cohen factorization theorem established in [1].

THEOREM 3.9. *Let B be an F -algebra.*

(i) *If B has a uniformly bounded left approximate identity, then, for any sequence $\{b_n\}$ in B with $b_n \rightarrow 0$, there exists an element $a \in B$ and a sequence $\{c_n\}$ in B with $c_n \rightarrow 0$ such that $b_n = ac_n$, $n = 1, 2, \dots$*

(ii) *If E is an F -space and also an essential topological left B -module, then, for each $x \in E$, there exist $a \in B$ and $y \in E$ such that $x = ay$.*

Note that this is a so-called left-hand version of the factorization theorem. Clearly its right-hand version also holds.

THEOREM 3.10. *Suppose X is an essential F -space, and let A be fundamental with $\{e_\lambda : \lambda \in I\}$ uniformly bounded. Then $X_e = AX = XA$.*

Proof. Clearly, $AX \subseteq X_e$. Now let $x \in X_e$. Since X is an essential topological left A -module, by Theorem 3.9 (ii), there exist $a \in A$ and $y \in X$ such that $x = ay$. Therefore, $x \in AX$. Hence $X_e = AX$. Similarly, using the right-hand version of Theorem 3.9 (ii), it follows that $X_e = XA$. ■

The following result is an analogue of ([19], Theorem 3.1) and ([21], Lemma 2.5).

THEOREM 3.11. *Let X and A be as in Theorem 3.10, and suppose that A is also commutative. Then the collection $\{N(a, U), a \in A, U \in \mathcal{W}\}$ is a base for β -neighborhoods of 0 in X .*

Proof. Let D be a finite set in X and $W \in \mathcal{W}$. Choose a balanced $U_1 \in \mathcal{W}$ with $U_1^2 \subseteq W$. Suppose $D = \{b_1, \dots, b_m\}$. Taking $b_{m+1} = b_{m+2} = \dots = 0$ in Theorem 3.9(i), there exist $a, c_1, \dots, c_m \in A$ such that $b_i = ac_i$, $i = 1, \dots, m$. Choose $r > 1$ with $\{c_1, \dots, c_m\} \subseteq rU_1$, and put $U = \frac{1}{r}U_1$. Then, using the commutativity of A , it is easy to see that $N(a, U) \subseteq N(D, W)$, as required. ■

REMARK. If A is a B^* -algebra, then the above theorem holds without the assumption of commutativity of A . Indeed, in view of the factorization result of Taylor ([21], Lemma 2.4), we can choose in the above proof elements

$a, c_1, \dots, c_m, c'_1, \dots, c'_m$ in A such that $b_i = ac_i = c'_i a$. This theorem also holds without commutativity if we modify the definition of β by setting either $N(D, W) = \{x \in X : Dx \subseteq W\}$ or $N(D, W) = \{x \in X : xD \subseteq W\}$ (cf. [19], §3).

4. Examples and Applications

In this section, we apply our results to special cases of the strict topology considered by other authors. We first elaborate two examples, one for the commutative case and the other for the general case, mentioned in the introduction.

EXAMPLE 4.1. For any locally compact Hausdorff space S , let $X = C_b(S)$ and $A = C_0(S)$. Then X is a commutative Banach algebra with sup norm and A is a closed ideal in X having a bounded approximate identity. The strict topology β on $C_b(S)$, first introduced by R.C. Buck [2] in 1958, is the locally convex topology generated by the seminorms $f \rightarrow \|\varphi f\|$ for $\varphi \in C_0(S)$. More generally, for any commutative Banach algebra A , Wang [23] considered the multiplier (or centralizer) algebra $X = M(A) := \{T : T \in CL(A), aT(b) = T(a)b \text{ for all } a, b \in A\}$ of A and defined the strict topology β on $M(A)$ as the one generated by the seminorms $T \rightarrow \|T(a)\|, a \in A$. He showed in particular that if A also has a bounded approximate identity $\{e_\lambda : \lambda \in I\}$, then: (1) A is β -dense in $M(A)$, (2) if $\lim_\lambda \|e_\lambda\| = 1$, the embedding $a \rightarrow L_a$ is an isometry of A into $M(A)$; hence A is a norm-closed ideal in $M(A)$, and (3) $M(A) = C_b(S)$ if $A = C_0(S)$.

EXAMPLE 4.2. Now consider (A, τ) itself an F -algebra having a bounded two-sided approximate identity. A pair (S, T) of mappings $S, T : A \rightarrow A$ is called a *double multiplier* (or a *double centralizer*) on A if $aS(b) = T(a)b$ for all $a, b \in A$. It follows from the faithfulness of A and the closed graph theorem that S and T are linear and continuous. Let $X = M_d(A)$ denote the algebra of all double multipliers (S, T) on A (see Johnson [6] and Busby [3] for detail). We define a topology σ (resp. s) on $M_d(A)$ as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$U_d(D, G) = \{(S, T) \in M_d(A) : S(D) \subseteq G, T(D) \subseteq G\}$$

where D is a bounded (resp. finite) subset of A and G is a neighborhood of 0 in A . Clearly $s \leq \sigma$. If we define a map $\mu_0 : A \rightarrow M_d(A)$ by $\mu_0(a) = (L_a, R_a)$, $a \in A$, then μ_0 is an algebra isomorphism with $\mu_0(A)$ a two-sided ideal in $M_d(A)$. Further: (i) $(M_d(A), \sigma)$ and $(M_d(A), s)$ are complete, (ii) A is s -dense in $M_d(A)$, and (iii) σ and s have the same bounded sets in $M_d(A)$ (see [3, 6, 8] for details).

REMARK 4.3. Note that A need not be σ -closed in $M_d(A)$ even if A is a metrizable locally C^* -algebra (see [12], p. 187) or a Banach algebra ([23], p. 1138). However, if A is a B^* -algebra or, more generally, an F -algebra whose topology is generated by a submultiplicative F -norm q such that $q(e_\lambda) = 1$ for all $\lambda \in I$, then it is easy to verify that $\mu_0 : A \rightarrow (M_d(A), \sigma)$ is an isometry; hence A is σ -closed in $M_d(A)$. This rectifies an error in ([8], Theorem 3.1.(b)).

Next, applying the notion of double multipliers as in Example 4.2 above, we consider the problem of completeness of (X, β) via an embedding of (X, β) into $(M_d(A), s)$. Define $\mu : X \rightarrow M_d(A)$ by $\mu(y) = (L_y, R_y), y \in X$. Then, as in ([3], Proposition 3.7), μ is a unique isomorphism satisfying $\mu(a) = \mu_0(a)$ for all $a \in A$.

The following result is given in [3, Proposition 3.7] for X a Banach algebra (see also [19], Theorem 3.5) and in [8] for X an F -algebra. For completeness, we include it here in a slightly improved form.

THEOREM 4.4. *Let μ and s be as above. Then*

- (a) $\mu : (X, \beta) \rightarrow (M_d(A), s)$ is continuous and open onto $\mu(X)$.
- (b) If A is complete and metrizable, then (X, β) is complete $\Leftrightarrow \mu(X) = M_d(A)$.

Proof. (a) Let $\{y_\alpha\}$ be a net in X with $y_\alpha \rightarrow^\beta y \in X$. Then, for each $a \in A$, $y_\alpha a \rightarrow^\tau ya$ and $ay_\alpha \rightarrow^\tau ay$ in A ; that is, $(L_{y_\alpha}, R_{y_\alpha}) \rightarrow^s (L_y, R_y)$ in $M_d(A)$. So μ is continuous. For any finite set $D \subseteq A$ and open $W \in \mathcal{W}$, we have

$$\begin{aligned} \mu(N(D, W)) &= \{(L_x, R_x) : x \in X \text{ with } L_x(D) \subseteq W \cap A, R_x(D) \subseteq W \cap A\} \\ &= \{\mu(x) : x \in X \text{ with } \mu(x) \in U_d(D, W \cap A)\} \end{aligned}$$

which is s -open in $\mu(X)$. Hence μ is open onto $\mu(X)$.

(b) If (X, β) is complete, then $\mu(X)$ is complete and hence a closed subset of $(M_d(A), s)$. Since $A \subseteq X$ and $\mu(A)$ is s -dense in $M_d(A)$, ([8], Theorem 3.2(b)), $\mu(X)$ is also s -dense in $M_d(A)$. By ([8], Theorem 3.2(a)), $(M_d(A), s)$ is complete. Thus (X, β) is complete $\Leftrightarrow \mu(X) = M_d(A)$. ■

REMARKS 4.5. (I). We can apply our results to extend an observation on inner derivation, due to Phillips [13]. Let X and A be as in Section 3, with X not necessarily an F -algebra. A linear map $D : A \rightarrow A$ is called a *derivation* on A if

$$D(ab) = D(a)b + aD(b), \quad a, b \in A.$$

For any $x \in X$, the map $\delta_x : A \rightarrow A$ defined by $\delta_x(a) = [x, a] = xa - ax$, $a \in A$, is a derivation on A , called the *inner derivation* on A . A derivation $D : A \rightarrow A$ is called *approximately inner* if there exists a net $\{a_\alpha : \alpha \in J\}$

in A such that $D(a) = \lim_{\alpha} \delta_{a_{\alpha}}(a)$ for all $a \in A$. We claim that, for any $x \in X$, δ_x is approximately inner. Indeed, by Theorem 3.1.(e), A is β -dense in X and so there exists a net $\{a_{\alpha} : \alpha \in J\}$ in A such that $a_{\alpha} \rightarrow^{\beta} x$. Then, for any $a \in A$,

$$\delta_x(a) = \lim_{\alpha} (a_{\alpha}a - aa_{\alpha}) = \lim_{\alpha} \delta_{a_{\alpha}}(a).$$

This was observed earlier in ([13], p. 244) in the case of $X = M_d(A)$.

REMARK (II). (1) If A is a closed ideal in a Banach algebra X , then clearly X is a Banach A -module. In this situation, we get the analogues of several results of Sentilles and Taylor [19] as well as of Busby [3, Propositions 3.1, 3.5-3.7]; see also Shapiro [20, Theorem B], Collins [4], Davenport [5, Proposition 2 and Lemma 4], Ruess [16, Propositions 2.1, 2.5, 2.6].

(2) If A is a locally C^* -algebra or a Fréchet algebra and $X = M_d(A)$, we obtain the work of Phillips ([12, Theorem 3.14], [13]) and the more recent one of Khan-Mohammad-Thaheem [8]; see also Busby [3] and Taylor [21, 22].

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