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STABILITY OF ITERATIVE PROCEDURES FOR MULTIVALUED MAPS IN METRIC SPACES

Abstract. Let (X, d) be a metric space and T a self-map of X . Let $x_{n+1} = f(T, x_n)$ denote some iterative procedure. Let $\{x_n\}$ be convergent to a fixed point u of T and $\{y_n\}$ be an arbitrary sequence in X . Set $\varepsilon_n = d[y_{n+1}, f(T, y_n)]$, $n = 0, 1, 2, \dots$, then the iterative procedure $f(T, x_n)$ is T -stable provided that $\lim_n \varepsilon_n = 0$ implies that $\lim_n y_n = u$. This definition has been extended by Singh and Chadha [34] to discuss the problem of stability for multivalued operators on metric spaces. The purpose of this paper is to present a fixed point theorem for generalized multivalued contractions on a setting more general than metric spaces. The same is utilized to discuss the problem of stability of iterative procedures in multivalued analysis. Some special cases due to Stefan Czerwik and others are discussed as special cases.

1. Introduction

Let (X, d) be a metric space and $T : X \rightarrow X$. Equations arising in many physical formulations can be expressed as a fixed point equation $Tx = x$. Let $\{x_n\}$ be a sequence obtained by some iterative procedure $f(T, x_n)$ that converges to a fixed point of T . In practical computations, the sequence $\{x_n\}$ is calculated in the following manner. First we choose the initial approximation $x_0 \in X$. Then we compute $x_1 = f(T, x_0)$ but, due to rounding off or discretization of the function, we do not get the exact value of x_1 , but a different value y_1 (say), which is close enough to x_1 , i.e., $y_1 \approx x_1$. Consequently, when computing $x_2 = f(T, x_1)$ we actually compute x_2 as $x_2 = f(T, y_1)$, and so, instead of the theoretical value x_2 , we actually obtain $y_2 \approx x_2$, and so on. In this way, instead of the actual sequence $\{x_n\}$ we obtain an approximate sequence $\{y_n\}$. The iterative procedure $f(T, x_n)$ is considered to be numerically stable if and only if the approximate sequence $\{y_n\}$ still converges to the desired solution of the equation $Tx = x$. This kind of problem

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for real-valued functions was first noticed by M. Urabe [37] in 1957. The study of stability of iterative procedures enjoys a celebrated place in applicable mathematics due to chaotic behavior of functions in discrete dynamics and other numerical computations using computer programs. For a detailed discussion on the role of stability of iterative procedures in mathematical sciences, especially in computational analysis, one may refer to Berinde [1], Chadha [2], Czerwik et. al. [10]-[11], Harder and Hicks [12]-[14], Ishikawa [15]-[16], Istratescu [17], Matkowski and Singh [20], Ortega and Rheinboldt [23], Osilike [24]-[26] and Rhoades [29]-[30]. We begin with the following classical theorem of stability on metric spaces due to A. M. Ostrowski [27].

THEOREM 1.1 (Ostrowski's stability theorem [27]). *Let (X, d) be a complete metric space and $T : X \rightarrow X$ a Banach contraction with contraction constant k , i.e. $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$, where $0 \leq k < 1$. Let $u \in X$ be the fixed point of T . Let $x_0 \in X$ and $x_{n+1} = Tx_n$, $n = 0, 1, 2, \dots$. Suppose that $\{y_n\}$ is a sequence in X and $\varepsilon_n = d(y_{n+1}, Ty_n)$. Then*

$$d(u, y_{n+1}) \leq d(u, x_{n+1}) + k^{n+1}d(x_0, y_0) + \sum_{r=0}^n k^{n-r}\varepsilon_r.$$

Moreover, $\lim_n y_n = u$ iff $\lim_n \varepsilon_n = 0$.

This result has been found very useful in numerical praxis and has been extended by several authors to various classes of single-valued operators (see Harder and Hicks [13]-[14], Jachymski [18], Liu [19], Osilike [24]-[26], Rhoades [29]-[30], Singh [34]-[35] and Zhou [38]). Inspired by the first result (Theorem 2.1 below) on the stability of Picard iterative procedures for multivalued contractions on metric spaces due to Singh and Chadha [34], Czerwik, Dlutek and Singh [11] studied the stability of iterative procedures in b-metric spaces. In Section 4, we obtain a fixed point theorem for generalized multivalued contractions in b-metric spaces. Section 5 deals with the stability of iterative procedures for multivalued maps on a general setting.

2. Stability for multivalued contractions

Let (X, d) be a metric space and $CL(X)$ the collection of all nonempty closed subsets of X . Then $(CL(X), H)$ stands for the well-known generalized Hausdorff metric space, where the generalized Hausdorff metric H is induced by the metric d (see, for instance, [22], [34] and [36]). A map $T : X \rightarrow CL(X)$ is called a multivalued contraction iff there exists a positive number $q < 1$ such that

$$(1) \quad H(Tx, Ty) \leq qd(x, y)$$

for all $x, y \in X$. Nadler, Jr. [21] (see also Rus et al. [32]) proved that

a multivalued contraction on a complete space has a fixed point. Ćirić [6] extended this result for generalized multivalued contractions on metric spaces. Singh and Chadha [34] extended Ostrowski's theorem to multivalued contractions using Nadler's theorem and introduced the following definition of stability of iterative procedures for multivalued maps. Let X be a metric space and $T : X \rightarrow CL(X)$. Let $x_0 \in X$ and $x_{n+1} \in f(T, x_n)$ denote the Picard iterative procedure for T . Let $\{x_n\}$ be convergent to a fixed point u of T and $\{y_n\}$ be an arbitrary sequence. Set $\varepsilon_n = H[y_{n+1}, f(T, y_n)]$, $n = 0, 1, 2, \dots$. The iterative procedure $f(T, x_n)$ is said to be T -stable provided that $\lim_n \varepsilon_n = 0$ implies that $\lim_n y_n = u$. This definition is due to Hicks and Harder [13, 14] (see also [12]) when T is a single-valued self-map of a metric space X . We cite here the first result on the stability of Picard iterative procedure for multivalued contractions.

THEOREM 2.1 ([34]). *Let X be a complete metric space and $T : X \rightarrow CL(X)$ satisfy the condition (1). Let x_0 be an arbitrary point in X and $\{x_n\}$ an orbit for T at x_0 such that $\{x_n\}$ is convergent to a fixed point u of T . Let $\{y_n\}$ be a sequence in X , and set $\varepsilon_n = H(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \dots$. Then*

$$d(u, y_{n+1}) \leq d(u, x_{n+1}) + q^{n+1}d(x_0, y_0) + \sum_{r=0}^n q^{n-r} \varepsilon_r.$$

Further, if Tu is singleton then $\lim_n y_n = u$ iff $\lim_n \varepsilon_n = 0$.

Czerwik, Dłutek and Singh [11] studied the stability of Picard iterative procedures for multivalued maps in b-metric spaces. In this paper, we obtain a fixed point theorem for generalized multivalued contractions in b-metric spaces and the same is used to study the stability of Picard iterative procedures for such maps.

3. Preliminaries

First we present some definitions and notations to be used in the sequel.

DEFINITION 3.1 (cf. [8]–[11]). Let X be a set and $s \geq 1$ a given real number. A function $d : X \times X \rightarrow \mathbb{R}^+$ is called a b-metric provided that for all $x, y, z \in X$,

- (bm-1) $d(x, y) = 0$ iff $x = y$,
- (bm-2) $d(x, y) = d(y, x)$,
- (bm-3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space. In all that follows, (X, d) is a b-metric space unless otherwise stated explicitly. As usual, the ordinary distance between a point $x \in X$ and a subset A of X will be denoted by $D(x, A)$.

Notice that the class of b-metric spaces is effectively larger than that of metric spaces, since a b-metric space is a metric space when $s = 1$ in (bm-3). The following example due to Czerwik [8, 9] shows that a b-metric on X need not be a metric on X .

EXAMPLE. Let $X = \{x_1, x_2, x_3\}$ and $d : X \times X \rightarrow \mathbb{R}^+$ such that

$$\begin{aligned} d(x_1, x_2) &= a > 2, d(x_1, x_3) = d(x_2, x_3) = 1 \text{ and } d(x_n, x_n) = 0, \\ d(x_n, x_k) &= d(x_k, x_n), d(x_n, x_k) \leq \frac{a}{2}[d(x_n, x_i) + d(x_i, x_k)], n, k, i = 1, 2, 3. \end{aligned}$$

Then (X, d) is a b-metric space.

DEFINITION 3.2 ([9]). Let (X, d) be a b-metric space. The generalized Hausdorff b-metric H on $CL(X)$, the collection of all non-empty closed subsets of (X, d) is defined as follows:

$$H(A, B) := \begin{cases} \max\{\sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A)\}, & \text{if the maximum exists,} \\ \infty, & \text{otherwise} \end{cases}$$

where $A, B \in CL(X)$.

We record the following properties to be used frequently.

LEMMA 3.1. For any $A, B, C \in CL(X)$,

- (i) $D(x, B) \leq d(x, y)$, for any $y \in B$,
- (ii) $D(A, B) \leq H(A, B)$
- (iii) $H(A, C) \leq s[H(A, B) + H(B, C)]$ (cf. Czerwik [9]–[11]).

We shall need the following result as well.

LEMMA 3.2 ([9]). Let A and B be distinct elements of $CL(X)$. Let $x \in A$. Then, for a given number $\alpha > 1$, there exists a point y in B such that

$$d(x, y) \leq \alpha H(A, B).$$

This result for a metric space is essentially due to Nadler, Jr. [21] and Ćirić [6], (see also Rus [31]).

THEOREM 3.1 ([9]). If (X, d) is a b-metric space, then the function $H : CL(X) \times CL(X) \rightarrow [0, \infty)$ is a generalized Hausdorff b-metric in $CL(X)$. Moreover, if (X, d) is complete, then $(CL(X), H)$ is also complete.

DEFINITION 3.3. A map $T : X \rightarrow CL(X)$ is said to be a generalized multi-valued contraction if and only if there exists a positive number $q < s^{-1}$ such that

$$(2) \quad H(Tx, Ty) \leq q \cdot \max\{d(x, y), D(x, Tx), D(y, Ty), \frac{1}{2}(D(x, Ty) + D(y, Tx))\}$$

holds for all $x, y \in X$.

An orbit of a multivalued map T at a point $x_0 \in X$ is a sequence $\{x_n : x_n \in Tx_{n-1}, n = 1, 2, \dots\}$. A space X is said to be T -orbitally complete iff every Cauchy sequence of the form $\{x_{n_i} : x_{n_i} \in Tx_{n_i-1}\}$ converges in X . Evidently, if X is a complete space then, X is T -orbitally complete. These definitions along with Definition 3.3 are essentially due to Ćirić [6] in metric spaces.

4. Fixed point theorems

THEOREM 4.1. *Let X be a complete b -metric space and T a generalized multivalued contraction on X . Then:*

(i) *for every $x_0 \in X$ there exists an orbit $\{x_n\}$ of T at x_0 and $u \in X$ such that $\lim_n x_n = u$;*

(ii) *the point u is fixed under T , i.e., $u \in Tu$.*

Proof. If $s = 1$ (cf. Def. 3.3) then all the conclusions follow from Ćirić [6]. So we need take $s > 1$. Pick $x_0 \in X$. Let $x_1 \in Tx_0$. We may assume that $H(Tx_0, Tx_1) > 0$, since otherwise $Tx_0 = Tx_1$, and $x_1 \in Tx_1$, i.e. x_1 is a fixed point of T . Let λ be a real number such that $\lambda > 0$ and $q^{1-\lambda} < 1/s$. Since $H(Tx_0, Tx_1) < q^{-\lambda}H(Tx_0, Tx_1)$ and $x_1 \in Tx_0$, by Lemma 3.2, there exists a point $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq q^{-\lambda}H(Tx_0, Tx_1).$$

Again assuming $Tx_1 \neq Tx_2$ and selecting a point x_3 in Tx_2 , we have

$$d(x_2, x_3) \leq q^{-\lambda}H(Tx_1, Tx_2).$$

Continuing the same way, we get a sequence $\{x_n : n \in \mathbb{N}\}$ of points of X such that $x_{n+1} \in Tx_n$ and

$$d(x_n, x_{n+1}) \leq q^{-\lambda}H(Tx_{n-1}, Tx_n), n = 1, 2, \dots$$

From (2),

$$\begin{aligned} d(x_n, x_{n+1}) &\leq q^{-\lambda}H(Tx_{n-1}, Tx_n) \\ &\leq q^{-\lambda} q \cdot \max\{d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), D(x_n, Tx_n), \frac{1}{2}D(x_{n-1}, Tx_n)\} \\ &\leq k \cdot \max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}d(x_{n-1}, x_{n+1})\}, \end{aligned}$$

where $k = q^{1-\lambda}$.

Since $k < 1$, the relation $d(x_n, x_{n+1}) \leq kd(x_n, x_{n+1})$ implies $d(x_n, x_{n+1}) = 0$, we may assume that $x_n \neq x_{n+1}$ and

$$d(x_n, x_{n+1}) \leq k \cdot \max\{d(x_{n-1}, x_n), \frac{1}{2}d(x_{n-1}, x_{n+1})\}.$$

So

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n), \text{ or else}$$

$$d(x_n, x_{n+1}) \leq \frac{ks}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

This yields

$$d(x_n, x_{n+1}) \leq \max \left\{ k, \frac{ks}{(2-ks)} \right\} d(x_{n-1}, x_n),$$

that is

$$d(x_n, x_{n+1}) \leq k' d(x_{n-1}, x_n), \quad \text{where } k' = \max \left\{ k, \frac{ks}{2-ks} \right\}.$$

Repeat this argument to obtain

$$d(x_n, x_{n+1}) \leq k' d(x_{n-1}, x_n) \leq \cdots \leq k'^n d(x_0, x_1).$$

Since $k' < 1$, it follows that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in Tx_n$, $n = 0, 1, 2, \dots$, and X is complete, there exists a point u in X such that $u = \lim_n x_n$. This proves (i).

By Lemma 3.1,

$$\begin{aligned} D(u, Tu) &\leq H(u, Tu) \leq sH(u, x_{n+1}) + sH(x_{n+1}, Tu) \\ &\leq sd(u, x_{n+1}) + sH(Tx_n, Tu) \\ &\leq sd(u, x_{n+1}) + sq \cdot \max\{d(x_n, u), D(x_n, Tx_n), D(u, Tu), \\ &\quad \frac{1}{2}[D(x_n, Tu) + D(u, Tx_n)]\} \\ &\leq sd(u, x_{n+1}) + sq \cdot \max\{d(x_n, u), d(x_n, x_{n+1}), D(u, Tu), \\ &\quad \frac{1}{2}[D(x_n, Tu) + d(u, x_{n+1})]\}. \end{aligned}$$

Making $n \rightarrow \infty$, this obtains $D(u, Tu) \leq sqD(u, Tu)$, and $u \in Tu$.

If T is a single-valued map in the above theorem then we obtain the following result.

THEOREM 4.2. *Let X be a complete b -metric space and $T : X \rightarrow X$ such that*

(3) $d(Tx, Ty) \leq q \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}$
for all $x, y \in X$ and $q < s^{-1}$. Then, T has a unique fixed point u such that $Tu = u$.

Proof. It follows from Theorem 4.1 that T has a fixed point. The unicity of the fixed point follows usually.

We remark that Theorem 4.1 with $s = 1$ is the main result of Ćirić [6] and the condition (3) with $s = 1$ is (21') discussed and compared by Rhoades [28].

We close this section with the following **open question**:

Is Theorem 4.1 with (2) replaced by

$$H(Tx, Ty) \leq q \cdot \max\{d(x, y), D(x, Tx), D(y, Ty), D(x, Ty), D(y, Tx)\}$$

true?

We remark that this question has an affirmative answer when T is a single-valued map on a complete metric space (cf. Ćirić [7]).

5. Stability results

In the sequel we shall need the following

LEMMA 5.1 ([10], see also [11], [30]). Let $\{\varepsilon_n\}$ be a sequence of nonnegative real numbers. Let

$$s_n = \sum_{r=0}^n (s\lambda)^{n-r} \varepsilon_r \text{ where } 0 \leq s\lambda < 1.$$

Then $\lim_n \varepsilon_n = 0$ iff $\lim_n s_n = 0$.

THEOREM 5.1. Let (X, d) be a complete b -metric space and $T : X \rightarrow CL(X)$ satisfy (2). Let $x_0 \in X$ and $\{x_n\}$ be an orbit for T at x_0 , i.e., $x_{n+1} \in Tx_n$, $n = 0, 1, 2, \dots$, and $\{x_n\}$ converge to a fixed point u of T . Let $\{y_n\}$ be a sequence in X and set $\varepsilon_n = H(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \dots$. If $s^2q < 1$ then

$$(4) \quad d(u, y_{n+1}) \leq sd(u, x_{n+1}) + s(s\lambda)^{n+1}d(x_0, y_0) \\ + s^3\lambda \sum_{r=0}^n (s\lambda)^{n-r} H_r + s^2 \sum_{r=0}^n (s\lambda)^{n-r} \varepsilon_r,$$

where $\lambda = sq/(1 - s^2q)$ and $H_r = H(x_r, Tx_r)$. If Tu is singleton, then $\lim_n y_n = u$ iff $\lim_n \varepsilon_n = 0$.

Proof. From (2), for any $x, y \in X$, one of the following holds:

$$\begin{aligned} H(Tx, Ty) &\leq qd(x, y) \\ H(Tx, Ty) &\leq qD(x, Tx) \leq qH(x, Tx); \\ H(Tx, Ty) &\leq qD(y, Ty) \leq qH(y, Ty) \leq sq\{H(y, x) + H(x, Ty)\} \\ &\leq sqd(x, y) + sq\{H(x, Tx) + H(Tx, Ty)\} \end{aligned}$$

i.e., $H(Tx, Ty) \leq \lambda\{d(x, y) + sH(x, Tx)\}$;

$$\begin{aligned} H(Tx, Ty) &\leq \frac{q}{2}\{D(x, Ty) + D(y, Tx)\} \leq \frac{q}{2}\{H(x, Ty) + H(y, Tx)\} \\ &\leq \frac{sq}{2}\{H(x, Tx) + H(Tx, Ty) + H(y, x) + H(x, Tx)\} \end{aligned}$$

i.e., $H(Tx, Ty) \leq sq/(2 - sq)\{2H(x, Tx) + d(y, x)\}$. Therefore, in all the cases,

$$(5) \quad H(Tx, Ty) \leq \lambda d(x, y) + s\lambda H(x, Tx).$$

We notice that, in view of Theorem 4.1, the conditions of Theorem 5.1 guarantee that T has a fixed point.

For any nonnegative integer n ,

$$d(x_{n+1}, y_{n+1}) \leq H(Tx_n, y_{n+1}) \leq s[H(Tx_n, Ty_n) + H(Ty_n, y_{n+1})].$$

Then, by virtue of (5), we obtain

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq s\lambda d(x_n, y_n) + s^2\lambda H(x_n, Tx_n) + s\varepsilon_n \\ &\leq s^2\lambda[\lambda d(x_{n-1}, y_{n-1}) + s\lambda H(x_{n-1}, Tx_{n-1}) + \varepsilon_{n-1}] \\ &\quad + s^2\lambda H_n + s\varepsilon_n \\ &\leq (s\lambda)^2 d(x_{n-1}, y_{n-1}) + s(s\lambda)^2 H_{n-1} + s(s\lambda)H_n \\ &\quad + s(s\lambda\varepsilon_{n-1} + \varepsilon_n). \end{aligned}$$

Repeating this process $(n-1)$ times, we get

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &\leq (s\lambda)^{n+1} d(x_0, y_0) + s^2\lambda \sum_{r=0}^n (s\lambda)^{n-r} H_r \\ &\quad + s \sum_{r=0}^n (s\lambda)^{n-r} \varepsilon_r. \end{aligned}$$

Consequently,

$$d(u, y_{n+1}) \leq sd(u, x_{n+1}) + sd(x_{n+1}, y_{n+1})$$

yields the inequality (4).

Assume that $\lim_n y_n = u$. Then by (5),

$$\begin{aligned} \varepsilon_n &= H(y_{n+1}, Ty_n) \leq sH(y_{n+1}, u) + sH(u, Ty_n) \\ &\leq sd(y_{n+1}, u) + sH(Tu, Ty_n) \\ &\leq sd(y_{n+1}, u) + s\lambda d(u, y_n) + s^2\lambda H(u, Tu) \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Suppose $\lim_n \varepsilon_n = 0$. Since $0 \leq s\lambda < 1$ and $\lim_n x_n = u$, first two terms on the right hand side of (4) vanish in the limit. Applying Lemma 5.1 to the last term of inequality (4), we see that $s^2 \lim_n \sum_{r=0}^n (s\lambda)^{n-r} \varepsilon_r = 0$.

Finally we show that, $s^3\lambda \lim_n \sum_{r=0}^n (s\lambda)^{n-r} H_r = 0$.

Let A denote the lower triangular matrix with entries $a_{nr} = (s\lambda)^{n-r}$. Then, $\lim_n a_{nr} = 0$ for each r , and

$$\lim_n \left(\sum_{r=0}^n a_{nr} \right) = \lim_n \left(\frac{1 - (s\lambda)^{n+1}}{1 - s\lambda} \right) = \frac{1}{1 - s\lambda}.$$

Therefore, A is multiplicative, i.e, for any convergent sequence $\{s_n\}$, $\lim_n A(s_n) = \frac{1}{1 - s\lambda} \lim_n s_n$. Thus, $s^3\lambda \lim_n \sum_{r=0}^n (s\lambda)^{n-r} H_r = 0$ (cf. [30], p. 692).

REMARK 5.1. If $s = 1$ and T is single-valued, then the following result is obtained from Theorem 5.1.

COROLLARY 5.1. *Let X be a complete metric space and $T : X \rightarrow X$ such that*

(6) $d(Tx, Ty) \leq q \cdot \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}(d(x, Ty) + d(y, Tx))\}$
 holds for all $x, y \in X$. Let u be the fixed point of T . Let $x_0 \in X$, $x_{n+1} = Tx_n$, $\{y_n\} \subset X$ and set $\varepsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, 2, \dots$. If $2q < 1$ then

$$d(u, y_{n+1}) \leq d(u, x_{n+1}) + q^{n+1}d(x_0, y_0) + mq \sum_{r=0}^n q^{n-r}d(x_r, x_{r+1}) + \sum_{r=0}^n q^{n-r}\varepsilon_r,$$

 where $m = (1 - q)^{-1}$. Then $\lim_n y_n = u$ iff $\lim_n \varepsilon_n = 0$.

We remark that the condition (6) is a slightly different from the following condition studied by Rhoades [30].

$$d(Tx, Ty) \leq c \cdot \max\{d(x, y), \frac{1}{2}(d(x, Tx) + d(y, Ty)), d(x, Ty), d(y, Tx)\}$$

where $0 \leq c < 1$. Indeed, Rhoades [op. cit.] has shown that the Picard iterative procedure for such a map T on a complete metric space is stable.

THEOREM 5.2. *Let the hypotheses of Theorem 5.1 hold, wherein ε_n is replaced by $\varepsilon_n = d(y_{n+1}, u_n)$, $u_n \in Ty_n$, $n = 0, 1, 2, \dots$; and $\lambda < s^{-2}$. Then*

$$(7) \quad d(u, y_{n+1}) \leq sd(u, x_{n+1}) + sg^{n+1}d(x_0, y_0) + s^2 \sum_{r=0}^n g^{n+1-r}H_r \\ + s^2 \sum_{r=0}^n g^{n-r}(sh_r + \varepsilon_r),$$

where $g = s^2\lambda$ and $h_r = H(x_{r+1}, Tx_r)$. Moreover, if Tu is singleton, then

$$\lim_n y_n = u \text{ iff } \lim_n \varepsilon_n = 0.$$

Proof. First we show the following

$$(8) \quad d(x_{n+1}, y_{n+1}) \leq g^{n+1}d(x_0, y_0) + s \sum_{r=0}^n g^{n+1-r}H_r \\ + s \sum_{r=0}^n g^{n-r}(sh_r + \varepsilon_r), \quad n = 0, 1, 2, \dots$$

We use (5) to have

$$d(x_1, y_1) \leq s[d(x_1, u_0) + d(u_0, y_1)] \leq sH(x_1, Ty_0) + s\varepsilon_0 \\ \leq s[sH(x_1, Tx_0) + sH(Tx_0, Ty_0)] + s\varepsilon_0,$$

that is,

$$d(x_1, y_1) \leq s^2\lambda d(x_0, y_0) + s(s^2\lambda)H_0 + s(sh_0 + \varepsilon_0).$$

This is the inequality (8) (for $n = 0$).

Now one may prove the inequality (8) by induction. Consequently, the relation (7) follows directly from

$$d(u, y_{n+1}) \leq sd(u, x_{n+1}) + sd(x_{n+1}, y_{n+1}).$$

Assume $\lim_n y_n = u$. Then $\varepsilon_n = d(y_{n+1}, u_n) \leq H(y_{n+1}, Ty_n)$ and, as in Theorem 5.1, $\lim_n \varepsilon_n = 0$. Suppose $\lim_n \varepsilon_n = 0$. Then from (5),

$$\begin{aligned} h_r &= H(x_{r+1}, Tx_r) \leq sH(x_{r+1}, Tu) + sH(Tu, Tx_r) \\ &\leq sd(x_{r+1}, u) + s\lambda d(u, x_r) + s^2\lambda H(u, Tu) \rightarrow 0 \text{ as } r \rightarrow \infty. \end{aligned}$$

Now, since $0 \leq g < 1$, in view of the proof of Theorem 5.1, $\lim_n y_n = u$. This completes the proof.

REMARK 5.2. Theorems 5.1 and 5.2 are extensions of the corresponding results of Czerwik, Dlutek and Singh [11].

REMARK 5.3. For $s = 1$, Theorems 5.1 and 5.2 include the similar results of Singh and Chadha [34].

REMARK 5.4. If we set $s = 1$ in Theorems 5.1 and 5.2, we obtain the similar results of Singh and Bhatnagar [33]. There is a scope to study stability problem for results presented in [4], [5], [15] and [16] on the setting of b -metric spaces.

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