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WEINGARTEN SURFACES OF REVOLUTION IN 3-DIMENSIONAL HYPERBOLIC SPACE

Abstract. The purpose of this paper is to construct a family of Weingarten surfaces of revolution satisfying the Weingarten relation $\kappa_1 = f(\kappa_2)$ in 3-dimensional hyperbolic space $\mathbb{H}^3$, where κ_1, κ_2 are principal curvatures and f is a some function.

1. Introduction

Geometers have been interested studying surfaces of constant mean curvature and constant Gaussian curvature in space forms for a long time. In this sense, Liebmann and Hilbert proved independently that the totally umbilical round spheres are the unique ovaloids with constant mean curvature and the unique compact surfaces with constant Gaussian curvature in Euclidean space. For the study of constant mean curvature or constant Gaussian curvature surfaces in 3-dimensional Lorentz-Minkowski space $\mathbb{L}^3$, J.-I. Hano and K. Nomizu ([4]) studied surfaces of revolution, F. Dillen et al., ([3]) classified ruled surfaces, and H. Liu ([7]) investigated translation surfaces. In particular, the present author and Y. H. Kim obtained a characterization of minimal ruled surfaces via the theory of finite type Gauss map ([5]) in $\mathbb{L}^3$. On the other hand, for the surfaces in 4-dimensional pseudo-Euclidean space H. Mori ([9]) constructed minimal rotation surfaces in 3-dimensional hyperbolic space $\mathbb{H}^3$, and in [8] H. Liu and G. Liu studied rotation surfaces of constant mean curvature in 4-dimensional pseudo-Euclidean space. On the other hand, constant mean curvature or constant Gaussian curvature surfaces are concerned with principal curvatures given on surfaces in space forms.

If we consider the class of Weingarten surface, that is, surfaces in 3-dimensional space forms whose principal curvatures κ_1, κ_2 satisfy the Wein-

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garten relation $\kappa_1 = f(\kappa_2)$ for some function f . So constant mean curvature surfaces are Weingarten surfaces.

In this article, we study surfaces of revolution in 3-dimensional hyperbolic space $\mathbb{H}^3$. Also, we construct Weingarten surfaces of revolution with Weingarten relation in $\mathbb{H}^3$.

2. Preliminaries

Let $\mathbb{L}^{n+1}$ be an $(n+1)$ -dimensional Lorentz-Minkowski space with the natural basis $e_1, \dots, e_{n+1}$, its metric $\langle, \rangle$ is given by

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{n+1} x_i y_i,$$

where $x = (x_1, x_2, \dots, x_{n+1}), y = (y_1, y_2, \dots, y_{n+1}) \in \mathbb{L}^{n+1}$. An n -dimensional de-Sitter space $\mathbb{S}_1^n$ and an n -dimensional hyperbolic space $\mathbb{H}^n$ is defined by

$$\mathbb{S}_1^n = \{x \in \mathbb{L}^{n+1} | \langle x, x \rangle = 1\}$$

and

$$\mathbb{H}^n = \{x \in \mathbb{L}^{n+1} | \langle x, x \rangle = -1, x_1 \geq 1\}.$$

It is well-known that $\mathbb{S}_1^n$ and $\mathbb{H}^n$ are the completely simply connected pseudo-Riemannian hypersurface and Riemannian hypersurface with constant sectional curvatures 1 and -1 in $\mathbb{L}^{n+1}$, respectively.

In 4-dimensional Lorentz-Minkowski space $\mathbb{L}^4$, let $C(u) = (x(u), y(u), z(u), 0), u \in I$, be any C^2 -curve in $\mathbb{H}^2$ which is parametrized by arc-length, whose domain of definition I is an open interval of real numbers including zero, and for which the following two equations are satisfies

$$(2.1) \quad \begin{cases} -x(u)^2 + y(u)^2 + z(u)^2 = -1, & x(u) \geq 1, \\ -x'(u)^2 + y'(u)^2 + z'(u)^2 = 1. \end{cases}$$

The surfaces

$$(2.2) \quad M : r(u, v) = (x(u), y(u), z(u) \cos v, z(u) \sin v), u \in I, v \in [0, 2\pi]$$

is by definition a surface of revolution in 3-dimensional hyperbolic space $\mathbb{H}^3$ obtained by rotating the curve $C(u)$ (See [2, 11]).

The first fundamental form of M in $\mathbb{H}^3$ is $du^2 + z(u)^2 dv^2$ and it is always space-like surface. By the regularity of the surface we assume that $z(u) > 0$ on I , and we put

$$(2.3) \quad \begin{aligned} \xi(u, v) = & (y'(u)z(u) - y(u)z'(u), x'(u)z(u) - x(u)z'(u), \\ & (x(u)y'(u) - x'(u)y(u)) \cos v, (x(u)y'(u) - x'(u)y(u)) \sin v). \end{aligned}$$

Then for the surface M , we have

$$\langle \xi, r_u \rangle = \langle \xi, r_v \rangle = \langle \xi, r \rangle = 0, \langle \xi, \xi \rangle = 1,$$

where $r_u = \frac{dr(u,v)}{du}$, $r_v = \frac{dr(u,v)}{dv}$. So we know that $\xi(u, v)$ is a unit normal vector field on M in $\mathbb{H}^3$.

We denote by $\tilde{\nabla}$ the covariant differentiation with respect to the induced pseudo-Riemannian metric of $\mathbb{L}^4$ and by $\bar{\nabla}$ and ∇ the covariant differentiations with respect to the induced metric of $\mathbb{H}^3$ and M , respectively. We denote by $\eta(x)$, $x \in \mathbb{H}^3$, the position vector of x with respect to the origin of $\mathbb{L}^4$ which is the normal vector field of $\mathbb{H}^3$ in $\mathbb{L}^4$. Then, considering that M is locally embedded in $\mathbb{H}^3$, we have the following Gauss's and Weingarten's formulas:

$$\begin{aligned} \tilde{\nabla}_X Y &= \bar{\nabla}_X Y + \langle X, Y \rangle \eta, \\ \bar{\nabla}_X Y &= \nabla_X Y + h(X, Y) \xi, \\ \bar{\nabla}_X \xi &= -A_\xi(X), \end{aligned} \quad (2.4)$$

where X and Y are tangent vector fields on M , and A_ξ is a field of type (1,1) tensor on M corresponding to ξ , that is $\langle A_\xi(X), Y \rangle = h(X, Y)$. From the above equations, we have

$$\begin{aligned} \langle A_\xi(r_u), r_u \rangle &= -\langle \bar{\nabla}_{r_u} \xi, r_u \rangle = \langle \xi, \bar{\nabla}_{r_u} r_u \rangle = \langle \xi, \tilde{\nabla}_{r_u} r_u \rangle \\ &= -x''(y'z - yz') + y''(x'z - xz') + z''(xy' - x'y), \\ \langle A_\xi(r_v), r_v \rangle &= -\langle \bar{\nabla}_{r_v} \xi, r_v \rangle = \langle \xi, \bar{\nabla}_{r_v} r_v \rangle = \langle \xi, \tilde{\nabla}_{r_v} r_v \rangle \\ &= -z(xy' - x'y), \\ \langle A_\xi(r_u), r_v \rangle &= -\langle \bar{\nabla}_{r_u} \xi, r_v \rangle = \langle \xi, \bar{\nabla}_{r_u} r_v \rangle = \langle \xi, \tilde{\nabla}_{r_u} r_v \rangle = 0. \end{aligned} \quad (2.5)$$

Since $\langle r_u, r_u \rangle = 1$, $\langle r_u, r_v \rangle = 0$, $\langle r_v, r_v \rangle = z^2$, then from (2.5) we obtain the principal curvatures κ_1 and κ_2 of M :

$$\begin{aligned} \kappa_1 &= z(x'y'' - x''y') - z'(xy'' - x''y) + z''(xy' - x'y), \\ \kappa_2 &= -z^{-1}(xy' - x'y). \end{aligned} \quad (2.6)$$

3. Weingarten surfaces of revolution in $\mathbb{H}^3$

In this section, we will construct a surfaces of revolution satisfying the Weingarten relation $\kappa_1 = f(\kappa_2)$ for some function f in 3-dimensional hyperbolic space $\mathbb{H}^3$. From the previous argument we see that the surface M has the following relations:

$$-x(u)^2 + y(u)^2 + z(u)^2 = -1, \quad x(u) \geq 1, \quad (3.1)$$

$$-x'(u)^2 + y'(u)^2 + z'(u)^2 = 1, \quad (3.2)$$

on the interval I . We now try to solve the above equations. From (3.1) we may put $x = x(u)$ and $y = y(u)$ by

$$(3.3) \quad \begin{cases} x(u) = (1 + z^2)^{\frac{1}{2}} \cosh \phi(u), \\ y(u) = (1 + z^2)^{\frac{1}{2}} \sinh \phi(u), \end{cases}$$

and then determine the function $\phi = \phi(u)$ satisfying (3.2). By using (3.2) and (3.3) we have

$$(3.4) \quad \phi'^2 = \frac{1 + z^2 - z'^2}{(1 + z^2)^2}.$$

We assume that $1 + z^2 - z'^2 > 0$ on I (when $1 + z^2 - z'^2 = 0$, ϕ is constant). Then, the function $\phi(u)$ is of the form

$$(3.5) \quad \phi(u) = \pm \int_0^u \frac{(1 + z(t)^2 - z'(t)^2)^{\frac{1}{2}}}{1 + z(t)^2} dt,$$

and without loss of generality we may assume that the signature is positive. From (3.3) and (3.5), we can show that

$$(3.6) \quad xy' - x'y = (1 + z^2 - z'^2)^{\frac{1}{2}},$$

$$(3.7) \quad xy'' - x''y = \frac{zz' - z'z''}{\sqrt{1 + z^2 - z'^2}}.$$

Differentiating (3.1) and (3.2) we obtain

$$\begin{aligned} xx' - yy' &= zz', \\ xx'' - yy'' &= zz'' + 1, \\ x'x'' - y'y'' &= z'z''. \end{aligned}$$

Solving the above equations for x'' and y'' we get

$$\begin{aligned} (xy' - x'y)x'' &= y'(zz'' + 1) - yz'z'', \\ (xy' - x'y)y'' &= x'(zz'' + 1) - xz'z'', \end{aligned}$$

which imply that we obtain from (3.6)

$$(3.8) \quad x'y'' - x''y' = \frac{-1 + z'^2 - zz''}{\sqrt{1 + z^2 - z'^2}}.$$

Substituting (3.6), (3.7) and (3.8) into (2.6), we have

$$(3.9) \quad \begin{aligned} \kappa_1 &= \frac{z'' - z}{\sqrt{1 + z^2 - z'^2}} = \frac{z - z''}{tz}, \\ \kappa_2 &= -z^{-1}\sqrt{1 + z^2 - z'^2} = t. \end{aligned}$$

Suppose that the surface of revolution M defined by (2.3) satisfies the Weingarten relation $\kappa_1 = f(\kappa_2)$ for some function f . Then from (3.9) we have

$$(3.10) \quad \frac{z - z''}{tz} = f(t).$$

By $t = -z^{-1}(1 + z^2 - z'^2)^{\frac{1}{2}}$, we obtain

$$(3.11) \quad \begin{aligned} z' &= \sqrt{1 + z^2 - z^2 t^2}, \\ z'' &= \frac{zz' - zz't^2 - z^2 tt'}{\sqrt{1 + z^2 - z^2 t^2}} = z - zt^2 - \frac{z^2 tt'}{z'}. \end{aligned}$$

From (3.10) and (3.11) we have

$$z \frac{dt}{dz} = f(t) - t$$

which implies

$$z = \exp \left(\int_0^u \frac{1}{f(t) - t} dt \right).$$

Consequently, we have

THEOREM 3.1. *Let M be a Weingarten surface of revolution in 3-dimensional hyperbolic space $\mathbb{H}^3$, the principal curvatures κ_1, κ_2 of M satisfy the Weingarten relation $\kappa_1 = f(\kappa_2)$, then it can be written as a part of the following surface:*

$$\left\{ \begin{array}{l} r(u, v) = (x(u), y(u), z(u) \cos v, z(u) \sin v), \quad u \in I, v \in [0, 2\pi] \\ x(u) = (1 + z^2)^{\frac{1}{2}} \cosh \phi(u) \\ y(u) = (1 + z^2)^{\frac{1}{2}} \sinh \phi(u) \\ \phi(u) = \int_0^u \frac{(1 + z(t)^2 - z'(t)^2)^{\frac{1}{2}}}{1 + z(t)^2} dt \\ z(u) = \exp \left(\int_0^u \frac{1}{f(t) - t} dt \right). \end{array} \right.$$

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