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PETRI NETS AS BITOPOLOGICAL SPACES

Abstract. In the paper a topological presentation of Petri nets has been presented. The presentation is based on a closure operation defined for an arbitrary transitive and dense binary relation. Such relations lead to a natural definition of Petri nets as a special bitopological spaces. In the paper some elementare properties of such spaces—the counter parts of some properties of Petri nets—are described. The paper may be seen as a generalization of [3].

1. Introduction

Petri nets are used as almost "standard" models of various concurrent systems. An important advantage of nets is that the user can in a sense "see" the shape of a process or system. A disadvantage of the mostly used two sorted presentation of nets is the fact that their (homo)morphisms are defined in a not "typically algebraic" way (see e.g. [1] or [8]). The presentation of nets used in this paper defines them as equationally definable (relational) algebras with one carrier-set only. In [4] it has been shown that Petri nets can be treated as triples of the form $N = (X, \text{entry}, \text{exit})$ with X being a set (of elements of the net N) called the *carrier* of the net N and $\text{entry}, \text{exit} \subseteq X \times X$ being binary relations defined on the whole set X and satisfying the conditions:

- (i) $\text{entry} \circ \text{entry} = \text{entry} \circ \text{exit} = \text{entry}$,
- (ii) $\text{exit} \circ \text{entry} = \text{exit} \circ \text{exit} = \text{exit}$,
- (iii) $\text{entry} \circ (\text{entry} \setminus \text{id}_X) = O$,
- (iv) $\text{exit} \circ (\text{exit} \setminus \text{id}_X) = O$.

So, a Petri net can be seen as a set with two transitive, dense (conditions (i) and (ii)) and nonextendable (conditions (iii) and (iv)) relations in the set X . The class of homomorphisms of such nets is richer than the class of "classical" homomorphisms of nets. If one wants to use net homomorphisms as a model of "aggregation of transitions into places" then one has to choose

between the simple two-sorted definition of nets and complicated definition of their homomorphisms, or a "non-typical" definition of nets (in this case of the above one-sorted presentation) and typical algebraic definition of their homomorphisms. In this paper the second possibility has been chosen. It seems that in this case, in contrast e.g. to programming, it is easier to work with one-sorted than with two-sorted algebras. The one sorted definition of Petri nets allows for a presentation of nets as some generalizations of topological spaces, the so called *bitopological spaces*. The main result of the paper is the introduction of such a bitopological representation of Petri nets.

In the paper the standard mathematical notation and terminology (see e.g. [6]) is used. Exemptions are the notation for the composition of relations and functions $((R \circ S)(x) = S(R(x)))$ and the fact that total relations are written in the form $R : X \longrightarrow Y$ instead of the more often used form $R \subseteq X \times Y$, that means if $R \subseteq X \times Y$ and for every $x \in X$ there exists $y \in Y$ with xRy then we write $R : X \longrightarrow Y$ instead of $R \subseteq X \times Y$. For all unexplained concepts of category theory and notation used in the paper the reader is referred to [6].

2. Topologies induced by relations

Let X be a set and $R : X \longrightarrow X$ be a relation in X satisfying the condition $R^2 = R \circ R = R$, i.e. R is transitive and dense. For any $A \subseteq X$ we define $C(A) = A \cup R(A)$.

PROPOSITION 1. *The operation $C : Pow(X) \longrightarrow Pow(X)$ is a topological closure operation in the set X .*

The topology and the closure operation determined in a set X by a transitive and dense relation $R : X \longrightarrow X$ will be called *induced* by R and denoted by τ_R and C_R respectively. Let X be a set, $R : X \longrightarrow X$ be a transitive and dense relation in the set X , C be the closure operation induced by the relation R , and R_{fix} be the set of all fix points of R .

PROPOSITION 2. *For every element x of X the one element set $\{x\}$ is closed in the topology induced by R iff $x \in R_{fix}$.*

Proof. Let $x \in X$ and $C(\{x\}) = \{x\}$. So we have $C(\{x\}) = \{x\} \cup R(\{x\}) = \{x\}$ which implies $R(\{x\}) = \emptyset$ or $R(\{x\}) = \{x\}$. The first equality is not possible because R is defined on the whole set X . So $R(\{x\}) = \{x\}$. The converse is straightforward. ■

The idea of our considerations is to obtain a topological presentation of Petri nets. We consider a bitopological space (X, τ_1, τ_2) consisting of a set X and topologies τ_1, τ_2 in the set X . One element subsets of X are elements of the corresponding Petri nets. If such a set, say $\{x\}$, is (in both topologies)

open then it represents a transition, if it is closed, a place of the Petri net. The closure operations determine relations $entry, exit : X \longrightarrow X$ defining the net. In order to illustrate a problem arising in this construction let us consider an example.

EXAMPLE 3. Let us consider a set $X = \{x, y, z\}$ and relations $en, ex \subseteq X \times X$ given by equalities:

$$en(x) = ex(x) = \{y, z\}, en(y) = ex(y) = \{y, z\}, en(z) = ex(z) = \{y, z\}.$$

Now we have

$$\begin{aligned} en(en(x)) &= ex(en(x)) = en(x) = \{y, z\}, \\ ex(en(y)) &= en(en(z)) = en(y) = \{y, z\}, \\ en(en(z)) &= ex(en(z)) = en(x) = \{y, z\}, \\ ex(en(x)) &= ex(en(x)) = ex(x) = \{y, z\}, \\ en(ex(y)) &= ex(ex(z)) = ex(y) = \{y, z\}, \\ ex(en(z)) &= ex(ex(z)) = ex(z) = \{y, z\}, \end{aligned}$$

i.e. the triple (X, en, ex) is a relational generalization of a graph seen as a one sorted algebra. But this "graph" has a very special property. It has no vertices. In fact there exists no element $x \in X$ with $en(x) = \{x\}$.

We have obtained this problem because in contrary to graphs the relations en and ex aren't functions. However such an assumption is in general too strong limitation.

DEFINITION 4. A relation $R : X \longrightarrow X$ is called *nonextendable* in a subset $X_0 \subseteq X$ iff $R \circ (R \setminus id_{X_0}) = \emptyset$. A family $(R_i : X \longrightarrow X)_{i \in I}$ of relations is *nonextendable* in X_0 iff for any $i, j \in I$ it holds $R_i \circ (R_j \setminus id_{X_0}) = \emptyset$.

The category of transitive, dense and nonextendable relations with morphisms transforming for any pairs (X, R) and (X', R') the relation R into $R \cup id_X$ will be denoted by **TDNRel**. Firstly we characterize the nonextendability of a relation in terms of its fixpoints.

PROPOSITION 5. A relation $R : X \longrightarrow X$ is nonextendable in the set $R(X)$ iff $R(X)$ is the set of all fix points of R .

Proof. $\implies$) Let us assume R is nonextendable in $R(X)$ and let $x \in R(X)$. If it was $R(\{x\}) \neq \{x\}$, then there would exist an element $y \in X$, such that $y \in R(\{x\})$ and $y \neq x$ (because $Dom R = X$). On the other hand there exists $x_0 \in X$ such that $x_0 R x$ (because $x \in R(X)$). Now we would have $x_0 R x$ and $x R y$ and $x \neq y$, i.e. $x_0 (R \circ (R \setminus id_X)) y$ which contradicts the assumption that $R \circ (R \setminus id_X) = \emptyset$.

$\impliedby$) Let $R(X)$ be the set of all fix points of R . So for every $x \in R(X)$ and $y \in X$ we have $x R y$ iff $x = y$, i.e. $R \circ (R \setminus id_X) = \emptyset$. ■

A relation $R : X \longrightarrow X$ (a family $(R_i : X \longrightarrow X)_{i \in I}$) of relations will be called *nonextendable* if it is nonextendable in the set $R(X)$.

COROLLARY 6. *Any transitive and dense function is nonextendable.*

Proof. Let $f : X \longrightarrow X$ be a transitive and dense function and $x \in f(X)$. So $x = f(y)$ for an element $y \in X$. Immediately from the definition of transitivity and density we obtain $f(x) = f(f(y)) = f(y) = x$, i.e. x is a fixpoint of f and we obtain the thesis by Proposition 5 above. ■

The above property can be exploited in order to obtain a bitopological presentation of graphs. Any graph can be seen as a triple (X, τ_1, τ_2) with τ_1 and τ_2 being some topologies in the set X (see [5]).

PROPOSITION 7. *For any set X , transitive, dense and nonextendable relation $R : X \longrightarrow X$ and $x \in X$ the set $\{x\}$ is open iff $x \in X \setminus R_{fix}$.*

Proof. Let $x \in X \setminus R(X)$ and let us consider the interior of $\{x\}$. Let $I : Pow(X) \longrightarrow Pow(X)$ be the interior operation in the topological space (X, τ_R) . From the property $I(A) \subseteq A$ it follows that $I(\{x\}) = \emptyset$ or $I(\{x\}) = \{x\}$. If it was $I(\{x\}) = \emptyset$ then $\{x\}$ would be a frontier set, i.e. it would be closed in τ_R , which is impossible because $x \in X \setminus R_{fix}$ (c.f. Proposition 2 above). So it must be $I(\{x\}) = \{x\}$, i.e. $\{x\}$ is an open set. The converse is evident. ■

The topological space (X, τ_R) may be very "irregular" and needn't satisfy even very weak separation conditions. On the other side, if τ_R is a T_1 topology, then R must be the identity relation in X and τ_R is the discrete topology. Some "between cases" are determined by the request that R has a non empty set of fix points. These fix points are exactly one element closed sets. All other one element sets are open. This property characterizes topological spaces induced by transitive, dense and nonextendable relations.

DEFINITION 8. A topological space (X, τ) will be called $T_{1/2}$ -space iff any one element set is either open or closed.

DEFINITION 9. For any $T_{1/2}$ -space (X, τ) and any points $x, y \in X$ we define $xR_\tau y \Leftrightarrow y \in C(\{x\}) \wedge C(\{y\}) = \{y\}$.

PROPOSITION 10. *The relation R_τ is transitive, dense and nonextendable.*

Proof. Let $x, y \in X$. We have

$$\begin{aligned}
 xR_\tau^2 y &\Leftrightarrow xR_\tau z \wedge zR_\tau y \text{ for some } z \in X && [\text{by def. of superposition}] \\
 &\Leftrightarrow z \in C(\{x\}) \wedge C(\{z\}) = \{z\} \wedge y \in C(\{z\}) \wedge C(\{y\}) = \{y\} && [\text{by def. of } R_\tau] \\
 &\Rightarrow z \in C(\{x\}) \wedge y = z \wedge C(\{z\}) = \{z\} [y \in C(\{z\}) \wedge C(\{z\}) = \{z\} \Rightarrow y = z] \\
 &\Rightarrow y \in C(\{x\}) \wedge C(\{y\}) = \{y\} && [y = z] \\
 &\Rightarrow xR_\tau y,
 \end{aligned}$$

So we have shown that $R_\tau^2 \subseteq R_\tau$. Now

$$\begin{aligned}
 xR_\tau y &\Leftrightarrow y \in C(\{x\}) \wedge C(\{y\}) = \{y\} && [\text{by def. of } R_\tau^2] \\
 \Rightarrow y \in C(\{x\}) \wedge C(\{y\}) = \{y\} \wedge y \in C(\{y\}) \wedge C(\{y\}) = \{y\} &&& [p \wedge q \Rightarrow p \wedge q \wedge q] \\
 \Rightarrow xR_\tau y \wedge yR_\tau y &&& [\text{by def. of } R_\tau] \\
 \Rightarrow \exists z xR_\tau z \wedge zR_\tau y &&& [P(x) \Rightarrow \exists_x P(x)] \\
 \Leftrightarrow x(R_\tau \circ R_\tau)y.
 \end{aligned}$$

which completes the proof of the first part of proposition.

b) If it was $xR_\tau \circ (R_\tau \setminus id_x)y$ for some $x, y \in X$ than we had

$$xR_\tau \circ (R_\tau \setminus id_x)y \Leftrightarrow \exists z xR_\tau z \quad \text{and} \quad z(R_\tau \setminus id_x)y.$$

Let $xR_\tau z_0$ and $z_0(R_\tau \setminus id_x)y$. Now we have

$$\begin{aligned}
 z_0 \in C(\{x\}) \wedge C(\{z_0\}) = \{z_0\} \wedge y \in C(\{z_0\}) \wedge y \neq z_0 &&& [\text{by def of } R_\tau] \\
 \Rightarrow z_0 = y \wedge z_0 \neq y &&& [C(\{z_0\}) = \{z_0\} \wedge y \in C(\{z_0\})].
 \end{aligned}$$

Thus it must be $R_\tau \circ (R_\tau \setminus id_x) = \emptyset$. ■

Let R and R' be transitive and dense relations in the sets X and X' respectively and $f : X \rightarrow X'$ be a function transforming R into $R' \cup id_{X'}$.

PROPOSITION 11. *The triple $f : (X, \tau_R) \rightarrow (X', \tau_{R'})$ is a continuous mapping from the topological space (X, τ_R) into $(X', \tau_{R'})$.*

Proof. If $(f \times f)(R) \subseteq R'$, i.e. if $R \circ f \subseteq f \circ (R' \cup id_{X'})$ then we have

$$\begin{aligned}
 f(C(A)) &= f(A \cup R(A)) && [\text{by def. of the closure operation}] \\
 &= f(A) \cup f(R(A)) && [f(A \cup B) = f(A) \cup f(B)] \\
 &= f(A) \cup (R \circ f)(A) && [f(R(A)) = (R \circ f)(A)] \\
 &\subseteq f(A) \cup (f \circ (R' \cup id_{X'}))(A) && [R \circ f \subseteq f \circ (R' \cup id_{X'})] \\
 &= f(A) \cup (f \circ R')(A) \cup (f \circ id_{X'})(A) = f(A) \cup R'(f(A)) \cup f(A) \\
 &= f(A) \cup R'(f(A)) \\
 &= C'(f(A)) && [\text{by def. of the closure operation}]. \quad \blacksquare
 \end{aligned}$$

LEMMA 12. *For any $T_{1/2}$ -space (X, τ) and any points $x, y \in X$ it holds*

$$x \neq y \wedge \{x\} \in \tau \wedge y \in C(\{x\}) \Rightarrow C(\{y\}) = \{y\}.$$

Proof. Let us assume $x \neq y$, $\{x\} \in \tau$ and $y \in C(\{x\})$. From the property $C(\{x\}) = \{x\} \cup Fr(\{x\})$, we infer that $y \in Fr(\{x\})$ that means for any $A \in \tau$ with $y \in A$ we have $A \cap \{x\} \neq \emptyset$. If the set $\{y\}$ would not be closed then it had to be open and from the fact that (X, τ) is of the type $T_{1/2}$ and $x \neq y$ we obtain $y \in \{y\} \in \tau$ and $\{y\} \cap \{x\} = \emptyset$ which contradicts the property $y \in Fr(\{x\})$. ■

LEMMA 13. For any $T_{1/2}$ -space $\Upsilon = (X, \tau)$ and points $x, y \in X$ it holds

$$x \neq y \wedge y \in C(x) \implies \{x\} \in \tau.$$

Proof. If it was $x \neq y$, $y \in C(x)$ and $\{x\} \notin \tau$ then it would be $C(\{x\}) = \{x\}$ because Υ is $T_{1/2}$ -space. Now it would be $y \in \{x\} = C(\{x\})$ and $y \neq x$ which is impossible. ■

COROLLARY 14. For any $T_{1/2}$ -space (X, τ) and any points $x, y \in X$ it holds

$$x \neq y \wedge y \in C(\{x\}) \implies C(\{y\}) = \{y\}.$$

Proof. Straightforward. ■

PROPOSITION 15. For any $T_{1/2}$ -spaces $T = (X, \tau)$ and $T' = (X', \tau')$ and any continuous mapping $f : T \longrightarrow T'$ we have

$$xR_{\tau}y \implies f(x)R_{\tau'}f(y) \text{ or } f(x) = f(y)$$

for all $x, y \in X$.

Proof. Let $x, y \in X$ and $xR_{\tau}y$. So we have $y \in C(\{x\})$ and $C(\{y\}) = \{y\}$. From the continuity of f we obtain

$$f(y) \in f(C(\{x\})) \subseteq C'(f(\{x\})) \wedge f(C(\{y\})) = \{f(y)\}.$$

Let us consider the set $\{f(y)\}$. If it is closed then $C'(\{f(y)\}) = \{f(y)\}$ and $f(x)R_{\tau'}f(y)$. If it is open then $f(x) = f(y)$ by corollary 14 above (if it was $f(x) \neq f(y)$ then it would be $C'(\{f(y)\}) = \{f(y)\}$ because $f(y) \in C'(f(\{x\}))$). ■

For any $T_{1/2}$ -space $\Upsilon = (X, \tau)$ let $\Gamma(\Upsilon) = (X, R_{\tau})$ and for any continuous mapping $f : \Upsilon \longrightarrow \Upsilon'$ let $\Gamma(f) : \Gamma(\Upsilon) \longrightarrow \Gamma(\Upsilon')$ be given by the assignment $\Gamma(f)(x) = f(x)$ for any $x \in X$. Then $\Gamma : \mathbf{Top}_{1/2} \longrightarrow \mathbf{TDNRel}$ is a functor from the category $\mathbf{Top}_{1/2}$ into the category $\mathbf{TDNRel}$. Let R and R' be transitive, dense and nonextendable relations in the sets X and X' respectively. Let the function $f : (X, R) \longrightarrow (X', R')$ transform the relation R into R' . Assigning to the pairs $\mathcal{R} = (X, R)$ and $\mathcal{R}' = (X', R')$ respectively the topological spaces (X, τ_R) and $(X', \tau_{R'})$ and to the triple $f : (X, R) \longrightarrow (X', R')$ the triple $f : (X, \tau_R) \longrightarrow (X', \tau_{R'})$ one obtains a functor $\Delta : \mathbf{TDNRel} \longrightarrow \mathbf{Top}_{1/2}$ from the category $\mathbf{TDNRel}$ into the category $\mathbf{Top}_{1/2}$.

COROLLARY 16. Categories $\mathbf{TDNRel}$ and $\mathbf{Top}_{1/2}$ are equivalent.

Proof. It is an immediate consequence of Propositions 11 and 15 above. ■

3. Bitopological spaces

By a *bitopological space* we mean a set endowed in two topologies, i.e. a triple of the form (X, τ_1, τ_2) , $((X, C_1, C_2)$ or (X, I_1, I_2) resp.) with τ_1, τ_2

being families of open sets (C_1, C_2 being topological closure operations and I_1, I_2 being topological interior operations resp.). Of course topologies in X may be defined in many other ways (e.g. by bases, subbases etc.). A bitopological space (X, τ_1, τ_2) will be called *open compatible* on a family $F \subseteq \text{Pow}(X)$ iff for any set $A \in F$ $A \in \tau_1 \iff A \in \tau_2$, *closed compatible* on F iff for any set $A \in F$ A is closed in $\tau_1 \iff A$ is closed in τ_2 and *compatible* iff it is open and closed compatible on F . A triple $f : (X, \tau_1, \tau_2) \longrightarrow (X', \tau'_1, \tau'_2)$ with $(X, \tau_1, \tau_2), (X', \tau'_1, \tau'_2)$ being bitopological spaces and $f : X \longrightarrow X'$ being a function will be called *bicontinuous* mapping iff both $f : (X, \tau_1) \longrightarrow (X', \tau'_1)$ and $f : (X, \tau_2) \longrightarrow (X', \tau'_2)$ are continuous mappings. For more detailed description of bitopological spaces the reader is referred to [3].

$T_{1/2}$ -bitopological spaces compatible on the family of one element sets will be called *net-spaces*, *N-spaces* for shortness. Let (X, C_1, C_2) be an *N-space*.

DEFINITION 17. For any $x, y, z \in X$ we define

- (a) $B^O(x, y, z) \iff y \in C_2(x) \cap C_1(z) \wedge \neg(x = y = z)$
- (b) $B^C(x, y, z) \iff x \in C_1(y) \wedge z \in C_2(y) \wedge \neg(x = y = z)$.

LEMMA 18. For any $x, y, z \in X$ it holds $B^O(x, y, z) \implies C(\{y\}) = \{y\}$ and $B^C(x, y, z) \implies \{y\} \in \tau$.

Proof. (a) Let us consider the following cases:

- (i) $x = z \neq y$. So we have $y \in C_1(\{z\})$ and $y \neq z$ and we obtain the thesis by Corollary 14.
- (ii) $x = y \neq z$. Analogously.
- (iii) $x \neq z = y$. Now $y \in C_2(x)$ and $y \neq x$ which implies $C(\{y\}) = \{y\}$ by Corollary 14.

In the remaining cases the reasoning is similar.

(b) If $B^C(x, y, z)$ then $x \in C_1(y)$ and $z \in C_2(y)$ which implies $\{y\} \in \tau$. Now if $x \neq y$ or $z \neq y$ then $\{y\} \in \tau$ by Corollary 14. In the remaining cases the reasoning is similar. ■

PROPOSITION 19. Bicontinuous mappings of *N-spaces* transform the relation B^O into $B'^O \cup \text{id}_{X'}$ and B^C into $B'^C \cup \text{id}_{X'}$.

Proof. Let $f : (X, C_1, C_2) \longrightarrow (X', C'_1, C'_2)$ be a bicontinuous mapping of *N-spaces* and $x, y, z \in X$.

- (1) Assume $B^O(x, y, z)$. So we have $y \in C_2(x)$ and $y \in C_1(z)$ which implies

$$f(y) \in f(C_2(x)) \subseteq C'_2(f(x)) \wedge f(y) \in f(C_1(z)) \subseteq C'_1(f(z)).$$

If $\neg(f(x) = f(y) = f(z))$ then we have $B^O(f(x), f(y), f(z))$ else

$$(f(x), f(y)), (f(x), f(z)), (f(y), f(z)) \in \text{id}_{X'}.$$

- (2) Assume $B^C(x, y, z)$. Now we have $x \in C_1(y)$ and $z \in C_2(y)$ which implies

$$f(x) \in f(C_1(y)) \subseteq C'_1(f(y)) \wedge f(z) \in f(C_2(y)) \subseteq C'_2(f(y)).$$

The remaining part is analogous to that in the point (1). ■

4. Petri nets as bitopological spaces

Immediately from Proposition 1 and Definition 8 we infer that every Petri net $N = (X, \text{entry}, \text{exit})$ induces a bitopological space $\Upsilon = (X, \tau_{\text{entry}}, \tau_{\text{exit}})$ in which both topologies $\tau_{\text{entry}}, \tau_{\text{exit}}$ are of the type $T_{1/2}$. Homomorphisms of Petri nets have been defined as relations compatible with the relations *entry* and *exit*, i.e. a relation $\rho \subseteq X \times X'$ is a *homomorphism* from a net $N = (X, \text{entry}, \text{exit})$ into $N' = (X', \text{entry}', \text{exit}')$ iff $\text{entry} \circ \rho \subseteq \rho \circ \text{entry}' \wedge \text{exit} \circ \rho \subseteq \rho \circ \text{exit}'$.

In Lemmas 20–22, Propositions 23, 24 and Corollary 25 below $\Upsilon = (X, \tau_1, \tau_2)$ denotes an arbitrary $T_{1/2}$ -space compatible on the family of one element sets. C_1 and C_2 denote the closure operations in Υ and Cl the family of all one element closed sets of Υ . R_1 and R_2 stand for the relations R_{τ_1} and R_{τ_2} as described in Definition 9 above.

LEMMA 20. 1. $R_1 \circ R_1 = R_1 \circ R_2 = R_1 \wedge R_2 \circ R_1 = R_2 \circ R_2 = R_2$

2. $R_i \circ id_{Cl} = R_i \wedge id_{Cl} \circ R_i = id_{Cl}$.

Proof. For $x, y, z \in X$ we have

$$xR_1yR_2z \iff y \in C_1(\{x\}) \wedge C_1(y) = \{y\} \wedge z \in C_2(\{y\}) \wedge C(z) = \{z\}.$$

From the compatibility of Υ on one element sets we obtain

$$z \in C_2(\{y\}) = C_1(\{y\}) = \{y\},$$

i.e. $y = z$. So xR_1z that means $R_1 \circ R_2 \subseteq R_1$. If xR_1y then

$$(y \in C_1(x) \wedge C_1(y) = \{y\}) \wedge (y \in C_2(y) \wedge C_2(y) = \{y\}),$$

i.e. $xR_1 \circ R_2y$, which implies $R_1 \subseteq R_1 \circ R_2$. Proofs of the remaining equalities are similar. ■

LEMMA 21. $R_i \circ (R_j \setminus id_{Cl}) = \emptyset$.

Proof. If it was $x(R_i \circ (R_j \setminus id_{Cl}))z$ then we would have

$$y \in C_i(x) \wedge C_i(y) = \{y\} \wedge z \in C_j(y) \wedge C_j(z) = \{z\} \wedge y \neq z.$$

By the compatibility of Υ on one element sets we obtain $C_j(y) = C_i(y) = \{y\}$ and consequently $z = y$ (because $z \in C_j(y)$) which contradicts the inequality $y \neq z$. ■

LEMMA 22. $B^O(x, y, z) \iff yR_2x \wedge yR_1z \wedge B^C(x, y, z) \iff xR_2y \wedge zR_1y$.

Proof. $B^O(x, y, z)$

$$\iff y \in C_2(x) \wedge y \in C_1(z) \wedge C_1(y) = \{y\} = C_2(y)$$

$$\iff (y \in C_2(x) \wedge C_2(y) = \{y\}) \wedge (y \in C_1(z) \wedge C_1(y) = \{y\})$$

$$\iff yR_2x \wedge yR_1z.$$

Proof of the second equivalence is similar. ■

PROPOSITION 23. *For any bitopological $T_{1/2}$ -space $\Upsilon = (X, \tau_1, \tau_2)$ compatible on one element sets the triple $\text{Net}(\Upsilon) = (X, R_1, R_2)$ is a Petri net (an e-net in [4]).*

Proof. It is an immediate consequence of Lemmas 20 and 21 above. ■

PROPOSITION 24. *For any bicontinuous mapping $f : (X, \tau_1, \tau_2) \longrightarrow (X', \tau'_1, \tau'_2)$ the triple $f : \text{Net}(X, \tau_1, \tau_2) \longrightarrow \text{Net}(X', \tau'_1, \tau'_2)$ is a homomorphism of Petri nets.*

Proof. It is an immediate consequence of Lemma 22 and Proposition 19 above. ■

If one restricts the class of homomorphisms of Petri nets (see [1], [4] or [8]) to those being functions then from corollary 16 we infer that the corresponding category of Petri nets is equivalent to the category of bitopological $T_{1/2}$ -spaces.

COROLLARY 25. *Assignments*

$$\text{Net}(X, \tau_1, \tau_2) = (X, R_1, R_2)$$

and

$$\text{Net}(f : (X, \tau_1, \tau_2) \longrightarrow (X', \tau'_1, \tau'_2)) = f : (X, R_1, R_2) \longrightarrow (X', R'_1, R'_2)$$

determine a functor $\text{Net} : \mathbf{Bitop} \longrightarrow \mathbf{e-nets}$ from the category of bitopological spaces compatible on one element sets into the category of e-nets. Conversely, assigning to any e-net $N = (X, \text{entry}, \text{exit})$ the bitopological space $\text{Top}(N) = (X, \tau_{\text{entry}}, \tau_{\text{exit}})$ and to any net homomorphism $f : N \longrightarrow N'$ the triple $f : \text{Top}(N) \longrightarrow \text{Top}(N')$ we obtain a functor from e-nets into $\mathbf{Bitop}$.

It can be proved that $\text{Top}(\text{Net}(\Upsilon)) = \Upsilon$ and $\text{Net}(\text{Top}(N)) = N$. The proof consists of simple but long calculations. Morphisms of the category of e-nets in general do not preserve the reachability relation. Those preserving this relation form the subcategory of closed continuous mappings and the subcategory of open continuous mappings is equivalent to the category of two sorted Petri nets.

4.1. The topological counter parts of some fundamental notions of Petri net theory. Let us illustrate how some typical notions of Petri nets theory can be expressed in the proposed formalism. Here only few fundamental and well known notions and properties will be described. The topo-

logical counter parts (definitions) of other notions and properties of Petri nets are also easy to formulated. Their proofs may be obtained immediately from the proofs of their "original" form as they have been formulated in the "classic" theory of Petri nets.

The "structural" properties of nets generating their partition into some "structural" classes can be characterized topologically as follows. Let $\Upsilon = (X, \tau_1, \tau_2)$ be a topological presentation of a Petri net, that means an N -space.

PROPOSITION 26. *Υ is a synchronisationsgraph iff for any one element closed set $\{x\}$ and open sets $\{t\}, \{t'\}$ it holds*

$$x \in C_1(t) \cap C_1(t') \text{ or } x \in C_2(t) \cap C_2(t') \implies t = t'.$$

In other words two different one element open sets have always the empty intersection of their frontiers in both topologies. Let us note that the intersection $C_i(t) \cap C_j(t')$ for $i \neq j$ needn't be empty.

PROPOSITION 27. *Υ is a state-machine iff for any one element open set $\{t\}$ and closed sets $\{x\}, \{x'\}$ it holds*

$$\{x, x'\} \subseteq C_1(t) \text{ or } \{x, x'\} \subseteq C_2(t) \implies x = x'.$$

In other words for any one element $\{t\}$ open set we have

$$\text{card}(\text{Fr}_1(t)) < 2 \text{ and } \text{card}(\text{Fr}_2(t)) < 2.$$

PROPOSITION 28. *Υ is a free choice net iff for any one element open sets $\{t\}$ and $\{t'\}$ and any closed set $\{x\}$ it holds*

$$x \in \text{Fr}_1(t) \cap \text{Fr}_1(t') \implies \text{Fr}_1(t) = \text{Fr}_1(t') = x.$$

The last property (free choiceness) may be generalized to the so called extended free-choiceness (see [7]).

PROPOSITION 29. *Υ is an extended free choice net iff for any open sets $\{t\}$ and $\{t'\}$ we have*

$$\text{Fr}_1(t) \cap \text{Fr}_1(t') \neq \emptyset \implies \text{Fr}_1(t) = \text{Fr}_1(t').$$

In order to speak about dynamic properties of nets we need a counterpart of the notion of marking. By a $\{0,1\}$ -marking in a Petri net $N = (P, T, \text{pre}, \text{post})$ one understands any subset of the set P of places of N . The natural topological counter part of this notion is the notion of frontier set. So, by a marked net we understand any pair (Υ, A) with $\Upsilon = (X, \tau_1, \tau_2)$ being an N -space and $A \subseteq X$ being a frontier set in both topologies τ_1 and τ_2 ⁽¹⁾. Now let $\Upsilon = (X, \tau_1, \tau_2)$ be an N -space and $A \subseteq X$.

⁽¹⁾ Let us note that markings may be defined as frontier sets in any of both topologies because the topologies τ_1 and τ_2 are compatible on closed sets.

DEFINITION 30. A is a *cycle* iff $Fr_1(A) \cap Fr_2(A) \neq \emptyset$.

An N-space Υ will be called *cycle-free* iff the only cycles of Υ are frontier sets. Cycle-free N-spaces correspond to the so called *occurrence-nets*, that means process-schemes.

DEFINITION 31. A continuous mapping of N-spaces

$$\pi : \Upsilon \longrightarrow \Upsilon'$$

will be called a *process* in the space τ' iff τ is cycle-free and for any one element open set $\{x\}$ both $\pi|_{C_1(\{x\})}$ and $\pi|_{C_2(\{x\})}$ are bijections (are homeomorphism).

Having the notion of a process one can define some operations on processes as follows. Let $\pi_0 : \Upsilon_0 \longrightarrow \Upsilon$ be a process in Υ .

- The origin and the end of π_0 are the processes

$$origin(\pi_0) = \pi|_{Fr_1(\tau)} \text{ and } end(\pi_0) = \pi|_{Fr_2(\tau)}.$$

- If $\pi_1 : \tau_1 \longrightarrow \tau$ is a process in τ such that $\pi_1(Fr_2(\tau)) = \pi_1(Fr_1(\tau_1))$ then the *sequential composition* of π and π_1 is the process

$$\pi_0 \circ \pi_1 : \tau_0 + \tau_1 \longrightarrow \tau$$

such that

$$(\pi_0 \circ \pi_1)((x, i)) = \begin{cases} \pi_0(x) & \text{if } i = 0 \\ \pi_1(x) & \text{if } i = 1 \end{cases}.$$

- If $\pi_0(\tau_0) \cap \pi_1(\tau_1) = \emptyset$ then the process

$$\pi_0 + \pi_1 : \tau_0 + \tau_1 \longrightarrow \tau$$

will be called the *parallel composition* of π_0 and π_1 .

The following properties of the algebra

$$(Processes_in_ \tau, origin, end, \circ, +)$$

can be easily proved.

1. The reduct $(Proc_in_ \tau, origin, end, \circ)$ is a category.
2. The reduct $(Proc_in_ \tau, +, \emptyset)$ is an abelian monoid.
3. $origin(\pi + \pi_2) \xrightarrow{=} origin(\pi) + origin(\pi')$ and $end(\pi + \pi_2) \xrightarrow{=} end(\pi) + end(\pi')$.
4. $\pi_1 \circ \pi_2 + \pi_3 \circ \pi_4 \xrightarrow{=} (\pi_1 + \pi_3) \circ (\pi_2 + \pi_4)$.

REMARK 32. The symbol $\xrightarrow{=}$ denotes the weak equality and should be read "if the left side is defined, the the rigth side is defined as well and both sides are equal".

5. Concluding remarks

There exists a lot of various "translations" of the language of relations (functions) into that of (families of) sets. Perhaps the most well known examples are the principle of abstraction or the relationships between tolerance relations and covering families of a set ⁽²⁾. There are also known some connections between relational systems and topology, e.g. topological spaces generated by semi- or partial orders. The result presented in the paper is of the same kind. The only essential difference here is the type of considered spaces; they are not "similar" to the "classical" topological spaces with a very "geometrical" origin. On the other side the origin of graphs is of a geometric character. In this sense the result of the paper can be seen as an illustration of the fact that various generalization of some geometrical ideas may leads to "non-geometric" notions. It may also be interesting how the properties of n-topologies (or at least bitopologies) are related to the properties of n-graphs and vice versa, e.g. how one can characterize the convergence in the language of graphs.

The origins of the one-sorted definition of graphs come probably from the "French school of category theory" (cf. [6] or [2] where some references can be found).

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⁽²⁾ This relationship play an important role in various theories of concurrency, e.g. in the theory of Petri nets. The sets of concurrent (independent) actions are exactly classes of a special tolerancy relation (the relation of mutual independence).