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STRONG VARIATIONAL PRINCIPLES AND GENERIC WELL-POSEDNESS

Abstract. The strong version of Ekeland's variational principle [JMAA, 47(1974), 324-353] due to Georgiev [JMAA, 131(1988), 1-21] is deductible in a direct way from the standard one. In addition, a substitution between them is possible for many generic well-posed optimization problems.

1. Introduction

Let (M, d) be a complete metric space; and $f : M \rightarrow R \cup \{\infty\}$, some function with

(1.1) f is proper ($\text{Dom}(f) \neq \emptyset$), bounded below ($\inf[f(M)] > -\infty$),

(1.2) f is lsc on M ($\{x \in M; f(x) \leq t\}$ is closed, for all $t \in R$).

The following 1974 statement in Ekeland [7] (referred to as Ekeland's variational principle (EVP) – the standard form) is well known.

THEOREM 1. *Let $\gamma > 0$ be arbitrary fixed, as well as some $u \in \text{Dom}(f)$. There exists then a point $z = z(u)$ in $\text{Dom}(f)$ with*

(1.3) $\gamma d(u, z) \leq f(u) - f(z)$ (hence $f(u) \geq f(z)$),

(1.4) $\gamma d(z, w) - (f(z) - f(w)) > 0$, for all $w \in M \setminus \{z\}$.

Hence, for each λ in $]0, 1[$ we necessarily have

(1.5) $\gamma d(z, w) - \lambda(f(z) - f(w)) > (1 - \lambda)\gamma d(z, w)$, for all $w \in M \setminus \{z\}$;

wherefrom, the convergence property below holds:

(1.6) $(w_n) \subseteq M, \gamma d(z, w_n) - \lambda(f(z) - f(w_n)) \rightarrow 0$ imply $w_n \rightarrow z$.

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This principle found some basic applications to convex and nonconvex analysis; we refer to the 1979 paper by Ekeland [8] for a survey of these. Consequently, it must be not surprising that, soon after its formulation, many versions of Theorem 1 were proposed. For example, a proof of this principle involving the denumerable chains of the quasi-ordered structure $(M, \leq)$, where

$$(1.7) \quad (x, y \in M): x \leq y \text{ iff } d(x, y) + f(y) \leq f(x)$$

was given in the 1976 paper by Brezis and Browder [2]; for various extensions of it we refer to Altman [1], Turinici [20] or Kang and Park [11]. Some pseudometric extensions of the same were proposed by Tataru [19] and Suzuki [18]; see also Isac [10]. Further efforts in this direction emerge from the fact that the very formulation of (1.3)+(1.4) makes our statement be viewed as a denumerable variant of Zorn's maximality principle (cf. Moore [14, ch. 4, Sect. 4]). So, the "abstract" contributions in this area due to Brunner [4] and Manka [13] may be added to this list; see also Brøndsted [3].

In the following, we shall concentrate on a certain "sharpened" local version of Theorem 1, also referred to as Ekeland's variational principle (EVP) – the strong form. (The specific assumptions (1.1)+(1.2) prevail).

THEOREM 2. *Let $\gamma, \delta > 0$ be arbitrary fixed, as well as some $u \in \text{Dom}(f)$. There exists then $z = z(\gamma, \delta; u) \in \text{Dom}(f)$ with the properties (1.4) and*

$$(1.8) \quad \gamma d(u, z) \leq f(u) - f(z) + \delta \text{ (hence } f(u) \geq f(z) - \delta),$$

$$(1.9) \quad (w_n) \subseteq M, \gamma d(z, w_n) - (f(z) - f(w_n)) \rightarrow 0 \text{ imply } w_n \rightarrow z.$$

This principle – which, technically speaking, is metric in nature – was obtained in 1988 by Georgiev [9] via geometric methods involving the Drop theorem (cf. Danes [5] and Penot [15]); another proof of the same, based on (generic) smooth variational techniques was provided in 1993 by Deville, Godefroy and Zizler [6]. But, as already suggested by (1.5)+(1.6), there must be a direct (metrical) way of deducing it from the standard EVP (subsumed to Theorem 1). It is our aim in the present exposition to show – by means of elementary methods – that this is indeed the case; details will be given in Section 2. The reduction process in question is not only methodological. Because, as we shall see from the study of generic well-posedness (cf. Section 3) the standard EVP gives, in these particular cases, the same amount of information like the strong EVP. This may have a theoretical impact upon the last quoted principle; but, from a practical perspective, the situation may be reversed. Further aspects will be delineated elsewhere.

2. Strong variational principles

We make now precise the assertions of the introductory part. Let (M, d) be a complete metric space; and $f : M \rightarrow R \cup \{\infty\}$, some function with the properties (1.1)+(1.2). The following completion of Theorem 2 is available.

THEOREM 3. *Let $\gamma, \delta > 0$ be arbitrary fixed, as well as some $u \in \text{Dom}(f)$. There exists then a number $\lambda = \lambda(\gamma, \delta; u)$ in $]0, 1[$ and some point $z = z(\gamma, \delta; u)$ in $\text{Dom}(f)$ with the properties (1.8) and*

$$(2.1) \quad \gamma d(z, w) - (f(z) - f(w)) > \lambda \gamma d(z, w), \text{ for all } w \in M \setminus \{z\};$$

so, necessarily, (1.9) holds too.

Proof. Denote for simplicity

$$M_u = \{x \in M; f(x) \leq f(u)\} \text{ (where } u \text{ is the above one).}$$

Clearly, $M_u \neq \emptyset$ (because $u \in M_u$); in addition, by the lsc assumption (1.2), M_u is closed (hence complete) in M . Let again f stand for the restriction of f over M_u ; note that (1.2) holds over M_u , as well as (1.1), because f is bounded on M_u and $\inf[f(M_u)] = \inf[f(M)]$. Finally, let $\lambda = \lambda(\delta; u)$ in $]0, 1[$ be taken in accordance with

$$(2.2) \quad \frac{\lambda}{1-\lambda}(f(u) - f_*) \leq \delta, \quad \text{where } f_* = \inf[f(M_u)];$$

and put $\gamma^* = (1-\lambda)\gamma$. By the remarks above, the standard EVP (subsumed to Theorem 1) is applicable to (M_u, d) , f and γ^* . So, for the starting point $u \in M_u$ there must be another point $z = z(\gamma^*; u)$ in M_u with the properties (1.3)+(1.4) [written for M_u and γ^* in place of M and γ]. The corresponding version of (1.3) yields, via (2.2),

$$\begin{aligned} \gamma d(u, z) &\leq \frac{1}{1-\lambda}(f(u) - f(z)) = f(u) - f(z) + \frac{\lambda}{1-\lambda}(f(u) - f(z)) \\ &\leq f(u) - f(z) + \frac{\lambda}{1-\lambda}(f(u) - f_*) \leq f(u) - f(z) + \delta; \end{aligned}$$

and from this, (1.8) is clear. On the other hand, the corresponding version of (1.4) gives (2.1) (by the very definition of M_u). The proof is complete. ■

Some remarks are in order. The difference between this result and the preceding one consists in the global evaluation (2.1). With an extra work however, it can be obtained via the reasoning of Theorem 2 (based, as above said, on (geometric type) drop techniques; cf. Georgiev [9]). Although rather involved when compared with the elementary argument used in Theorem 3, this raises a lot of interesting technical aspects; cf. Turinici [21]. On the other hand, the very proof above shows that

$$(2.3) \quad \text{Theorem 1} \implies \text{Theorem 3} \quad (\text{from a logical viewpoint}).$$

The question of the reciprocal implication being also true remains open; we conjecture that the answer is negative.

3. Generic well-posedness

Let (X, d) be a complete metric space. Denote by $\mathcal{K}(X)$ the family of all (nonempty) closed parts of X . Clearly, this object depends only on the topology induced by the ambient metric. So, it remains the same when d is to be substituted by the (topological) equivalent metric

$$(3.1) \quad e(x, y) = \min\{1, d(x, y)\}, \quad x, y \in X;$$

which, in addition, is bounded (over X). The natural metric on $\mathcal{K}(X)$ is the Hausdorff one

$$(3.2) \quad \Delta(A, B) = \max[\sup\{e(x, B); x \in A\}, \sup\{e(y, A); y \in B\}];$$

here, $e(\cdot, Z)$ is the distance to Z function (where $\emptyset \neq Z \subseteq X$). Note that

$$(3.3) \quad (\mathcal{K}(X), \Delta) \text{ is complete (cf. Kuratowski [12, ch 3, Sect 29])}.$$

Further, denote by $\mathcal{C}_s(X)$ the class of all bounded from below lsc functions from X to R . Like before, this object depends only on the topology induced by d (cf. (1.2)). So, it remains the same when d is to be substituted by e above. The natural metric on $\mathcal{C}_s(X)$ is

$$(3.4) \quad \Gamma(f, g) = \sup\{\sigma(f(x), g(x)); x \in X\};$$

here, $\sigma(\cdot, \cdot)$ is the metric on R attached to the standard metric $\rho(\cdot, \cdot)$ over the same under the model of (3.1) above:

$$\sigma(s, t) = \min\{1, \rho(s, t)\} \quad (\text{where } \rho(s, t) = |s - t|).$$

In addition to this, one has (directly)

$$(3.5) \quad (\mathcal{C}_s(X), \Gamma) \text{ is complete (in the usual sense)}.$$

Having these precised, put $\mathcal{M} = \mathcal{K}(X) \times \mathcal{C}_s(X)$ and give this object the product metric $\Delta \times \Gamma$ (which, by (3.3) and (3.5) is complete too). For each couple (A, f) in $\mathcal{M}$, consider the minimization problem

$$(P(A, f)) \text{ find } a \in A \text{ with } f(a) = \inf[f(A)].$$

Following Revalski [17], we say that this problem is *well posed* when it has at least one solution $a \in A$ with the regularity property

$$(3.6) \quad (x_n) \subseteq X, d(x_n, A) \rightarrow 0, f(x_n) \rightarrow f(a) \text{ imply } x_n \rightarrow a.$$

Denote by $\mathcal{M}_0$ the associated subclass of $\mathcal{M}$; i.e.,

$$\mathcal{M}_0 = \{(A, f) \in \mathcal{M}; (P(A, f)) \text{ is well posed}\}.$$

The basic question to be solved is that of determining the "size" of $\mathcal{M}_0$ in $\mathcal{M}$. An appropriate answer to this is contained in

THEOREM 4. *Under the above conventions*

(3.7) $\mathcal{M}_0$ is dense in $\mathcal{M}$.

That is: for each $(A, f) \in \mathcal{M}$ and each ε in $]0, 1[$ there exists $(A_\varepsilon, f_\varepsilon) \in \mathcal{M}_0$ in such a way that $\Delta(A, A_\varepsilon) \leq \varepsilon$, $\Gamma(f, f_\varepsilon) \leq \varepsilon$.

Proof. Take the numbers $\mu > 1$, $\gamma > 0$ in accordance with $\mu\gamma < \varepsilon$. Further, put $M = \{x \in X; e(x, A) \leq \varepsilon\}$; and fix some $a \in A$. Clearly, the standard EVP (subsumed to Theorem 1) is applicable to (M, e) , f and γ ; so, for the starting point a in $A \subseteq M$ there must be another one $a(\varepsilon)$ in M with the properties (1.3)+(1.4) (written for the precised data). The latter of these becomes

(3.8) $\gamma e(a(\varepsilon), x) > f(a(\varepsilon)) - f(x)$, for all $x \in M \setminus \{a(\varepsilon)\}$.

Denote for simplicity

$$A_\varepsilon = A \cup \{a(\varepsilon)\}; \quad f_\varepsilon(x) = f(x) + \mu\gamma e(a(\varepsilon), x), \quad x \in X.$$

We claim that this couple is the desired one so as to satisfy the conclusions in the statement.

(i) For the moment, it is clear that $(A_\varepsilon, f_\varepsilon) \in \mathcal{M}$. Moreover, we actually have $(A_\varepsilon, f_\varepsilon) \in \mathcal{M}_0$; i.e., $(P(A_\varepsilon, f_\varepsilon))$ is well posed. Indeed, $f_\varepsilon(a(\varepsilon)) = \inf[f_\varepsilon(A_\varepsilon)]$; because (cf. (3.8))

(3.9) $f_\varepsilon(x) - f_\varepsilon(a(\varepsilon)) \geq (\mu - 1)\gamma e(a(\varepsilon), x) \geq 0$, for all $x \in M$.

In addition, let (x_n) be a sequence in X with

(3.10) $d(x_n, A_\varepsilon) \rightarrow 0$, $f_\varepsilon(x_n) \rightarrow f_\varepsilon(a(\varepsilon))$, as $n \rightarrow \infty$.

If $d(x_n, a(\varepsilon)) \rightarrow 0$, we are done. The alternative to this yields

(3.11) $\inf_n d(y_n, a(\varepsilon)) > 0$, for a certain subsequence (y_n) of (x_n) .

Combining with the evident relation

$$d(x, A_\varepsilon) = \min\{d(x, a(\varepsilon)), d(x, A)\}, \quad x \in X,$$

gives, by means of (3.10) (the first half)

$$d(y_n, A) \rightarrow 0; \text{ hence } y_n \in M, \quad \forall n \geq n(\varepsilon), \text{ where } n(\varepsilon) \text{ is large enough.}$$

But then (3.10) (the second half) yields [via (3.9)]

$$e(y_n, a(\varepsilon)) \rightarrow 0 \text{ (hence } d(y_n, a(\varepsilon)) \rightarrow 0); \text{ in contradiction with (3.11).}$$

Consequently, this alternative cannot hold; and the claim follows.

(ii) By the very choice of $a(\varepsilon)$ (as element of M) one has $\Delta(A_\varepsilon, A) = e(a(\varepsilon), A) \leq \varepsilon$. In addition, the relation

$$|f_\varepsilon(x) - f(x)| = \mu\gamma e(a(\varepsilon), x) \leq \mu\gamma < \varepsilon, \quad x \in X$$

yields (by definition) $\Gamma(f_\varepsilon, f) \leq \varepsilon$. The proof is complete. ■

A different way of getting the same conclusion is that of using the strong EVP (subsumed to Theorem 2); we refer to Georgiev [9] for details. But, as results from the proof in the statement above, this device is avoidable. Moreover, a close analysis shows that in all density results of the quoted paper, an argument based on the standard EVP is available too. An interesting open question is that of such a conclusion having a general character; we conjecture that the answer is positive. Further aspects may be found in Phelps [16, Sect. 5].

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