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GENERALIZED CR-SUBMANIFOLDS OF THE TRANS HYPERBOLIC SASAKIAN MANIFOLD

Abstract. The purpose of the present paper is to study generalized CR- submanifolds of the trans hyperbolic Sasakian manifold and obtained their basic properties. The conditions under which the distribution of generalized CR-submanifolds of the trans hyperbolic Sasakian manifold is integrable have been obtained. Totally geodesic generalized CR- submanifolds of the trans hyperbolic Sasakian manifold have also been studied.

1. Introductution and preliminaries

Bejancu defined and studied CR-submanifolds of Kaehlerian manifolds [1]. CR-submanifolds of Sasakian manifolds were studied by Kobayashi [8]. Bejancu and Papaghuic defined almost semi – invariant submanifold of Sasakian manifold [3]. Oubina studied a new class of almost contact Riemannian manifold known as Trans Sasakian manifold [6], which generalizes both α -Sasakian and β -Kenmotsu structure. Ion Mihai introduced a new class of submanifolds called genealised CR-submanifolds of Sasakian manifolds [4]. M. D. Upadhyay studied an almost contact hyperbolic (f, g, η, ξ) -structure [9]. Bhatt and Dube studied on a CR-submanifolds of a trans hyperbolic Sasakian manifold [7]. Submanifolds of hyperbolic contact metric manifold were studied by R. B. Pal [11]. N. K. Joshi and K. K. Dube studied a semi invariant submanifold of an almost r-contact hyperbolic metric manifold [10]. The purpose of the present paper is to study the generalised CR-submanifold of a trans Hyperbolic Sasakian manifold.

Let V_n be an n-dimensional almost hyperbolic contact metric manifold with the almost hyperbolic contact metric structure (F, U, u, g) , where F is a tensor field of type $(1, 1)$, U is a vector field, u is a 1-form and g is the Riemannian metric on V_n . Then the following conditions [9] are satisfied:

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$$(1.1) \quad F^2X = X + u(X)U, \quad FU = 0, \quad u \circ F = 0, \quad u(U) = -1,$$

$$(1.2) \quad g(FX, FY) = -g(X, Y) - u(X)u(Y),$$

$$(1.3) \quad g(FX, Y) = -g(X, FY),$$

for vector fields X, Y tangent to V_n . An almost hyperbolic contact metric structure (F, U, u, g) on V_n is called trans hyperbolic Sasakian [7] if and only if

$$(1.4) \quad (\bar{\nabla}_X F)Y = \alpha\{g(X, Y)U - u(Y)FX\} + \beta\{g(FX, Y)U - u(Y)FX\},$$

for all X, Y tangent to V_n and α, β non zero constant. Where ∇ is the Riemannian connexion with respect to g . From (1.4) it follows that

$$(1.5) \quad \bar{\nabla}_X U = -\alpha FX + \beta(X - u(X)U),$$

for any vector field X tangent to V_n . Now if V_m be an m -dimensional submanifold isometrically immersed in a trans hyperbolic Sasakian manifold V_n such that the structure vector field U of V_n is tangent to the submanifold V_m . We denote by $\{U\}$ the 1-dimensional distribution spanned by U on V_m and $\{U\}^\perp$ the complementary orthogonal distribution to $\{U\}$ in $T(V_m)$. For any $X \in T(V_m)$ we have $g(FX, U) = 0$. Then we let us put,

$$(1.6) \quad FX = bX + cX,$$

where $bX \in \{U\}^\perp$ and $cX \in T^\perp(V_m)$. Thus $X \rightarrow bX$ is an endomorphism of the tangent bundle $T(V_m)$ and $X \rightarrow cX$ is a normal bundle valued 1-form on V_m .

1.1. DEFINITION. A submanifold V_m of an almost hyperbolic contact metric manifold V_n with almost hyperbolic contact metric structure (F, U, u, g) is said to be generalized CR-submanifold if $D_X^\perp = T_X(V_m) \cap FT_X^\perp(V_m)$; $x \in V_m$ defines a differentiable subbundle of $T_x(V_m)$. Thus for $X \in D^\perp$ one has $bX = 0$. We denote by D the complementary orthogonal subbundle to $D^\perp \oplus \{U\}$ in $T(V_m)$.

For any $X \in D$, $bX \neq 0$. Also we have $bD = D$. Thus for generalised CR-submanifold V_m we have the orthogonal decomposition

$$(1.7) \quad T(V_m) = D \oplus D^\perp \oplus \{U\}.$$

2. Lemmas

For generalized CR-submanifold V_m of the trans hyperbolic Sasakian manifold V_n , we denote by g both the Riemannian metrics on V_n and V_m .

For each $X \in T(V_m)$ we put

$$(2.1) \quad X = PX + QX - u(X)U,$$

where PX and QX belong to the distribution D and $D^\perp$ respectively. For any $N \in \perp_X(V_m)$ we put

$$(2.2) \quad N = tN + fN,$$

where tN is tangential part of FN and fN is the normal part of FN . By using (1.2) we have

$$g(FX, cY) = g(FX, bY + cY) = g(FX, FY) = -g(X, Y) = 0,$$

for $X \in D_X^\perp$ and $Y \in D_X$. Therefore,

$$(2.3) \quad g(FD_X^\perp, cD_X) = 0.$$

We denote by v the orthogonal complementary vector bundle to $FD^\perp \oplus cD$ in $T^\perp(V_m)$. Thus we have

$$(2.4) \quad T^\perp(V_m) = FD^\perp \oplus cD \oplus v.$$

2.1. LEMMA. *The morphism t and f satisfy*

$$(2.5) \quad t(fD^\perp) = D^\perp$$

and

$$(2.6) \quad t(cD) \subset D.$$

Proof. For $X \in D^\perp$ and $Y \in D$, $g(tFX, Y) = g(tFX + fFY, Y) = g(F^2X, Y) = g(X, Y) = 0$.

Also $g(tFX, U) = g(F^2X, U) = g(FX, U) = 0$. Therefore $t(fD^\perp) \subset D^\perp$. For $X \in D^\perp$, we have, $X = F^2X = tFX + fFX$ which implies $X = tFX$.

Consequently, $D^\perp \subset t(FD^\perp)$. Hence the relation (2.5) follows. The relation (2.6) is trivial.

Now we denote by $\bar{\nabla}$ (resp. ∇) the Riemannian connexion on V_n (resp. V_m) with respect to the Riemannian metric g . The linear connection induced by $\bar{\nabla}$ on the normal bundle $T^\perp(V_m)$ is denoted by $\nabla^\perp$. Then the equation of Gauss and Wiengarten are given by

$$(2.7) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.8) \quad \bar{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

for $X, Y \in T(V_m)$ and $N \in T^\perp(V_m)$. Where h is the second fundamental form of V_m and A_N is the fundamental tensor of Wiengarten with respect to the normal section N . These tensor fields are related by

$$(2.9) \quad g(h(X, Y), N) = g(A_N X, Y),$$

for $X, Y \in T(V_m)$ and $N \in T^\perp(V_m)$.

We denote

$$W(X, Y) = \nabla_X bPY - A_{CPY}X - A_{FQY}X.$$

2.2. LEMMA. Let V_m be a generalized CR-submanifold of the trans hyperbolic Sasakian manifold V_n , then we have

$$(2.10) \quad P(W(X, Y)) - bP\nabla_X Y - Pth(X, Y) = -\alpha u(Y)PX - \beta u(Y)PbX$$

$$(2.11) \quad Q(W(X, Y)) - Qth(X, Y) = -\alpha u(Y)QX - \beta u(Y)QbX$$

$$(2.12) \quad u(W(X, Y)) = \alpha g(FX, FY) + \beta g(FbX, FbY)$$

$$(2.13) \quad h(X, bPY) + \nabla_X^\perp cPY + \nabla_X^\perp fQY - cP\nabla_X Y - fh(X, Y) = -\beta u(Y)cX$$

for $X, Y \in T(V_m)$.

Proof. Using (1.6), (2.1), (2.2), (2.7) and (2.8) in (1.4) we have, for $X, Y \in T(V_m)$

$$\bar{\nabla}_X FY - \bar{\nabla}_X FY = \alpha\{g(X, Y)U - u(Y)FX\} + \beta\{g(FX, Y)U - u(Y)FX\},$$

$$\bar{\nabla}_X FPY + \bar{\nabla}_X FQY - F\nabla_X Y - fh(X, Y)$$

$$= \alpha\{g(X, Y)U - u(Y)FX\} + \beta\{g(FX, Y)U - u(Y)FX\},$$

$$\nabla_X bPY + h(X, bPY) - A_{cPY}X + \nabla_X^\perp cPY - A_{FQY}X$$

$$+ \nabla_X^\perp FQY - b\nabla_X Y - c\nabla_X Y - FPh(X, Y) - FQh(X, Y)$$

$$= \alpha\{g(X, Y)U - u(Y)FX\} + \beta\{g(FX, Y)U - u(Y)FX\},$$

$$\nabla_X bPY + h(X, bPY) - A_{cPY}X + \nabla_X^\perp cPY - A_{FQY}X + \nabla_X^\perp FQY - bP\nabla_X Y$$

$$- cQ\nabla_X Y - Qb\nabla_X Y - cP\nabla_X Y - Pth(X, Y)$$

$$- Pfh(X, Y) - Qth(X, Y) - Qfh(X, Y)$$

$$= \alpha\{g(X, Y)U - u(Y)FX\} + \beta\{g(FX, Y)U - u(Y)FX\},$$

$$\nabla_X bPY + h(X, bPY) - A_{cPY}X + \nabla_X^\perp cPY - A_{FQY}X + \nabla_X^\perp FQY$$

$$- bP\nabla_X Y - cP\nabla_X Y - FQ\nabla_X Y - Pth(X, Y) - fh(X, Y) - Qth(X, Y)$$

$$= \alpha\{g(X, Y)U - u(Y)FX\} + \beta\{g(FX, Y)U - u(Y)FX\}.$$

Taking components on each of the vector bundle D , $D^\perp$, $\{U\}$ and $T^\perp(V_m)$ respectively, we get the required results.

2.3. LEMMA. For generalized CR-submanifolds V_m of the trans hyperbolic Sasakian manifold V_n , the following relations holds:

$$(2.14) \quad P(t\nabla_X^\perp N + A_{fN}X - \nabla_X tN) = bPA_NX,$$

$$(2.15) \quad Q(t\nabla_X^\perp N + A_{fN}X - \nabla_X tN) = 0,$$

$$(2.16) \quad u(A_{fN}X - \nabla_X tN) = -\beta g(cX, N),$$

$$(2.17) \quad h(X, tN) + FQA_NX + \nabla_X^\perp fN + cPA_NX = f\nabla_X^\perp N,$$

for $X \in T(V_m)$ and $N \in T^\perp(V_m)$.

Proof. For $X \in T(V_m)$ and $N \in T^\perp(V_m)$ by using (1.6), (2.1), (2.2) and (2.7), (2.8) in (1.4) we get,

$$\begin{aligned}
(\bar{\nabla}_X F)Y &= \alpha\{g(X, Y)U - u(Y)FX\} + \beta\{g(FX, Y)U - u(Y)FX\}, \\
\bar{\nabla}_X FY - F\bar{\nabla}_X Y &= \beta g(cX, N)U, \\
\nabla_X FY + h(X, FY) - F\bar{\nabla}_X(PY + QY - u(Y)U) &= \beta g(cX, N)U, \\
\nabla_X tN + h(X, tN) + \nabla_X fN + h(X, fN) - FP\bar{\nabla}_X Y - FQ\bar{\nabla}_X Y &= \beta g(cX, N)U, \\
P\nabla_X tN + Q\nabla_X tN - u(\nabla_X tN)U + h(X, tN) - PA_{fN}X - QA_{fN}X - u(A_{fN}X)U \\
&\quad + \nabla_X^\perp fN + bPA_NX - fP\nabla_X^\perp N + cPA_NX - FQ\nabla_X^\perp N + FQA_NX \\
&= \beta g(cX, N)U, \\
P\nabla_X tN + Q\nabla_X tN - u(\nabla_X tN)U + h(X, tN) - PA_{fN}X - QA_{fN}X \\
&\quad - u(A_{fN}X)U + \nabla_X^\perp fN + bPA_NX - Pt\nabla_X^\perp N - Qt\nabla_X^\perp N + cPA_NX \\
&\quad - f\nabla_X^\perp N + FQA_NX = \beta g(cX, N)U.
\end{aligned}$$

Taking components on each of the vector bundles D , $D^\perp$, $\{U\}$ and $T^\perp(V_m)$ respectively, we get the required results.

2.4. LEMMA. *For generalized CR-submanifold V_m of the trans hyperbolic Sasakian manifold V_n , we have the following:*

$$(2.18) \quad \nabla_X U = -\alpha bX + \beta X; h(X, U) = -\alpha cX \quad \text{for } X \in D,$$

$$(2.19) \quad \nabla_Y U = \beta Y; h(Y, U) = -\alpha FY, \quad \text{for } Y \in D^\perp,$$

$$(2.20) \quad \nabla_U U = 0; h(U, U) = 0.$$

Proof. The proof follows from (1.5) by using (1.6) (2.1) and (2.7), we have

$$\begin{aligned}
\bar{\nabla}_X U &= -\alpha bX - \alpha cX + \beta X, \\
\nabla_X U + h(X, U) &= -\alpha bX - \alpha cX + \beta X, \\
\nabla_X U &= -\alpha bX + \beta X, \quad h(X, U) = -\alpha cX.
\end{aligned}$$

Hence (2.18) follows.

$$\begin{aligned}
\bar{\nabla}_Y U &= -\alpha FY + \beta(Y - u(Y)U), \\
\nabla_Y U + h(Y, U) &= -\alpha FY + \beta Y - u(Y)\beta U, \\
\nabla_Y U &= \beta Y; \quad h(Y, U) = -\alpha FY, \quad \text{for } Y \in D^\perp.
\end{aligned}$$

Hence (2.19) follows.

$$\begin{aligned}
\bar{\nabla}_U U &= -\alpha FU + \beta(U - u(X)U), \\
\nabla_U U + h(U, U) &= -\alpha FU + \beta(U - u(X)U), \\
\nabla_U U &= 0, \quad h(U, U) = 0.
\end{aligned}$$

Hence (2.20) follows.

2.5. LEMMA. *Let V_m be a generalized CR-submanifold of the trans hyperbolic Sasakian manifold V_n . We have the following:*

$$(2.21) \quad A_{FX}Y = A_{FY}X, \quad \text{for } X, Y \in D^\perp.$$

Proof. By using (1.3), (1.4), (2.7) and (2.9) we get

$$\begin{aligned} g(A_{FX}Y, Z) &= g(h(Y, Z)FX) = g(\bar{\nabla}_Z Y, FX) = g(F\bar{\nabla}_Z Y, X), \\ &= g(\bar{\nabla}_Z FY, X) = g(FY, \bar{\nabla}_Z X) = g(h(Z, X), FY), \\ &= g(h(X, Z), FY) = g(A_{FY}X, Z) \end{aligned}$$

for $X, Y \in D^\perp$ and $Z \in T(V_m)$. Hence the lemma follows.

2.6. LEMMA. *Let V_m be a generalized CR-submanifold of the trans hyperbolic Sasakian manifold. We have the following:*

$$(2.22) \quad V \notin D^\perp, \quad \text{for } V \in D^\perp$$

$$(2.23) \quad \nabla_U S \notin D, \quad \text{for } S \in D.$$

Proof. Let us take $X = U$ and $V = FN$ in (2.14) where $N \in FD$. Taking account that $tN = FN$, $fN = 0$ we get $Pt\nabla_U^\perp N - P\nabla_U V = bPA_N U$

$$(2.24) \quad Pt\nabla_U^\perp N - bPA_N U = PA_U V.$$

The second relation of (2.18) gives

$$g(PN_N U, S) = g(A_N U, S) = g(h(S, U)N) = -\alpha g(cS, N) = 0, \quad \text{for } S \in D.$$

Hence (2.24) becomes

$$(2.25) \quad Pt\nabla_U^\perp N = P\nabla_U V.$$

On the other hand (2.17) implies

$$(2.26) \quad h(U, V) = f\nabla_U^\perp N - FQA_N U$$

for $V \in D^\perp$, $h(U, V) = h(V, U) = -\alpha FV \in FD^\perp$ by (2.19).

Now for $X \in D^\perp$, by virtue of Lemma (2.5) and of equation (2.9) we have

$$\begin{aligned} g(h(U, V), FX) &= g(h(V, U), FX) = g(A_{FX}V, U) = g(A_{FV}X, U), \\ &= g(h(X, U), FV) = g(h(X, U), N) = g(A_N U, X) = g(FA_N U, FX), \\ &= g(FPA_N U, FX) + g(FQA_N U, FX) = g(FQA_N U, FX), \end{aligned}$$

since $cD^\perp \in F^\perp$.

Therefore $h(U, V) = FQA_N U$, which together with (2.26) implies $f\nabla_U^\perp N = 2FQA_N U$, hence $\nabla_U^\perp N \notin FD^\perp$, since F is an automorphism of $cD \oplus v$. Thus $t\nabla_U^\perp N \notin D^\perp$ and from (2.25) it follows that

$$(2.27) \quad P\nabla_U V \neq 0, \quad \text{for all } V \in D^\perp.$$

Next from (2.16) we have

$$(2.28) \quad u(\nabla_U V) \neq 0,$$

for all $V = FN \in D^\perp$, where $N \in FD^\perp$.

Hence (2.22) follows from (2.27) and (2.28).

Finally by using (2.1), (2.20) and (2.22), we have

$$g(\nabla_U S, X) \neq g(\nabla_U S, PX) \quad \text{for } X \in TV_m$$

and $S \in D$. Thus we have $\nabla_U S \notin D$, for $S \in D$ and this completes the proof.

3. Integrability of distributions

3.1. THEOREM. *For generalized CR-submanifold V_m of the trans hyperbolic Sasakian manifold V_n , the distribution $D^\perp$ is always involutive.*

Proof. For $X, Y \in D^\perp$ by using (2.19) we get,

$$\begin{aligned} (3.1) \quad g([X, Y], U) &= g(\nabla_X Y, U) - g(\nabla_Y X, U) \\ &= g(X, \nabla_Y U) - g(Y, \nabla_X U) = 0. \end{aligned}$$

On the other hand from (2.10) we have

$$(3.2) \quad bP\nabla_X Y = -PA_{FY}X - Pth(X, Y), \quad \text{for } X, Y \in D^\perp$$

and then using lemma (2.5) we get from (3.2)

$$(3.3) \quad bP[X, Y] = 0, \quad \text{for } X, Y \in D^\perp.$$

As b is an automorphism of D , the theorem follows from (3.1) and (3.3).

3.2. THEOREM. *For generalized CR-submanifold V_m of the trans hyperbolic Sasakian manifold V_n , the distribution D is never involutive.*

Proof. For $X, Y \in D$ by using (2.18), we have

$$\begin{aligned} (3.4) \quad g([X, Y], U) &= g(\nabla_X Y, U) - g(\nabla_Y X, U) = g(X, \nabla_Y U) - g(Y, \nabla_X U) \\ &= -\alpha g(X, bY) + \alpha g(Y, BX) = 2\alpha g(Y, bX). \end{aligned}$$

Taking $X \neq 0$ and $Y = bX$ in (3.4), it follows that D is never involutive.

3.3. THEOREM. *For generalized CR-submanifold V_m of the trans hyperbolic Sasakian manifold V_n , the distribution $D^\perp$ is totally geodesic in V_m iff*

$$(3.5) \quad h(X, bZ) + \nabla_X^\perp cZ \in cD \oplus v \quad \text{for } X \in D^\perp \text{ and } Z \in D \oplus \{U\}.$$

Proof. For $X, Y \in D^\perp$, $Z \in D \oplus \{U\}$, and using (1.3), (1.4), (2.7) and (2.8), we get

$$\begin{aligned} (3.6) \quad g(\nabla_X Y, Z) &= -g(Y, \bar{\nabla}_X Z) = -g(\bar{\nabla}_X Z, Y) = -g(F\bar{\nabla}_X Z, FY) \\ &= -g(\bar{\nabla}_X bZ + \bar{\nabla}_X cZ, FY) \\ &= -g(\nabla_X bZ + h(X, bZ) - A_{cZ}X + \nabla_X^\perp cZ, FY) \\ g(\nabla_X Y, Z) &= -g(h(X, bZ) + \nabla_X^\perp cZ, FY). \end{aligned}$$

Hence the theorem follows from (3.6).

3.4. THEOREM. *Let V_m be a generalized CR-submanifold of the trans hyperbolic Sasakian manifold V_n . Then the distribution $D \oplus \{U\}$ is involutive*

and totally geodesic in V_m if and only if,

$$(3.7) \quad h(X, bY) + \nabla_X^\perp cY \in cD \oplus v \quad \text{for } X, Y \in D \oplus \{U\}.$$

Proof. For $X, Y \in D \oplus \{U\}$ and $Z \in D^\perp$ by using (1.3), (1.4), (1.6), (2.7) and (2.8), we obtain

$$(3.8) \quad \begin{aligned} g(\nabla_X Y, Z) &= g(\bar{\nabla}_X Y, Z) = g(F\bar{\nabla}_X Y, FZ), \\ g(\nabla_X Y, Z) &= g(\nabla_X bY + h(X, bY) - A_{cY}X + \nabla_X^\perp cY, FZ). \end{aligned}$$

Hence the theorem follows from (3.8).

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