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GENERALIZED EULER METHOD FOR NONLINEAR FIRST ORDER PARTIAL DIFFERENTIAL FUNCTIONAL EQUATIONS

Abstract. Classical solutions of the local Cauchy problem on the Haar pyramid are approximated in the paper by solutions of suitable quasilinear systems of difference equations. The proof of the stability of the difference problem is based on a comparison technique with nonlinear estimates of the Perron type.

This new approach to the numerical solving of nonlinear functional differential equations is generated by a quasilinearization method for initial problems.

Numerical examples are given.

1. Difference systems corresponding to nonlinear equations

For any metric spaces X and Y we denote by $C(X, Y)$ the class of all continuous functions from X into Y . We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Let E be the Haar pyramid

$$E = \{(t, x) \in \mathbb{R}^{1+n} : t \in [0, a], -b + Mt \leq x \leq b - Mt\}$$

where $x = (x_1, \dots, x_n)$, $a > 0$, $M = (M_1, \dots, M_n) \in \mathbb{R}_+^n$, $\mathbb{R}_+ = [0, +\infty)$, $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ and $b \geq Ma$. Write $E_0 = [-b_0, 0] \times [-b, b] \subset \mathbb{R}^{1+n}$ where $b_0 \in \mathbb{R}_+$ and $\Omega = E \times \mathbb{R} \times \mathbb{R}^n$. Suppose that the functions

$$f : \Omega \rightarrow \mathbb{R}, \quad \varphi : E_0 \rightarrow \mathbb{R}, \quad V : C(E_0 \cup E, \mathbb{R}) \rightarrow C(E, \mathbb{R})$$

are given. We consider the problem consisting of the functional differential equation

$$(1) \quad \partial_t z(t, x) = f(t, x, V[z](t, x), \partial_x z(t, x))$$

and the initial condition

$$(2) \quad z(t, x) = \varphi(t, x) \text{ for } (t, x) \in E_0$$

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where $\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z)$. A function $z : E_0 \cup E \rightarrow \mathbb{R}$ is called a classical solution of the above problem if

- (i) $v \in C(E_0 \cup E, \mathbb{R})$ and v is of class C^1 on E ,
- (ii) v satisfies (1) on E and initial condition (2) holds.

The Haar pyramid is a natural domain of the existence and uniqueness of classical or generalized solutions to initial problems for differential or functional differential equations with first order partial derivatives ([5], [9], [15], [16]). Write

$$E_t = (E_0 \cup E) \cap ([-b_0, t] \times \mathbb{R}^n), \quad 0 \leq t \leq a.$$

We will say that the operator V satisfies the Volterra condition if for each $(t, x) \in E$ and for $z, \bar{z} \in C(E_0 \cup E, \mathbb{R})$ such that $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in E_t$ we have $V[z](t, x) = V[\bar{z}](t, x)$ where $x \in [-b + Mt, b - Mt]$. Note that the Volterra condition means that the value of $V[z]$ at the point (t, x) depends on the restriction of z to the set E_t only.

Let us fix our notations on vectors and matrices. We will denote by $M_{n \times n}$ the space of all $n \times n$ matrices with real elements. For $x \in \mathbb{R}^n$, $U \in M_{n \times n}$ where

$$x = (x_1, \dots, x_n), \quad U = [u_{ij}]_{i,j=1,\dots,n},$$

we put

$$\|x\| = \sum_{j=1}^n |x_j| \text{ and } \|U\| = \max \left\{ \sum_{j=1}^n |u_{ij}| : 1 \leq i \leq n \right\}.$$

The product of two matrices is denoted by $\star$. If $U \in M_{n \times n}$ then U^T is the transpose matrix. We use the symbol $\circ$ to denote the scalar product in $\mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, we write $x \diamond y = (x_1 y_1, \dots, x_n y_n)$.

For functions $z \in C(E_0 \cup E, \mathbb{R})$, $u \in C(E_0 \cup E, \mathbb{R}^n)$, $u = (u_1, \dots, u_n)$, and for a point $t \in [0, a]$ we put

$$\|z\|_t = \max\{|z(\tau, y)| : (\tau, y) \in E_t\} \text{ and } \|u\|_t = \max\{\|u(\tau, y)\| : (\tau, y) \in E_t\}.$$

Let us denote by $\mathbb{F}(X, Y)$ the class of all functions defined on X and taking values in Y , where X and Y are arbitrary sets. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers respectively.

We formulate a difference problem corresponding to (1), (2). We define a mesh on the set $E_0 \cup E$ in the following way. Suppose that (h_0, h') , $h' = (h_1, \dots, h_n)$, stand for steps of the mesh. For $h = (h_0, h')$ and for $(r, m) \in \mathbb{Z}^{1+n}$ where $m = (m_1, \dots, m_n)$, we define nodal points as follows $t^{(r)} = rh_0$, $x^{(m)} = m \diamond h'$, $x^{(m)} = (x_1^{(m_1)}, \dots, x_n^{(m_n)})$. Denote by Δ the set of all $h = (h_0, h')$ such that there is $N = (N_1, \dots, N_n) \in \mathbb{N}^n$ with the

properties: $N \diamond h' = b$ and $h' \leq Mh_0$. Let $K \in \mathbb{N}$ be defined by $Kh_0 \leq a < (K+1)h_0$. Write

$$\mathbb{R}_h^{1+n} = \{(t^{(r)}, x^{(m)}) : (r, m) \in \mathbb{Z}^{1+n}\}$$

and

$$E_{h,0} = E_0 \cap \mathbb{R}_h^{1+n}, \quad E_h = E \cap \mathbb{R}_h^{1+n}, \quad I_h = \{t^{(r)} : 0 \leq r \leq K\},$$

$$E_{h,r} = (E_{h,0} \cup E_h) \cap ([-b_0, t^{(r)}] \times \mathbb{R}^n), \quad 0 \leq r \leq K.$$

For functions $\eta : I_h \rightarrow \mathbb{R}$, $z : E_{h,0} \cup E_h \rightarrow \mathbb{R}$, $u : E_{h,0} \cup E_h \rightarrow M_{n \times n}$ we write $\eta^{(r)} = \eta(t^{(r)})$, $z^{(r,m)} = z(t^{(r)}, x^{(m)})$, $u^{(r,m)} = u(t^{(r)}, x^{(m)})$ and

$$\|z\|_{h,i} = \max\{|u^{(r,m)}| : (t^{(r)}, x^{(m)}) \in E_{h,i}\},$$

$$\|u\|_{h,i} = \max\{\|u^{(r,m)}\| : (t^{(r)}, x^{(m)}) \in E_{h,i}\}$$

where $0 \leq i \leq K$. Let $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^n$, 1 standing on the j -th place.

Approximate solutions of problem (1), (2) are functions z_h defined on the mesh $E_{h,0} \cup E_h$. On the other hand, equation (1) contains the functional variable $V[z]$ which is an element of the space $C(E, \mathbb{R})$. Therefore we need the interpolating operator $T_h : F(E_{h,0} \cup E_h, \mathbb{R}) \rightarrow C(E, \mathbb{R})$. We formulate assumptions on V and T_h .

Assumption $H[V, T_h]$. Suppose that the operator $V : C(E_0 \cup E, \mathbb{R}) \rightarrow C(E, \mathbb{R})$ satisfies the Volterra condition and there is an operator $T_h : F(E_{h,0} \cup E_h, \mathbb{R}) \rightarrow C(E, \mathbb{R})$ such that

1) there is $L_0 \in \mathbb{R}_+$ such that for $z, \bar{z} \in F(E_{h,0} \cup E_h, \mathbb{R})$ we have

$$(3) \quad \|T_h[z] - T_h[\bar{z}]\|_{t^{(r)}} \leq L_0 \|z - \bar{z}\|_{h,r}, \quad 0 \leq r \leq K,$$

2) there is $\mu > 0$ such that for each function $v : E_0 \cup E \rightarrow \mathbb{R}$ which is of class C^1 there is $c_0 \in \mathbb{R}_+$ such that

$$(4) \quad \|V[v] - T_h[v_h]\|_{t^{(r)}} \leq c_0 h_0^\mu, \quad 0 \leq r \leq K,$$

where v_h is the restriction of v to the set $E_{h,0} \cup E_h$.

Note that condition 1) of Assumption $H[V, T_h]$ implies that T_h satisfies the following Volterra condition: if $z, \bar{z} \in F(E_{h,0} \cup E_h, \mathbb{R})$ and $z(\tau, y) = \bar{z}(\tau, y)$ for $(\tau, y) \in E_{h,r}$ then $T_h[z](t^{(r)}, x) = T_h[\bar{z}](t^{(r)}, x)$ where $x \in [-b + Mt^{(r)}, b - Mt^{(r)}]$.

We denote by δ_0 the difference operator with respect to variable t and by $\delta = (\delta_1, \dots, \delta_n)$ the difference operator for spatial variables $(x_1, \dots, x_n) = x$. Write

$$\delta_0 z^{(r,m)} = \frac{1}{h_0} (z^{(r+1,m)} - z^{(r,m)})$$

and

$$(5) \quad \delta_i z^{(r,m)} = \frac{1}{h_i} (z^{(r,m+e_i)} - z^{(r,m)}) \text{ for } 1 \leq i \leq \kappa,$$

$$(6) \quad \delta_i z^{(r,m)} = \frac{1}{h_i} (z^{(r,m)} - z^{(r,m-e_i)}) \text{ for } \kappa + 1 \leq i \leq n,$$

where $0 \leq \kappa \leq n$ is fixed. Note that δ is given by (5) for $\kappa = n$ and δ is defined by (6) for $\kappa = 0$. A classical Euler method for problem (1), (2) consists in replacing partial derivatives in (1) by the above difference operators and by replacing the operator V by the interpolating operator T_h . This leads to the difference equation

$$(7) \quad \delta_0 z^{(r,m)} = f(t^{(r)}, x^{(m)}, T_h[z]^{(r,m)}, \delta z^{(r,m)})$$

with the initial condition

$$(8) \quad z^{(r,m)} = \varphi_h^{(r,m)} \text{ on } E_{h,0}$$

where $\delta z^{(r,m)} = (\delta_1 z^{(r,m)}, \dots, \delta_n z^{(r,m)})$ and $\varphi_h : E_{h,0} \rightarrow \mathbb{R}$ is a given function. Sufficient conditions for the convergence of method (7), (8) are given in the following theorem.

THEOREM 1.1. *Suppose that Assumption $H[V, T_h]$ is satisfied and*

- 1) *the function $f : \Omega \rightarrow \mathbb{R}$ of the variables (t, x, p, q) is continuous and the derivatives $(\partial_{q_1} f, \dots, \partial_{q_n} f) = \partial_q f$ exist on Ω and $\partial_q f \in C(\Omega, \mathbb{R}^n)$,*
- 2) *there is $A \in \mathbb{R}_+$ such that*

$$(9) \quad |f(t, x, p, q) - f(t, x, \bar{p}, q)| \leq A|p - \bar{p}| \text{ on } \Omega$$

and

$$(10) \quad \partial_{q_i} f(t, x, p, q) \geq 0 \text{ on } \Omega \text{ for } 1 \leq i \leq \kappa,$$

$$(11) \quad \partial_{q_i} f(t, x, p, q) \leq 0 \text{ on } \Omega \text{ for } \kappa + 1 \leq i \leq n,$$

- 3) *the function $\varphi : E_0 \rightarrow \mathbb{R}$ is of class C^1 and $v : E_0 \cup E \rightarrow \mathbb{R}$ is a solution of (1), (2) and v is of class C^1 ,*
- 4) *$h \in \Delta$ and*

$$(12) \quad 1 - h_0 \sum_{j=1}^n \frac{1}{h_j} |\partial_{q_j} f(t, x, p, q)| \geq 0 \text{ on } \Omega,$$

- 5) *$z_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}$ is a solution of (7), (8) and there is $\alpha_0 : \Delta \rightarrow \mathbb{R}_+$ such that*

$$|\varphi_h^{(r,m)} - \varphi_h^{(r,m)}| \leq \alpha_0(h) \text{ on } E_{h,0} \text{ and } \lim_{h \rightarrow 0} \alpha_0(h) = 0.$$

Then there is a function $\alpha : \Delta \rightarrow \mathbb{R}_+$ such that

$$\|v_h - z_h\|_{h,r} \leq \alpha(h) \text{ for } 0 \leq r \leq K \text{ and } \lim_{h \rightarrow 0} \alpha(h) = 0,$$

where v_h is the restriction of v to the set $E_{h,0} \cup E_h$.

The above theorem may be proved by methods used in [1], [7], [10], [13]. It is easy to see that the Lipschitz condition (9) may be replaced in Theorem 1.1 by a nonlinear estimate of the Perron type. Note that the definitions of difference operators (5), (6) are generated by conditions (10), (11). The following condition is important in these considerations. Write

$$(13) \quad \text{sign } \partial_q f = (\text{sign } \partial_{q_1} f, \dots, \text{sign } \partial_{q_n} f).$$

We have assumed in Theorem 1.1 that the function $\text{sign } \partial_q f$ is constant on Ω .

Now we formulate a new class of difference problems corresponding to (1), (2). We will need the following assumptions.

Assumption $H_0[f]$. The function $f \in C(\Omega, \mathbb{R})$ is such that

- 1) the partial derivatives

$$(\partial_{x_1} f(P), \dots, \partial_{x_n} f(P)) = \partial_x f(P), \quad \partial_p f(P),$$

$$(\partial_{q_1} f(P), \dots, \partial_{q_n} f(P)) = \partial_q f(P)$$

exist for $P = (t, x, p, q) \in \Omega$ and $\partial_x f, \partial_q f \in C(\Omega, \mathbb{R}^n)$, $\partial_p f \in C(\Omega, \mathbb{R})$,

- 2) there is $A \in \mathbb{R}_+$ such that for $P \in \Omega$ we have

$$\|\partial_x f(P)\|, |\partial_p f(P)|, \|\partial_q f(P)\| \leq A.$$

Assumption $H[V, T_h, L_h]$. Suppose that the operator $V : C(E_0 \cup E) \rightarrow C(E, \mathbb{R})$ satisfies the Volterra condition and

- 1) if $z : E_0 \cup \mathbb{R} \rightarrow \mathbb{R}$ is a function of class C^1 then there exist partial derivatives

$$(\partial_{x_1} V[z](t, x), \dots, \partial_{x_n} V[z](t, x)) = \partial_x V[z](t, x), \quad (t, x) \in E,$$

and $\partial_x V[z] \in C(E, \mathbb{R}^n)$,

- 2) there is an operator $T_h : \mathbb{F}(E_{h,0} \cup E_h, \mathbb{R}) \rightarrow C(E, \mathbb{R})$ such that

(i) the Lipschitz condition (3) is satisfied on $\mathbb{F}(E_{h,0} \cup E_h, \mathbb{R})$,

(ii) there is $\mu > 0$ such that for each function $v : E_0 \cup E \rightarrow \mathbb{R}$ which is of class C^2 there is $c_0 \in \mathbb{R}_+$ such that estimate (4) is satisfied where v_h is the restriction of v to the set $E_{h,0} \cup E_h$,

- 3) there is an interpolating operator $L_h : \mathbb{F}(E_{h,0} \cup E_h, \mathbb{R}^{1+n}) \rightarrow C(E, \mathbb{R}^n)$ with the properties

- (i) there is $L_1 \in \mathbb{R}_+$ such that for $(z, u), (\bar{z}, \bar{u}) \in \mathbb{F}(E_{h,0} \cup E_h, \mathbb{R}^{1+n})$ we have

$$\|L_h[z, u] - L_h[\bar{z}, \bar{u}]\|_{t(r)} \leq L_1 [\|z - \bar{z}\|_{h,r} + \|u - \bar{u}\|_{h,r}], \quad 0 \leq r \leq K,$$

- (ii) there is $\nu > 0$ such that for each function $v : E_0 \cup E \rightarrow \mathbb{R}$ which is of class C^2 there is c_1 with the property

$$\|L_h[v_h, \partial_x v_h] - \partial_x V[v]\|_{t(r)} \leq c_1 h_0^\nu, \quad 0 \leq r \leq K,$$

where v_h and $\partial_x v_h$ are the restrictions of v and $\partial_x v$ to the set $E_{h,0} \cup E_h$.

REMARK 1.2. It follows from condition 3) of Assumption $H[V, T_h, L_h]$ that the operator L_h satisfies the Volterra condition.

For functions $(z, u) \in \mathbb{F}(E_{h,0} \cup E_h, \mathbb{R}^{1+n})$ we write $L_h[z, u] = (L_{h,1}[z, u], \dots, L_{h,n}[z, u])$. Now we formulate a system of difference equations corresponding to (1), (2). Let us denote by (z, u) , $u = (u_1, \dots, u_n)$ the unknown functions of the variables $(t^{(r)}, x^{(m)})$. Write

$$\begin{aligned} P^{(r,m)}[z, u] &= (t^{(r)}, x^{(m)}, T_h[z]^{(r,m)}, u^{(r,m)}), \\ Q^{(r,m)}[z, u] &= (t^{(r)}, x^{(m)}, T_h[z]^{(r,m)}, u^{(r+1,m)}). \end{aligned}$$

We consider the system of difference equations

$$(14) \quad \delta_0 z^{(r,m)} = f(Q^{(r,m)}[z, u]) + \partial_q f(Q^{(r,m)}[z, u]) \circ (\delta z^{(r,m)} - u^{(r,m)})$$

and

$$(15) \quad \begin{aligned} \delta_0 u^{(r,m)} &= \partial_x f(P^{(r,m)}[z, u]) + \partial_p f(P^{(r,m)}[z, u]) L_h[z, u]^{(r,m)} \\ &\quad + \partial_q f(P^{(r,m)}[z, u]) \star [\delta u^{(r,m)}]^T \end{aligned}$$

with the initial condition

$$(16) \quad z^{(r,m)} = \varphi_h^{(r,m)}, \quad u^{(r,m)} = \psi_h^{(r,m)} \text{ for } (t^{(r)}, x^{(m)}) \in E_{h,0},$$

and

$$\begin{aligned} \delta z^{(r,m)} &= (\delta_1 z^{(r,m)}, \dots, \delta_n z^{(r,m)}), \quad \delta_0 u^{(r,m)} = (\delta_0 u_1^{(r,m)}, \dots, \delta_0 u_n^{(r,m)}), \\ \delta u^{(r,m)} &= [\delta_j u_i^{(r,m)}]_{i,j=1,\dots,n}. \end{aligned}$$

The difference operators δ_0 and $\delta = (\delta_1, \dots, \delta_n)$ are defined in the following way. Write

$$(17) \quad \delta_0 z^{(r,m)} = \frac{1}{h_0} (z^{(r+1,m)} - z^{(r,m)})$$

and

$$(18) \quad \delta_0 u_i^{(r,m)} = \frac{1}{h_0} (u_i^{(r+1,m)} - u_i^{(r,m)}), \quad 1 \leq i \leq n.$$

We define the difference operators with respect to spatial variables as follows:

$$(19) \quad \text{if } \partial_{q_j} f(Q^{(r,m)}[z, u]) \geq 0 \text{ then } \delta_j z^{(r,m)} = \frac{1}{h_j} (z^{(r,m+e_j)} - z^{(r,m)}),$$

$$(20) \quad \text{if } \partial_{q_j} f(Q^{(r,m)}[z, u]) < 0 \text{ then } \delta_j z^{(r,m)} = \frac{1}{h_j} (z^{(r,m)} - z^{(r,m-e_j)}).$$

Moreover we put:

$$(21) \quad \text{if } \partial_{q_j} f(P^{(r,m)}[z, u]) \geq 0 \\ \text{then } \delta_j u_i^{(r,m)} = \frac{1}{h_j} (u_i^{(r,m+e_j)} - u_i^{(r,m)}), \quad 1 \leq i \leq n,$$

$$(22) \quad \text{if } \partial_{q_j} f(P^{(r,m)}[z, u]) < 0 \\ \text{then } \delta_j u_i^{(r,m)} = \frac{1}{h_j} (u_i^{(r,m)} - u_i^{(r,m-e_j)}), \quad 1 \leq i \leq n.$$

The difference problem consisting of system (14), (15) with initial condition (16) where the difference operators δ_0 , δ are defined by (17) - (22) is called a generalized Euler method for problem (1), (2). Initial problem (14) - (16) has exactly one solution (z_h, u_h) where $z_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}$ and $u_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}^n$, $u_h = (u_{h,1}, \dots, u_{h,n})$. Indeed, if we assume that the solution (z_h, u_h) of (14) - (16) is defined on the set $E_{h,r}$, $0 \leq r < K$, and $(t^{(r+1)}, x^{(m)}) \in E_h$ then condition $h \in \Delta$ implies that

$$(t^{(r)}, x^{(m+e_j)}), (t^{(r)}, x^{(m-e_j)}) \in E_{h,r} \text{ for } 1 \leq j \leq n.$$

It follows that $u_h^{(r+1,m)}$ may be calculated from (15), (16) and (18), (21), (22). Then $z_h^{(r+1,m)}$ may be calculated from (14), (16) and (17), (19), (20) and consequently (z_h, u_h) is defined on $E_{h,r+1}$. Then by induction, there exists exactly one solution of the generalized Euler method. Write

$$E'_h = \{(t^{(r)}, x^{(m)}) \in E_h : (t^{(r+1)}, x^{(m)}) \in E_h\}.$$

The motivation for the definition of the set E'_h is the following. Solutions of initial problem (14) - (16) are functions (z_h, u_h) defined on $E_{h,0} \cup E_h$. We will write difference functional system (14), (15) at each point of the set E'_h . Then, using initial condition (16) we calculate all the values of (z_h, u_h) on E_h .

There are the following differences between the classical Euler method presented in Theorem 1.1 and our method.

I. We have omitted the assumption that the function (13) is constant on Ω .

II. If we apply difference method (7), (8) then we approximate the spatial derivatives of z in (1) with the using of difference operators (5), (6). In

our method we approximate the derivative $\partial_x z$ in (1) by using difference equation (15) which is generated by (1).

III. Suppose that we calculate the number $z_h^{(r+1,m)}$ by using method (7), (8). Then we apply the vector δz_h considered at the point $(t^{(r)}, x^{(m)})$. In our method we calculate $z^{(r+1,m)}$ by means of (14) and we use the vector u_h at the point $(t^{(r)}, x^{(m)})$ and also at the point $(t^{(r+1)}, x^{(m)})$.

IV. We approximate classical solutions of nonlinear equation (1) by solutions of the difference functional problem which is quasilinear.

Numerical results obtained by (14) - (16) are better than those obtained by classical methods. We show adequate examples.

Difference system (14), (15) is obtained in the following way. Suppose that Assumption $H_0[f]$ is satisfied and that the function $\varphi : E_0 \rightarrow \mathbb{R}$ is of class C^1 . Existence theory of classical or generalized solutions to initial problem (1), (2) is based on a method of quasilinearization. The method consists in replacing problem (1), (2) with the following quasilinear functional differential problem. We first introduce an additional unknown function $u = \partial_x z$, $u = (u_1, \dots, u_n)$, in equation (1). Then we consider the linearization of (1) with respect to u :

$$(23) \quad \partial_t z(t, x) = f(U[z, u; t, x]) + \partial_q f(U[z, u; t, x]) \circ (\partial_x z(t, x) - u(t, x)),$$

where $U[z, u; t, x] = (t, x, V[z](t, x), u(t, x))$. By virtue of (1) we obtain the following differential system for the unknown function u :

$$(24) \quad \partial_t u(t, x) = \partial_x f(U[z, u; t, x]) + \partial_p f(U[z, u; t, x]) \partial_x V[z](t, x) \\ + \partial_q f(U[z, u; t, x]) \star [\partial_x u(t, x)]^T.$$

It is natural to consider the following initial condition for system (23), (24):

$$(25) \quad z(t, x) = \varphi(t, x), \quad u(t, x) = \partial_x \varphi(t, x) \quad \text{for } (t, x) \in E_0.$$

Difference problem (14) - (16) is a discretization of (23) - (25).

The auxiliary system (23), (24) has the following property: differential equations of bicharacteristics for (23) and for each equation of system (24) are the same and they have the form

$$\eta'(t) = -\partial_q f(t, \eta(t), V[z](t, \eta(t)), u(t, \eta(t)))$$

where $\eta = (\eta_1, \dots, \eta_n)$. This property of quasilinear system (23), (24) is important in the investigation of the stability of the generalized Euler method.

Note that each equation of system (23), (24) depends on the unknown functions (z, u) and it contains partial derivatives of only one scalar function. Under natural assumptions on given functions, the following properties of (1), (2) and (23) - (25) may be proved ([8], Chapter IV):

(A) if $\tilde{v} : E_0 \cup E \rightarrow \mathbb{R}$ is a classical solution of (1), (2) and $\tilde{v}$ is of class C^2 on E the functions $(\tilde{v}, \partial_x \tilde{v})$ satisfy (23), (24),

(B) if $(\tilde{v}, \tilde{u}) : E_0 \cup E \rightarrow \mathbb{R}^{1+n}$, $\tilde{u} = (\tilde{u}_1, \dots, \tilde{u}_n)$, is a classical solution of (23)–(25) then $\partial_x \tilde{v} = \tilde{u}$ and $\tilde{v}$ satisfies (1), (2).

The above method of quasilinearization and the theory of bicharacteristics were first considered in nonfunctional setting by S. Cinquini and S. Cinquini Cibrario ([3], [4]). It was extended in [8], [11] on nonlinear functional differential problems. We use these ideas in the paper for numerical solving of (1), (2). The generalized Euler method for equations without a functional dependence was considered in [1].

Existence results for problem (1), (2) are given in [8], [9]. Difference methods for problem (1), (2) are based on general ideas for finite difference equations which were introduced in [6], [12], [14]. For further bibliographical information concerning numerical methods for partial functional differential equations see the survey paper [2] and the monograph [8].

2. Convergence of the generalized Euler method

The following assumptions on f will be needed throughout this section.

Assumption $H[f]$. The function $f : \Omega \rightarrow \mathbb{R}$ satisfies Assumption $H_0[f]$ and there is $L \in \mathbb{R}_+$ such that the expressions

$$\begin{aligned} & \|\partial_x f(t, x, p, q) - \partial_x f(t, x, \bar{p}, \bar{q})\|, \quad |\partial_p f(t, x, p, q) - \partial_p f(t, x, \bar{p}, \bar{q})|, \\ & \|\partial_q f(t, x, p, q) - \partial_q f(t, x, \bar{p}, \bar{q})\| \end{aligned}$$

can be estimated from above by $L[|p - \bar{p}| + \|q - \bar{q}\|]$.

THEOREM 2.1. Suppose that Assumption $H[f]$ and $H[V, T_h, L_h]$ are satisfied and

- 1) $h \in \Delta$ and estimate (12) is satisfied,
- 2) the function $\varphi : E_0 \rightarrow E$ is of class C^2 and $v : E_0 \cup E \rightarrow \mathbb{R}$ is the solution of problem (1), (2) and v is of class C^2 on $E_0 \cup E$,
- 3) $(z_h, u_h) : E_{h,0} \cup E_h \rightarrow \mathbb{R}^{1+n}$, $u_h = (u_{h,1}, \dots, u_{h,n})$, is the solution of difference problem (14)–(16) with δ_0 and $\delta = (\delta_1, \dots, \delta_n)$ defined by (17)–(22) and there is $\alpha_0 : \Delta \rightarrow \mathbb{R}_+$ such that

$$(26) \quad |\varphi^{(r,m)} - \varphi_h^{(r,m)}| + \|\partial_x \varphi^{(r,m)} - \psi_h^{(r,m)}\| \leq \alpha_0(h) \quad \text{on } E_{h,0}$$

$$\text{and } \lim_{h \rightarrow 0} \alpha_0(h) = 0.$$

Then there is $\alpha : \Delta \rightarrow \mathbb{R}_+$ such that

$$(27) \quad \|v_h - z_h\|_{h,r} + \|\partial_x v_h - u_h\|_{h,r} \leq \alpha(h), \quad 0 \leq r \leq K,$$

and $\lim_{h \rightarrow 0} \alpha(h) = 0$, where v_h and $\partial_x v_h$ are the restrictions of v and $\partial_x v$ to the set $E_{h,0} \cup E_h$.

Proof. Write $w = \partial_x v$ and $w = (w_1, \dots, w_n)$. Then the functions $(v, w) : E_0 \cup E \rightarrow \mathbb{R}^{1+n}$ satisfy quasilinear system (23), (24) and initial condition (25). Let us denote by $\xi_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}$ and $\lambda_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}^n$ the functions

$$(28) \quad \xi_h^{(r,m)} = v_h^{(r,m)} - z_h^{(r,m)}, \quad \lambda_h^{(r,m)} = w_h^{(r,m)} - u_h^{(r,m)}$$

and $\lambda_h = (\lambda_{h,1}, \dots, \lambda_{h,n})$ where w_h is the restriction of w to the set $E_{h,0} \cup E_h$. Put

$$(29) \quad \omega_{h,0}^{(r)} = \|\xi_h\|_{h,r}, \quad \omega_{h,1}^{(r)} = \|\lambda_h\|_{h,r}, \quad 0 \leq r \leq K,$$

and $\omega_h = \omega_{h,0} + \omega_{h,1}$. We will write a difference inequality for the function ω_h .

We first prove some properties of $\omega_{h,0}$. Set

$$U^{(r,m)}[v, w] = (t^{(r)}, x^{(m)}, V[v]^{(r,m)}, w^{(r,m)}).$$

Let us denote by $\Gamma_{h,0}, \Lambda_{h,0} : E'_h \rightarrow \mathbb{R}$ the functions given by

$$(30) \quad \Gamma_{h,0}^{(r,m)} = \delta_0 v_h^{(r,m)} - \partial_t v^{(r,m)} + \partial_q f(U^{(r,m)}[v, w]) \circ (\partial_x v^{(r,m)} - \delta v_h^{(r,m)})$$

and

$$(31) \quad \Lambda_{h,0}^{(r,m)} = f(U^{(r,m)}) - f(Q^{(r,m)}[z_h, u_h]) - \partial_q f(U^{(r,m)}[v, w]) \circ w^{(r,m)} \\ + \partial_q f(Q^{(r,m)}[z_h, u_h]) \circ u_h^{(r,m)} + [\partial_q f(U^{(r,m)}[v, w]) - \partial_q f(Q^{(r,m)}[z_h, u_h])] \circ \delta v_h^{(r,m)}.$$

It follows from (14) and (23) that

$$(32) \quad \xi^{(r+1,m)} = \xi_h^{(r,m)} + h_0 \partial_q f(Q^{(r,m)}[z_h, u_h]) \circ \delta \xi_h^{(r,m)} + h_0 [\Gamma_{h,0}^{(r,m)} + \Lambda_{h,0}^{(r,m)}]$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. Put

$$J^+[r, m] = \{j \in \{1, \dots, n\} : \partial_{q_j} f(Q^{(r,m)}[z_h, u_h]) \geq 0\},$$

$$J^-[r, m] = \{1, \dots, n\} \setminus J^+[r, m].$$

It follows from (19), (20) and (12) that

$$(33) \quad \left| \xi_h^{(r,m)} + h_0 \partial_q f(Q^{(r,m)}[z_h, u_h]) \circ \delta \xi_h^{(r,m)} \right| \\ \leq |\xi_h^{(r,m)}| \left[1 - h_0 \sum_{j=1}^n \frac{1}{h_j} |\partial_{q_j} f(Q^{(r,m)}[z_h, u_h])| \right] \\ + h_0 \sum_{j \in J^+[r, m]} \frac{1}{h_j} \partial_{q_j} f(Q^{(r,m)}[z_h, u_h]) |\xi_h^{(r,m+e_j)}| \\ - h_0 \sum_{j \in J^-[r, m]} \frac{1}{h_j} \partial_{q_j} f(Q^{(r,m)}[z_h, u_h]) |\xi_h^{(r,m-e_j)}| \\ \leq \omega_{h,0}^{(r)} \text{ for } (t^{(r)}, x^{(m)}) \in E'_h.$$

There is $\bar{c} \in \mathbb{R}_+$ such that

$$(34) \quad \|\partial_x v(t, x)\|, \|\partial_x V[v](t, x)\| \leq \bar{c} \text{ for } (t, x) \in E.$$

It follows from Assumptions $H[f]$ and $H[V, T_h, L_h]$ that

$$\begin{aligned} & \left| f(U^{(r,m)}[v, w]) - f(Q^{(r,m)}[z_h, u_h]) \right| \\ & \leq A[c_0 h_0^\mu + \bar{c} h_0 + L_0 \omega_{h,0}^{(r)} + \omega_{h,1}^{(r+1,m)}], \quad (t^{(r)}, x^{(m)}) \in E'_h, \end{aligned}$$

and a similar estimate for the expression

$$\|\partial_q f(U^{(r,m)}[v, w]) - \partial_q f(Q^{(r,m)}[z_h, u_h])\|$$

holds on E'_h . We thus get

$$(35) \quad |\Lambda_{h,0}^{(r,m)}| \leq (A + 2\bar{c}L)[c_0 h_0^\mu + \bar{c} h_0 + L_0 \omega_{h,0}^{(r)} + \omega_{h,1}^{(r+1)}] + A\omega_{h,1}^{(r)}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. There is $\gamma_0 : \Delta \rightarrow \mathbb{R}_+$ such that

$$(36) \quad |\Gamma_{h,0}^{(r,m)}| \leq \gamma_0(h) \text{ on } E'_h \text{ and } \lim_{h \rightarrow 0} \gamma_0(h) = 0.$$

We conclude from (32) - (36) that

$$(37) \quad \begin{aligned} \omega_{h,0}^{(r+1)} & \leq \omega_{h,0}^{(r)} + h_0(A + 2\bar{c}L)[c_0 h_0^\mu + \bar{c} h_0 + L_0 \omega_{h,0}^{(r)} + \omega_{h,1}^{(r+1)}] \\ & \quad + h_0 A \omega_{h,1}^{(r)} + h_0 \gamma_0(h), \quad 0 \leq r \leq K-1. \end{aligned}$$

Now we write a difference inequality for the function $\omega_{h,1}$. Let the functions $\Gamma_h, \Lambda_h : E'_h \rightarrow \mathbb{R}^n$ be defined by

$$(38) \quad \Gamma_h^{(r,m)} = \delta_0 w_h^{(r,m)} - \partial_t w^{(r,m)} + \partial_q f(U^{(r,m)}[v, w]) \star [\partial_x w^{(r,m)} - \delta w_h^{(r,m)}]^T$$

and

$$(39) \quad \begin{aligned} \Lambda_h^{(r,m)} & = \partial_x f(U^{(r,m)}[v, w]) - \partial_x f(P^{(r,m)}[z_h, u_h]) \\ & \quad + \partial_p f(U^{(r,m)}[v, w]) \partial_x V[v]^{(r,m)} - \partial_p f(P^{(r,m)}[z_h, u_h]) L_h[z_h, u_h]^{(r,m)} \\ & \quad + [\partial_q f(U^{(r,m)}[v, w]) - \partial_q f(P^{(r,m)}[z_h, u_h])] \star [\delta w_h^{(r,m)}]^T. \end{aligned}$$

It follows from (15) and (24) that

$$(40) \quad \lambda_h^{(r+1,m)} = \lambda_h^{(r,m)} + h_0 \partial_q f(P^{(r,m)}[z_h, u_h]) \star [\delta \lambda_h^{(r,m)}]^T + h_0 [\Gamma_h^{(r,m)} + \Lambda_h^{(r,m)}]$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. Write

$$\begin{aligned} I^+[r, m] & = \{j \in \{1, \dots, n\} : \partial_{q_j} f(P^{(r,m)}[z_h, u_h]) \geq 0\}, \\ I^-[r, m] & = \{1, \dots, n\} \setminus I^+[r, m] \end{aligned}$$

and

$$S_h^{(r,m)} = \lambda_h^{(r,m)} + h_0 \partial_q f(P^{(r,m)}[z_h, u_h]) \star [\delta \lambda_h^{(r,m)}]^T.$$

It follows from (12), (21), (22) that

$$(41) \quad \|S_h^{(r,m)}\| \leq \|\lambda_h^{(r,m)}\| \left[1 - h_0 \sum_{j=1}^n \frac{1}{h_j} |\partial_{q_j} f(P^{(r,m)}[z_h, u_h])| \right] \\ + h_0 \sum_{j \in I^+[r,m]} \frac{1}{h_j} \partial_{q_j} f(P^{(r,m)}[z_h, u_h]) \|\lambda_h^{(r,m+e_j)}\| \\ - h_0 \sum_{j \in I^-[r,m]} \frac{1}{h_j} \partial_{q_j} f(P^{(r,m)}[z_h, u_h]) \|\lambda_h^{(r,m-e_j)}\| \leq \omega_{h,1}^{(r)}$$

for $(t^{(r)}, x^{(m)}) \in E'_h$. It follows from Assumption $H[f]$ and $H[T_h, L_h]$ that

$$\|\partial_x f(U^{(r,m)}[v, w]) - \partial_x f(P^{(r,m)}[z_h, u_h])\| \leq L[c_0 h_0^\mu + L_0 \omega_{h,0}^{(r)} + \omega_{h,1}^{(r)}]$$

and the same estimate we obtain for the derivatives $\partial_p f$ and $\partial_q f$. We thus get

$$(42) \quad \|\Lambda_h^{(r,m)}\| \leq L(1 + 2\tilde{c})[c_0 h_0^\mu + L_0 \omega_{h,0}^{(r)} + \omega_{h,1}^{(r)}] + A[c_1 h_0^\nu + L_1 \omega_{h,1}^{(r)}],$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. There is $\gamma: \Delta \rightarrow \mathbb{R}_+$ such that

$$(43) \quad \|\Gamma^{(r,m)}\| \leq \gamma(h) \text{ on } E'_h \text{ and } \lim_{h \rightarrow 0} \gamma(h) = 0.$$

According to (40)-(43), we have

$$(44) \quad \omega_{h,1}^{(r+1)} \leq \omega_{h,1}^{(r)} + h_0 L(1 + 2\tilde{c})[c_0 h_0^\mu + L_0 \omega_{h,0}^{(r)} + \omega_{h,1}^{(r)}] \\ + h_0 A[c_1 h_0^\nu + L_1 \omega_{h,1}^{(r)}] + h_0 \gamma(h), \quad 0 \leq r \leq K-1.$$

Combining (37) with (44) we can assert that there are $\tilde{L}, \tilde{A} > 0$ such that

$$\omega_h^{(r+1)} \leq \omega_h^{(r)}(1 + h_0 \tilde{L}) + h_0[\gamma_0(h) + \gamma(h) + \tilde{A}(h_0 + h_0^\mu + h_0^\nu)], \quad 0 \leq r \leq K-1.$$

and $\omega_0^{(0)} \leq \alpha_0(h)$. This gives

$$\omega_h^{(r)} \leq \alpha(h) \text{ for } 0 \leq r \leq K,$$

with

$$\alpha(h) = \alpha_0(h) \exp\{\tilde{L}a\} + [\gamma_0(h) + \gamma(h) + \tilde{A}(h_0 + h_0^\mu + h_0^\nu)] \frac{\exp\{\tilde{L}a\} - 1}{\tilde{L}}.$$

This proves the theorem.

3. Convergence of the difference method with nonlinear estimates

In this part of the paper we assume nonlinear estimates of the Perron type for the functions $\partial_x f$, $\partial_p f$, $\partial_q f$ instead of the Lipschitz conditions with respect to (p, q) . Therefore we need the following modification of the

difference system. We consider the generalized Euler method consisting of equation (15) and

$$(45) \quad \delta_0 z^{(r,m)} = f(P^{(r,m)}[z, u]) + \partial_q f(P^{(r,m)}[z, u]) \circ [\delta z^{(r,m)} - u^{(r,m)}]$$

with initial condition (16). Difference expressions $\delta_0 z^{(r,m)}$, $\delta_0 u^{(r,m)}$ and $\delta u^{(r,m)}$ are given by (17), (18) and (21), (22). Difference operator $\delta z^{(r,m)} = (\delta_1 z^{(r,m)}, \dots, \delta_n z^{(r,m)})$ is defined in the following way:

$$(46) \quad \text{if } \partial_{q_j} f(P^{(r,m)}[z, u]) \geq 0 \text{ then } \delta_j z^{(r,m)} = \frac{1}{h_j} (z^{(r,m+e_j)} - z^{(r,m)}),$$

$$(47) \quad \text{if } \partial_{q_j} f(P^{(r,m)}[z, u]) < 0 \text{ then } \delta_j z^{(r,m)} = \frac{1}{h_j} (z^{(r,m)} - z^{(r,m-e_j)}).$$

Assumption $H[\sigma, f]$. Suppose that Assumption $H_0[f]$ is satisfied and there is a function $\sigma : [0, a] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

- 1) σ is continuous and it is nondecreasing with respect to both variables,
- 2) $\sigma(t, 0) = 0$ for $t \in [0, a]$ and for each $c \geq 1$ the maximal solution of the Cauchy problem

$$\eta'(t) = c\eta(t) + c\sigma(t, c\eta(t)), \quad \eta(0) = 0,$$

is $\bar{\eta}(t) = 0$ for $t \in [0, a]$,

- 3) the terms

$$\|\partial_x f(t, x, p, q) - \partial_x f(t, x, \bar{p}, \bar{q})\|, \quad |\partial_p f(t, x, p, q) - \partial_p f(t, x, \bar{p}, \bar{q})|, \\ \|\partial_q f(t, x, p, q) - \partial_q f(t, x, \bar{p}, \bar{q})\|$$

are bounded from above by $\sigma(t, |p - \bar{p}| + \|q - \bar{q}\|)$.

THEOREM 3.1. Suppose that Assumptions $H[V, T_h, L_h]$ and $H[\sigma, f]$ are satisfied and

- 1) $h \in \Delta$ and condition (12) is satisfied,
- 2) the function $\varphi : E_0 \rightarrow \mathbb{R}$ is of class C^2 and $v : E_0 \cup E \rightarrow \mathbb{R}$ is the solution of problem (1), (2) and v is of class C^2 on $E_0 \cup E$,
- 3) $(z_h, u_h) : E_{h,0} \cup E_h \rightarrow \mathbb{R}^{1+n}$, $u_h = (u_{h,1}, \dots, u_{h,n})$, is the solution of difference problem (15), (16), (45) with δ_0 and $\delta = (\delta_1, \dots, \delta_n)$ defined by (17), (18) and (21), (22), (46), (47),
- 4) there is $\alpha_0 : \Delta \rightarrow \mathbb{R}_+$ such that condition (26) is satisfied and $\lim_{h \rightarrow 0} \alpha_0(h) = 0$.

Then there is a number $\varepsilon_0 > 0$ and a function $\alpha : \Delta \rightarrow \mathbb{R}_+$ such that condition (27) holds for $\|h\| < \varepsilon_0$ and $\lim_{h \rightarrow 0} \alpha(h) = 0$.

Proof. Write $w = \partial_x v$. Let $\xi_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}$ and $\lambda_h : E_{h,0} \cup E_h \rightarrow \mathbb{R}^n$ be the functions defined by (28) where v_h and w_h are the restrictions of v and w to the set $E_{h,0} \cup E_h$. We write a difference inequality for the function

$\omega_h = \omega_{h,0} + \omega_{h,1}$ where $\omega_{h,0}$ and $\omega_{h,1}$ are given by (29). We can proceed analogously to the proof of Theorem 2.1. We deduce that the functions $\omega_{h,0}$ and $\omega_{h,1}$ satisfy the recurrent inequalities

$$(48) \quad \omega_{h,0}^{(r+1)} \leq \omega_{h,0}^{(r)} + 2\bar{c}h_0\sigma(t^{(r)}, c_0h_0^\mu + \bar{b}\omega_h^{(r)}) \\ + Ah_0\omega_{h,1}^{(r)} + h_0\gamma_0(h) + Ah_0[c_0h_0^\mu + \bar{b}\omega_h^{(r)}], \quad 0 \leq r \leq K-1,$$

and

$$(49) \quad \omega_{h,1}^{(r+1)} \leq \omega_{h,1}^{(r)} + (2\bar{c} + 1)h_0\sigma(t^{(r)}, c_0h_0^\mu + \bar{b}\omega_h^{(r)}) \\ + h_0\gamma(h) + Ah_0[c_1h_0^\nu + L_1\omega_h^{(r)}], \quad 0 \leq r \leq K-1,$$

where $\bar{b} = \max\{1, L_0\}$. The functions γ_0 and γ are given by (36) and (43) respectively, the constant $\bar{c} \in \mathbb{R}_+$ is such that estimates (34) hold. The functions $\Gamma_{h,0}$, Γ_h and Λ_h are defined by (30), (38) and (39) respectively, whereas the function $\Lambda_{h,0}$ is given by (31) with $P^{(r,m)}[z_h, u_h]$ instead of $Q^{(r,m)}[z_h, u_h]$. Adding inequalities (48) and (49) we conclude that the function ω_h satisfies the recurrent inequality

$$(50) \quad \omega_h^{(r+1)} \leq (1 + \bar{c}h_0)\omega_h^{(r)} + \bar{d}h_0\sigma(t^{(r)}, c_0h_0^\mu + \bar{b}\omega_h^{(r)}) + h_0\tilde{\gamma}(h),$$

$0 \leq r \leq K-1$, where

$$\bar{d} = 4\bar{c} + 1, \quad \bar{a} = A \max\{c_1, c_0\}, \quad \bar{c} = A(1 + \bar{b} + L_1), \\ \tilde{\gamma}(h) = \gamma_0(h) + \gamma(h) + \bar{a}(h_0^\mu + h_0^\nu).$$

Consider the Cauchy problem

$$(51) \quad \eta'(t) = \bar{c}\eta(t) + \bar{d}\sigma(t, c_0h_0^\mu + \bar{b}\eta(t)) + \tilde{\gamma}(h), \quad \eta(0) = \alpha_0(h).$$

It follows from condition 2) of Assumption $H[\sigma, f]$ that there is $\varepsilon_0 > 0$ such that for $\|h\| < \varepsilon_0$ there exists the maximal solution η_h of problem (51) and η_h is defined on $[0, a]$. Moreover we have

$$\lim_{h \rightarrow 0} \eta_h(t) = 0 \quad \text{uniformly on } [0, a].$$

It follows from condition 1) of Assumption $H[\sigma, f]$ that the function η_h satisfies the recurrent inequality

$$(52) \quad \eta_h^{(r+1)} \geq (1 + \bar{c}h_0)\eta_h^{(r)} + \bar{d}\sigma(t^{(r)}, c_0h_0^\mu + \bar{b}\eta_h^{(r)}) + h_0\tilde{\gamma}(h), \quad 0 \leq r \leq K-1.$$

Since $\omega_h^{(0)} \leq \eta_h^{(0)}$, relations (50) and (52) show that $\omega_h^{(r)} \leq \eta_h^{(r)}$ for $0 \leq r \leq K$. The we obtain estimate (27) for $\alpha(h) = \eta_h(a)$. This proves the theorem.

Now we give an error estimate for the method (15), (16), (45) with δ_0 and δ defined by (17), (18) and (21), (22), (46), (47).

LEMMA 3.2. Suppose that

- 1) the assumptions of Theorem 3.1 are satisfied with $\sigma(t, p) = Lp$, $(t, p) \in [0, a] \times R_+$, where $L \in \mathbb{R}_+$, (i. e. the functions $\partial_x f, \partial_p f, \partial_q f$ satisfy the Lipschitz condition with respect to (p, q)),
- 2) the function $v : E_0 \cup E \rightarrow R$ is of class C^3 and $\tilde{C} \in \mathbb{R}_+$ is such a constant that

$$(53) \quad |\partial_{tt}v(t, x)|, |\partial_{tx_i}(t, x)|, |\partial_{x_i x_j}v(t, x)| \leq \tilde{C} \text{ for } i, j = 1, \dots, n$$

and

$$(54) \quad |\partial_{ttt}v(t, x)|, |\partial_{ttx_i}(t, x)|, |\partial_{tx_i x_j}(t, x)|, |\partial_{x_i x_j x_k}v(t, x)| \leq \tilde{C}$$

for $i, j, k = 1, \dots, n$ where $(t, x) \in E$.

Then

$$(55) \quad \|v_h - z_h\|_{h,r} + \|\partial_x v_h - u_h\|_{h,r} \leq \bar{\alpha}(h), \quad 0 \leq r \leq K,$$

where

$$\bar{\alpha}(h) = \alpha_0(h)e^{\bar{L}a} + \bar{\gamma}(h)\frac{e^{\bar{L}a} - 1}{\bar{L}} \text{ if } \bar{L} > 0,$$

$$\bar{\alpha}(h) = \alpha_0(h) + a\bar{\gamma}(h) \text{ if } \bar{L} = 0,$$

and

$$\bar{L} = \bar{c} + \bar{d}\bar{b}L, \quad \bar{\gamma}(h) = \bar{a}h_0^\mu + \bar{a}h_0^\nu + \bar{b}h_0,$$

$$\bar{a} = 1 + \bar{d}\bar{L}c_0, \quad \bar{b} = \tilde{C}(1 + A\|M\|).$$

Proof. It follows from (53), (54) that

$$|\Gamma_{h,0}^{(r,m)}|, \|\Gamma_h^{(r,m)}\| \leq h_0 \frac{\tilde{C}}{2}(1 + A\|M\|)$$

on E'_h . The Cauchy problem (51) is equivalent to the following one

$$\eta'(t) = \bar{L}\eta(t) + \bar{\gamma}(h), \quad \eta(0) = \alpha_0(h).$$

Then assertion (55) follows from Theorem 3.1.

REMARK 3.3. In the results on error estimates we need estimates for the derivatives of the solution v of problem (1), (2). One may obtain them by the method of differential inequalities, see [8], Chapter I.

EXAMPLE 3.4. Suppose that the functions $\psi_0 : [0, a] \rightarrow \mathbb{R}$, $\psi = (\psi_1, \dots, \psi_n) : E \rightarrow \mathbb{R}^n$ are given. We assume that

- 1) $\psi_0 \in C([0, a], \mathbb{R})$ and $-b_0 \leq \psi_0(t) \leq t$ for $t \in [0, a]$,
- 2) $\psi \in C(E, \mathbb{R}^n)$ and the partial derivatives

$$[\partial_{x_j}\psi_i(t, x)]_{i,j=1,\dots,n} = \partial_x \psi(t, x)$$

exist on E and $\partial_x \psi \in C(E, M_{n \times n})$,

- 3) $(\psi_0(t), \psi(t, x)) \in E_0 \cup E$ for $(t, x) \in E$.

Let the operator $V : C(E_0 \cup E, E) \rightarrow C(E, \mathbb{R})$ be given by $V[z](t, x) = z(\psi_0, t), \psi(t, x))$. Then V satisfies the Volterra condition and equation (1) is equivalent to the equation with deviated variables

$$\partial_t z(t, x) = f(t, x, z(\psi_0(t), \psi(t, x)), \partial_x z(t, x)).$$

It is easy to construct the operators T_h and $L_h = (L_{h,1}, \dots, L_{h,n})$ using the interpolating operator on the Haar pyramid introduced in [8] (Chapter III), see also [7].

Now we consider differential integral equations. Suppose that $\kappa \in \mathbb{N}$, $1 \leq \kappa \leq n$, is fixed. For each $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we write $x = (x', x'')$ where $x' = (x_1, \dots, x_\kappa)$, $x'' = (x_{\kappa+1}, \dots, x_n)$. We have $x' = x$ if $\kappa = n$.

EXAMPLE 3.5. Consider the operator $V : C(E_0 \cup E, \mathbb{R}) \rightarrow C(E, \mathbb{R})$ given by

$$V[z](t, x) = \int_{-b_0 - x'}^t \int_{x'}^{x'} z(\tau, s', x'') ds' d\tau,$$

where $s' = (s_1, \dots, s_\kappa)$. Then (1) is equivalent to the differential integral equation

$$\partial_t z(t, x) = f\left(t, x, \int_{-b_0 - x'}^t \int_{x'}^{x'} z(\tau, s', x'') ds' d\tau, \partial_x z(t, x)\right).$$

We can construct the operators T_h and $L_h = (L_{h,1}, \dots, L_{h,n})$ using the interpolating operator on the Haar pyramid considered in [8] (Chapter III), see also [7].

4. Numerical experiments

Put $n = 1$ and $E = \{(t, x) \in \mathbb{R}^2 : t \in [0, a], -b + Mt \leq x \leq b - Mt\}$ where $a, M > 0$ and $b \geq Ma$. Consider the differential integral equation

$$(56) \quad \partial_t z(t, x) = \sin[t + \partial_x z(t, x)] + \cos\left[t^2 + 1.5 \int_{-x}^x z(t, s) ds\right] + f(t, x)$$

with the initial condition

$$(57) \quad z(0, x) = 0 \quad \text{for } x \in [-b, b]$$

where

$$f(t, x) = \frac{1}{3}x^3 + 2tx^2 - \sin[t(1 + x^2 + 2tx)] - \cos[t^2(1 + x^3)].$$

The function $v(t, x) = \frac{1}{3}tx^3 + t^2x^2$ is the solution of (56), (57). The classical difference method has the form

$$(58) \quad \begin{aligned} z^{(r+1, m)} &= \frac{1}{2}[z^{(r, m+1)} + z^{(r, m-1)}] + h_0 \sin[t^{(r)} + \delta z^{(r, m)}] \\ &\quad + h_0 \cos[(t^{(r)})^2 + 1.5I^{(r, m)}] + h_0 f^{(r, m)} \end{aligned}$$

and

$$(59) \quad z^{(0,m)} = 0 \text{ for } x^{(m)} \in [-b, b]$$

where $\delta z^{(r,m)} = (2h_1)^{-1}(z^{(r,m+1)} - z^{(r,m-1)})$ and

$$(60) \quad I^{(r,m)} = \int_{-x^{(m)}}^{x^{(m)}} T_h(t^{(r)}, s) ds$$

with T_h defined in [8], Chapter 3 for $n = 1$. We need the following relations for $h = (h_0, h_1)$: $h_1 = Mh_0$ and $h_1 \geq h_0$. It follows from Theorem 3.21 in [8] that the above difference method is convergent. Note that Theorem 1.1 is not applicable to (56), (57). Let us denote by $\tilde{z}_h : E_h \rightarrow R$ the solution of (58), (59). Consider the generalized Euler method to problem (56), (57) and its solution $(z_h, u_h) : E_h \rightarrow \mathbb{R}^2$. Let $k_r \in \mathbb{N}$, $0 \leq r \leq K$, be defined by $b - Mt^{(r)} = x^{(k_r)}$ and

$$\begin{aligned} \tilde{\varepsilon}^{(r,m)} &= |v^{(r,m)} - \tilde{z}_h^{(r,m)}|, \quad \varepsilon^{(r,m)} = |v^{(r,m)} - z_h^{(r,m)}| \\ \tilde{\eta}^{(r)} &= \frac{1}{2k_r + 1} \sum_{m=-k_r}^{k_r} \tilde{\varepsilon}^{(r,m)}, \quad \eta^{(r)} = \frac{1}{2k_r + 1} \sum_{m=-k_r}^{k_r} \varepsilon^{(r,m)}. \end{aligned}$$

The numbers $\tilde{\eta}^{(r)}$ and $\eta^{(r)}$ are the arithmetical means of the errors with fixed $t^{(r)}$. We have solved problem (58), (59) and the generalized Euler method for the following sets of parameters:

- (i) $b = 4, a = 0.4, M = 10$,
- (ii) $b = 4, a = 0.8, M = 5$.

In the Table we give experimental values of the errors $\tilde{\eta}^{(r)}$ and $\eta^{(r)}$ for the following steps of the mesh:

$$\tilde{h} = (5 \cdot 10^{-3}, 5 \cdot 10^{-2}), \quad \bar{h} = (5 \cdot 10^{-4}, 5 \cdot 10^{-3}), \quad h^* = (10^{-2}, 5 \cdot 10^{-2}).$$

Table of errors $\frac{\tilde{\eta}}{\eta}$

$t^{(r)}$	$\tilde{h}$	$\bar{h}$	$t^{(r)}$	h^*
0.225	$\frac{0.0109}{0.0008}$	$\frac{0.0011}{0.0001}$	0.45	$\frac{0.0230}{0.0037}$
0.275	$\frac{0.0121}{0.0009}$	$\frac{0.0012}{0.0001}$	0.55	$\frac{0.0265}{0.0040}$
0.325	$\frac{0.0114}{0.0009}$	$\frac{0.0011}{0.0001}$	0.65	$\frac{0.0301}{0.0039}$
0.375	$\frac{0.0123}{0.0009}$	$\frac{0.0012}{0.0001}$	0.75	$\frac{0.0457}{0.0066}$

Note that $\eta^{(r)} < \tilde{\eta}^{(r)}$ for all the values r .

Now we consider a differential equation with deviated variables. Put $n = 2$ and

$$E = \{(t, x, y) \in \mathbb{R}^3 : t \in [0, a], -b + Mt \leq x \leq b - Mt, -b + Mt \leq y \leq b - Mt\}$$

where $a, M > 0$ and $b \geq Ma$. Let us denote by z the unknown function of the variables (t, x, y) . Consider the differential equation

$$(61) \quad \partial_t z(t, x, y) = \sin[\partial_x z(t, x, y) - 2txz(t, x, y)] \\ + \cos[\partial_y z(t, x, y) + 2tyz(t, x, y)] \\ + (x^2 - y^2)z(t, x, y) - z(t, 0.5x, 0.5y) + f(t, x, y)$$

with the initial condition

$$(62) \quad z(0, x, y) = 1, \quad x, y \in [-b, b],$$

where $f(t, x, y) = \exp\{0.25(x^2 - y^2)\} - 1$. The solution of the problem is known, it is $w(t, x, y) = \exp\{t(x^2 - y^2)\}$. The classical difference method for (61), (62) has the form

$$\delta_0 z^{(r,m)} = \sin[\delta_1 z^{(r,m)} - 2t^{(r)}x^{(m_1)}z^{(r,m)}] + \cos[\delta_2 z^{(r,m)} + 2t^{(r)}y^{(m_2)}z^{(r,m)}] \\ - [(x^{(m_1)})^2 - (y^{(m_2)})^2]z^{(r,m)} - Az^{(r,m)} + f^{(r,m)}$$

and

$$z^{(0,m)} = 1 \quad \text{for } x^{(m_1)}, y^{(m_2)} \in [-b, b],$$

where

$$\delta_0 z^{(r,m)} = \frac{1}{h_0} \left[z^{(r+1,m)} - \frac{1}{4} (z^{(r,m_1+1,m_2)} + \right. \\ \left. + z^{(r,m_1-1,m_2)} + z^{(r,m_1,m_2+1)} + z^{(r,m_1,m_2-1)}) \right], \\ \delta_1 z^{(r,m)} = \frac{1}{2h_1} [z^{(r,m_1+1,m_2)} - z^{(r,m_1-1,m_2)}], \\ \delta_2 z^{(r,m)} = \frac{1}{2h_2} [z^{(r,m_1,m_2+1)} - z^{(r,m_1,m_2-1)}]$$

and

$$Az^{(r,m)} = T_h(t^{(r)}, 0.5x^{(m_1)}, 0.5y^{(m_2)})$$

with T_h defined in [8] (Chapter III) for $n = 2$. Let us denote by $\tilde{z}_h : E_h \rightarrow \mathbb{R}$ the solution of the above problem. Consider also the generalized Euler method to problem (61), (62) and its solution $(z_h, u_h, v_h) : E_h \rightarrow \mathbb{R}^3$.

We have solved the classical difference problem and the generalized Euler method for the following sets of parameters:

- (i) $b = 2, M = 5, a = 0.4,$
- (ii) $b = 2, M = 2, a = 1.$

We have assumed the $h_1 = h_2$. Let $k_r \in \mathbb{N}$, $0 \leq r \leq K$ be defined by $2 - Mt^{(r)} = x^{(k_r)} = y^{(k_r)}$. Write

$$\tilde{\varepsilon}_h^{(r,m)} = |w^{(r,m)} - \tilde{z}_h^{(r,m)}|, \quad \varepsilon_h^{(r,m)} = |w^{(r,m)} - z_h^{(r,m)}|$$

and

$$\tilde{\eta}^{(r)} = \frac{1}{(2k_r + 1)^2} \sum_{m=(-k_r, -k_r)}^{(k_r, k_r)} \tilde{\varepsilon}^{(r,m)}, \quad \eta^{(r)} = \frac{1}{(2k_r + 1)^2} \sum_{m=(-k_r, -k_r)}^{(k_r, k_r)} \varepsilon^{(r,m)},$$

where $0 \leq r \leq K$. The numbers $\tilde{\eta}^{(r)}$ and $\eta^{(r)}$ are the arithmetical means of the errors with fixed $t^{(r)}$. In the Table we give experimental values of the errors $\tilde{\eta}^{(r)}$ and $\eta^{(r)}$ for the following steps of the mesh:

$$\tilde{h} = (5 \cdot 10^{-3}, 1 \cdot 10^{-2}), \quad \bar{h} = (10^{-3}, 5 \cdot 10^{-3}), \quad h^* = (10^{-2}, 2 \cdot 10^{-2}).$$

Table of errors $\frac{\tilde{\eta}}{\eta}$				
$t^{(r)}$	$\tilde{h}$	$\bar{h}$	$t^{(r)}$	h^*
0.2	$\frac{0.3293}{0.0013}$	$\frac{0.1298}{0.0003}$	0.3	$\frac{0.3669}{0.0027}$
0, 25	$\frac{0.4549}{0.0019}$	$\frac{0.0932}{0.0003}$	0.5	$\frac{0.2983}{0.0047}$
0.3	$\frac{0.3636}{0.0025}$	$\frac{0.0561}{0.0002}$	0.7	$\frac{0.1976}{0.0049}$
0.35	$\frac{0.3587}{0.0032}$	$\frac{0.0307}{0.0001}$	0.9	$\frac{0.2195}{0.0043}$

Note that $\eta^{(r)} < \tilde{\eta}^{(r)}$ for all the values r .

Thus we see that the errors of the classical difference method are larger than the errors of the generalized Euler method. This is due to the fact that the approximation of the spatial derivatives of z in the new method is better than the respective approximation of $\partial_x z$ in a classical case. Methods described in Theorems 2.1 and 3.1 have the potential for applications in the numerical solving of first order nonlinear differential equations.

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