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A NOTE ON HOMOMORPHISMS OF INNER PRODUCT SPACES

Abstract. Recently, Buhagiar and Chetcuti [1] have shown that if V_1 and V_2 are two separable, real inner product spaces such that the modular ortholattices of their finite and cofinite subspaces are algebraically isomorphic, then V_1 and V_2 are isomorphic as inner product spaces. Their proof is based on the properties of inner product spaces, in particular it makes use of Gleason's theorem. In this note we show, using techniques of projective geometry, that their result holds for any inner product spaces, real, complex or quaternionic, of dimension at least three, not necessarily separable. We also consider the case when the algebraic isomorphism is replaced by a homomorphism, and the case when the underlying fields of V_1 and V_2 are not the same.

1. Projective geometries and inner product spaces

The most usual definition of a projective geometry is the following. A *projective geometry* is a couple $(G; *)$, where G is a nonempty set with an operator $*$: $G \times G \rightarrow \mathcal{P}(G)$, where $\mathcal{P}(G)$ denotes the set of all subsets of G , satisfying the following axioms (we write $a*b$ for the image of (a, b) under $*$):

- (P1) $a * a = \{a\}$ for all $a \in G$;
- (P2) $a \in b * a$ for all $a, b \in G$;
- (P3) $a \in b * p$ and $p \in c * d$ and $a \neq c$ implies $(a * c) \cap (b * d) \neq \emptyset$.

For other possible axiomatizations see [4].

A subset E of a projective geometry G is a *subspace* if $a, b \in E$, $c \in a * b$ implies $c \in E$.

An *isomorphism* of projective geometries is a bijection $g : G_1 \rightarrow G_2$ such that

$$a, b, c \in G, a \in b * c \Leftrightarrow ga \in gb * gc.$$

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In [4], morphisms of projective geometries are introduced in terms of partial maps. In the sequel, we follow the notations in [4]. A *partial map* from a set X to a set Y is a map $f : X \setminus N \rightarrow Y$ defined on the complement of a set $N \subseteq X$. We write $f : X \dashrightarrow Y$, or also $f : X \rightarrow Y$ if we know that $N = \emptyset$.

The set N is called the *kernel* of f and is denoted by $\text{Ker } f$.

The set $X \setminus N$ is called the *domain* of f and is denoted by $\text{Dom } f$.

A *morphism* from a projective geometry G_1 to a projective geometry G_2 is a partial map $g : G_1 \dashrightarrow G_2$ satisfying the following axioms:

- (M1) $N := \text{Ker } g$ is a subspace of G_1 ;
- (M2) if $a, b \notin N$, $c \in N$ and $a \in b * c$, then $ga = gb$;
- (M3) if $a, b, c \notin N$ and $a \in b * c$, then $ga \in gb * gc$.

Notice that every isomorphism $g : G_1 \rightarrow G_2$ is a morphism with empty kernel. It is proved in [4] that projective geometries as objects together with morphisms form a category.

A morphism $g : G_1 \dashrightarrow G_2$ of projective geometries is called a *homomorphism* [4, Def. 6.5.1] if it satisfies the additional conditions:

- (M4) $a, b \in \text{Dom } g$, $ga \neq gb$ implies $g(a * b) = ga * gb$.
- (M5) if $a, b \in \text{Dom } g$, $a \neq b$ and $ga = gb$, then $(a * b) \cap \text{Ker } g \neq \emptyset$.

Let V be any vector space. Define $P(V)$ as the set of all one-dimensional subspaces of V . Then $P(V)$ becomes a projective geometry if one defines, for $X, Y, Z \in P(V)$, $Z \in X * Y$ iff $Z \subseteq X + Y$. For $x \in V$, $x \neq 0$, let $[x]$ denote the one-dimensional subspace generated by x . Denote $V^0 := V \setminus \{0\}$. It is sometimes convenient to consider the mapping $x \mapsto [x]$ from V^0 to $P(V)$ as a partial map $\pi : V \dashrightarrow P(V)$ with $\text{Ker } \pi = \{0\}$.

Let V_1, V_2 be vector spaces over K_1, K_2 . We call a map $f : V_1 \rightarrow V_2$ *semilinear* (cf [4, Def. 6.3.1]) if it is additive (i.e. $f(x + y) = f(x) + f(y)$ for $x, y \in V_1$) and if there exists a homomorphism $\sigma : K_1 \rightarrow K_2$ such that $f(\lambda x) = \sigma(\lambda)f(x)$ for $\lambda \in K_1$ and $x \in V_1$.

A semilinear map $f : V_1 \rightarrow V_2$ is called *quasilinear* ([4, Def. 6.6.10]) if σ is bijective, i.e. an isomorphism between the fields.

There are the following relations between semilinear maps and morphisms of projective geometries, and between quasilinear maps and homomorphisms.

THEOREM 1.1. ([4, Th. 6.3.5]) *Let V_1, V_2 be vector spaces over K_1, K_2 and $f : V_1 \rightarrow V_2$ an additive map. Assume that $f(V_1)$ contains at least two linearly independent vectors. Then the following conditions are equivalent:*

- (1) $f(K_1 \cdot x) \subseteq K_2 \cdot f(x)$ for all $x \in V_1$;
- (2) f is semilinear;

- (3) there is a partial map $g : P(V_1) \dashrightarrow P(V_2)$ with $g \circ \pi_1 = \pi_2 \circ f$, where $\pi_i : V_i \dashrightarrow P(V_i)$ denotes the corresponding mapping.

Moreover, the partial map g is unique, and it is a morphism of projective geometries.

The map g in the above theorem is denoted $g = Pf$, and one says that g is induced by the semilinear map f .

THEOREM 1.2. [4, Prop. 6.6.12] Let $f : V_1 \rightarrow V_2$ be a semilinear map.

¹ If f is quasilinear, then Pf is a homomorphism.

² If Pf is a homomorphism and non-constant, then f is quasilinear.

An *orthogeometry* is a projective geometry G together with a relation $\perp$, called *orthogonality*, which satisfies the following axioms ([4, Def. 14.1.1]):

- (O1) $a \perp b$ implies $b \perp a$;
- (O2) if $a \perp p$, $b \perp p$ and $c \in a * b$, then $c \perp p$;
- (O3) if $a, b, c \in G$ and $b \neq c$ then there exists $p \in b * c$ with $p \perp a$;
- (O4) for every $a \in G$ there exists $b \in G$ with $a \not\perp b$.

Let G_1 and G_2 be two orthogeometries and $g : G_1 \dashrightarrow G_2$ a morphism of projective geometries. One says that g is *orthogonal* if ([4, Def. 14.3.1])

- (1) $a, b \in \text{Dom } g$ and $a \perp b$ imply $ga \perp gb$.

Let V be a vector space over a field K . A map $\Phi : V \times V \rightarrow K$ is a *Hermitian form* if there exists an involution $\sigma : K \rightarrow K$ (i.e., an anti-isomorphism of order 2), such that the following axioms are satisfied (cf [4, Def. 14.1.5]):

- (1) $\Phi(x_1 + x_2, y) = \Phi(x_1, y) + \Phi(x_2, y)$ and $\Phi(\lambda x, y) = \lambda \Phi(x, y)$;
- (2) $\Phi(x, y) = \sigma \Phi(y, x)$ for all $x, y \in V$.

The form Φ is *non-singular* if $\Phi(x, y) = 0$ for all $y \in V$ implies $x = 0$.

By [4, Prop. 14.1.6], if $\Phi : V \times V \rightarrow K$ is a nonsingular Hermitian form, then the projective geometry $P(V)$ together with the relation $\perp$ defined by $\pi(x) \perp \pi(y)$ iff $\Phi(x, y) = 0$ is an orthogeometry.

Moreover, if $(V, \langle \cdot, \cdot \rangle)$ is an inner product space (or a pre-Hilbertian space) over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then the mapping $\Phi(x, y) := \langle x, y \rangle$ is a non-singular Hermitian form. The associated orthogeometry is *pure* in the sense that for every point $a \in P(V)$, $a \not\perp a$ holds.

From [4, Th. 14.3.4], using the fact that $P(V_2)$ is pure, we obtain the following statement.

THEOREM 1.3. Let V_1 and V_2 be inner product spaces over K_1 and K_2 , respectively, $\Phi_i : V_i \times V_i \rightarrow K_i$, $i = 1, 2$ their respective inner products. Assume that $\dim V_1 \geq 3$. Let $g : P(V_1) \dashrightarrow P(V_2)$ be a nonempty

orthogonal morphism. Then there exists a semilinear map $f : V_1 \rightarrow V_2$ which induces g and preserves linearly independent subsets of V_1 , such that $\Phi_1(x, y) = 0$ if and only if $\Phi_2(fx, fy) = 0$.

Notice that by [4, Prop. 14.3.3] and [4, Prop. 7.2.5], the orthogonal morphism g in Theorem 1.3 is injective and has an empty kernel.

In addition, we have the following statements.

THEOREM 1.4. [4, Th. 14.3.6] *Let V_1 and V_2 be two inner product spaces of dimension ≥ 3 over the same field $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and $g : P(V_1) \dashrightarrow P(V_2)$ a non-empty orthogonal morphism. Then g is induced by a quasilinear isometry $f : V_1 \rightarrow V_2$, i.e., one has $\Phi_2(fx, fy) = \tau\Phi_1(x, y)$ for all $x, y \in V_1$ where τ is the automorphism associated to f . In the complex case τ is either the identity or the conjugation.*

The following result is known as Wigner's theorem [4, Cor. 14.3.7].

THEOREM 1.5. *Let V_1 and V_2 be two inner product spaces of dimensions ≥ 3 over the same field $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and $g : P(V_1) \dashrightarrow P(V_2)$ a surjective map satisfying $a \perp b$ iff $ga \perp gb$. Then g is induced by a bijective quasilinear isometry $f : V_1 \rightarrow V_2$.*

2. Homomorphisms of the family of finite and cofinite subspaces of an inner product space

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space (over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$), and let $F(V)$ denote the family of all finite- and all cofinite-dimensional linear subspaces of V . With orthocomplementation defined by $A^\perp = \{x \in V : \langle x, y \rangle = 0 \forall y \in A\}$, $F(V)$ is an atomic modular ortholattice, where infimum of the subspaces A, B is formed by their intersection $A \cap B$, and supremum of the subspaces A, B is the subspace $(A \cup B)^{\perp\perp}$. A subspace M of V is orthoclosed if $M^{\perp\perp} = M$. All elements in $F(V)$ are orthoclosed subspaces. Moreover, for any $A, B \in F(V)$, $(A \cup B)^{\perp\perp} = A + B$, where $A + B$ is the linear subspace generated by A and B . In particular, $A + A^\perp = V$, that is every $A \in F(V)$ is a *splitting* subspace.

For more information about inner product spaces and subfamilies of their subspaces see [2].

Let V_1 and V_2 be inner product spaces. A map $\psi : F(V_1) \rightarrow F(V_2)$ will be called a *homomorphism* if it satisfies the following conditions*:

- (i) $\psi(A^\perp) = \psi(A)^\perp$ for all $A \in F(V_1)$;
- (ii) $\psi(A \vee B) = \psi(A) \vee \psi(B)$ whenever $A, B \in F(V_1)$ and $A \subseteq B^\perp$.

*We are indebted to the referee for the following observation: Axiom (i) could be weakened to $\psi(A^\perp) \leq \psi(A)^\perp$. This would entail only a minor revision of the following considerations.

Observe that (i) and (ii) imply $\psi(V_1) = V_2$. Indeed, for every $A \in F(V_1)$ we have $\psi(V_1) = \psi(A \vee A^\perp) = \psi(A) \vee \psi(A^\perp) = \psi(A) \vee \psi(A)^\perp = V_2$. Moreover, ψ is monotone. Indeed, $A, B \in F(V_1)$, $A \leq B$ implies $\psi(A) \leq \psi(A) \vee \psi(A^\perp \cap B) = \psi(A \vee A^\perp \cap B) = \psi(B)$.

A homomorphism ψ is an *isomorphism* if ψ is a bijection and ψ^{-1} satisfies (i) and (ii). Notice that every isomorphism maps atoms to atoms.

LEMMA 2.1. *Let V_1 and V_2 be inner product spaces over the fields K_1 and K_2 , respectively. Let $\psi : F(V_1) \rightarrow F(V_2)$ be a homomorphism which maps atoms of $F(V_1)$ to atoms of $F(V_2)$. If $x, y \in F(V_1)$ are atoms and $\psi(x) \neq \psi(y)$ then*

$$\psi(x \vee y) = \psi(x) \vee \psi(y).$$

Proof. Assume $\psi(x) \neq \psi(y)$, then $x \neq y$, and by orthomodularity, $y \vee x = y \vee [(y \vee x) \wedge y^\perp]$, and by covering property in $F(V_1)$, $(y \vee x) \wedge y^\perp =: z$ is an atom in $y^\perp$. Therefore $\psi(y \vee x) = \psi(y) \vee \psi(z) \geq \psi(y) \vee \psi(x)$, and by covering property in $F(V_2)$, the last inequality is an equality, which completes the proof. ■

LEMMA 2.2. *Let $\psi : F(V_1) \rightarrow F(V_2)$ be a homomorphism which maps atoms to atoms. The following statements are equivalent:*

- (i) *For every atoms $p, q, r \in F(V_1)$, $r \leq p \vee q$ implies $\psi(r) \leq \psi(p) \vee \psi(q)$.*
- (ii) *For every atoms $p, q \in F(V_1)$, $p \not\leq q$ implies $\psi(p) \not\leq \psi(q)$.*
- (iii) *For every atoms $p, q \in F(V_1)$, $p \neq q$ implies $\psi(p) \neq \psi(q)$.*
- (iv) *The homomorphism ψ is injective.*

Proof. (i) implies (ii): Assume that $a \not\leq b$ and $\psi(a) \leq \psi(b)$. Then $a \neq b$, so that $p := (a \vee b) \cap b^\perp$ is an atom such that $p \perp b$, $p \neq a$, and $p \leq a \vee b$ implies $b \leq a \vee p$. But then $\psi(b) \leq \psi(p)$ and $\psi(a) \leq \psi(b)$ imply $\psi(b) \leq \psi(b)$, a contradiction.

(ii) implies (iii): Assume that $a \neq b$ and $\psi(a) = \psi(b)$. Clearly, $a \not\leq b$. Put $p := (a \vee b) \cap b^\perp$. Then $p \perp b$, $p \not\leq a$ implies by (ii) that $\psi(p) \not\leq \psi(b)$ and $\psi(p) \not\leq \psi(a)$, a contradiction.

(iii) implies (i) follows by Lemma 2.1.

(i) implies (iv): Let $A, B \in F(V_1)$, and assume that $A \neq B$ and $\psi(A) = \psi(B)$. With no loss of generality, we may assume that there is an atom $a \in A$, $a \notin B$. Define $p := (a \vee B) \cap B^\perp$. Choose an atom $b \in B$ such that $p \leq a \vee b$. Then $\psi(p) \leq \psi(a) \vee \psi(b) \leq \psi(A) \vee \psi(B) = \psi(B)$. But also $\psi(p) \not\leq \psi(B)$, a contradiction.

(iv) implies (iii) is trivial. ■

Let V be an inner product space. For $0 \neq v_1, v_2, \dots, v_n$ let $[v_1, v_2, \dots, v_n]$ denote the subspace generated by $v_1, v_2, \dots, v_n$. We may identify points in $P(V)$ with one-dimensional subspaces (atoms) of $F(V)$, and let lines

correspond to two-dimensional subspaces. Hence for every $0 \neq v, w \in V$, $[v] = \pi(v)$, and $\pi(v) * \pi(w) = [v, w] = [v] \vee [w]$.

PROPOSITION 2.3. *Let V_1, V_2 be inner product spaces and let $\psi : F(V_1) \rightarrow F(V_2)$ be a homomorphism which maps atoms to atoms. Define the mapping $\phi : P(V_1) \rightarrow P(V_2)$ by $\phi(x) := \psi(x)$. Then ϕ is an orthogonal morphism if and only if ψ satisfies (one of) the equivalent conditions of Lemma 2.2.*

Proof. Clearly, the mapping ϕ has an empty kernel. Assume that the conditions of Lemma 2.2 are satisfied. Let $x, y, z \in P(V_1)$, $z \in x * y$, then by (i) of Lemma 2.2, $\phi(z) = \psi(z) \leq \psi(x) \vee \psi(y) = \phi(x) * \phi(y)$. It is easy to check that (M1), (M2) and (M3) are satisfied. If $x \perp y$, then $\phi(x) = \psi(x) \perp \psi(y) = \phi(y)$. From this it follows that $\phi : P(V_1) \rightarrow P(V_2)$ is an orthogonal morphism.

Conversely, since by the remarks after Theorem 1.3, every nontrivial orthogonal morphism $\phi : P(V_1) \rightarrow P(V_2)$ has an empty kernel and is injective, condition (iii) of Lemma 2.2 is satisfied. ■

The following theorem is an analogue of the Wigner theorem.

THEOREM 2.4. *Let $(V_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, \langle \cdot, \cdot \rangle_2)$ be two inner product spaces of dimension ≥ 3 over the same field $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, and let $\psi : F(V_1) \rightarrow F(V_2)$ be an isomorphism. Then there is bijective quasilinear isometry (i.e., an isomorphism of inner product spaces) $f : V_1 \rightarrow V_2$. In the complex case, the isometry is either linear or antilinear.*

Proof. If ψ is an isomorphism, then it maps atoms to atoms. Moreover, it is surjective and for all atoms in $F(V_1)$, $x \perp y$ iff $\psi(x) \perp \psi(y)$. These properties are inherited by the mapping $\phi : P(V_1) \rightarrow P(V_2)$. The result follows by applying Theorem 1.5. ■

In what follows, we study the case when ψ is a homomorphism.

Let V be an inner product space, and let H be a finite-dimensional subspace. The interval $[0, H]$ in $F(V)$ is a modular ortholattice with respect to the relative orthocomplementation $X \mapsto X^\perp \cap H$, $0 \leq X \leq H$. If we restrict the inner product $\langle \cdot, \cdot \rangle$ of V to H , then H becomes an inner product space, and as it is finite-dimensional, it is a Hilbert space. Evidently, the modular ortholattice $L(H)$ of all subspaces of H is isomorphic with $[0, H]$. Owing to $A + A^\perp = V$, for every $v \in V$, $v = v_1 + v_2$ with $v_1 \in A$, $v_2 \in A^\perp$. So we may define the projection $P_A : V \rightarrow V$ by $P_A v = v_1$, $v \in V$. If $A \in [0, H]$, $v \in H$, then $v, v_1 \in H$ implies $v_2 \in H$, hence $v_2 \in A^\perp \cap H$. Hence the restriction of P_A to H becomes a projection in the inner product space H endowed with the inner product inherited from V .

Recall that a *state* on $L(H)$ is a map $m : L(H) \rightarrow [0, 1] \subset \mathbb{R}$ such that $m(H) = 1$ and $m(A \vee B) = m(A) + m(B)$ whenever $A, B \in L(H)$,

$A \perp B$. In particular, for every $v \in H$, $\|v\| = 1$, the mapping $m_v(A) = \langle P_A v, v \rangle$ where P_A is the projection in H onto A , is a state on $L(H)$. By Gleason's theorem, every state on $L(H)$ extends to a linear functional m^* on the set $B(H)$ of all operators on H such that $m(A) = m^*(P_A)$ for all $A \in L(H)$. Moreover, every state on $L(H)$ has the Jauch-Piron property, namely $m(A) = 1 = m(B)$ implies that $m(A \cap B) = 1$.

The proof of the following proposition is based on the ideas from [3]. It makes use of some Hilbert space properties including the Gleason theorem.

PROPOSITION 2.5. *Let V_1 and V_2 be inner product spaces with dimensions at least three over the same field $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let $F(V_1)$ and $F(V_2)$ be their respective sets of all finite and all cofinite subspaces. Let $\psi : F(V_1) \rightarrow F(V_2)$ be a homomorphism which maps atoms to atoms. Then ψ is injective.*

Proof. Let H be any finite-dimensional subspace of V_1 with $\dim H \geq 3$.

The homomorphism $\psi : F(V_1) \rightarrow F(V_2)$ maps H into $\psi(H)$, and taking into account the orthogonal base in H , and the fact that ψ maps atoms to atoms, we see that $\psi(H)$ is finite-dimensional with $\dim \psi(H) \geq 3$. The interval $[0, H] \subseteq F(V_1)$ is mapped into the interval $[0, \psi(H)] \subseteq F(V_2)$. The latter is isomorphic with the modular ortholattice $L(\psi(H))$ of all subspaces of the Hilbert space $\psi(H)$. If $A \in L(H)$, the corresponding projection is P_A , and the projection onto its image $\psi(A)$ is $P_{\psi(A)}$.

For every state m' on $L(\psi(H))$, the composite mapping $m' \circ \psi : L(H) \rightarrow [0, 1]$ is a state on $L(H)$. Notice that for all $A \in L(H)$, $m' \circ \psi(A) = m'^*(P_{\psi(A)})$. For example, for every $v' \in \psi(H)$, $\|v'\| = 1$, $A \mapsto m_{v'}(\psi(A)) = \langle P_{\psi(A)} v', v' \rangle$ is a state on $L(H)$.

Now let p, q be atoms (one-dimensional subspaces) in $F(V_1)$ and assume that $p \neq q$ and $\psi(p) = \psi(q)$. Let H be a finite dimensional subspace of V_1 with $\dim H \geq 3$ which contains p and q . Then for any $v' \in \psi(p)$, $\|v'\| = 1$, we have

$$m_{v'} \circ \psi(p) = \langle P_{\psi(p)} v', v' \rangle = 1 = m_{v'} \circ \psi(q),$$

hence by the Jauch-Piron property, $m_{v'} \circ \psi(p \cap q) = 1$, contradicting the fact that $p \cap q = 0$. This implies that for every atoms $p, q \in F(V_1)$, $p \neq q \implies \psi(p) \neq \psi(q)$, and according Lemma 2.2, ψ is injective. ■

From Theorem 1.3 and Theorem 1.4, applying Proposition 2.5, we obtain the following results.

THEOREM 2.6. *Let $(V_1; \langle \cdot, \cdot \rangle_1)$ and $(V_2; \langle \cdot, \cdot \rangle_2)$ be inner product spaces over the fields K_1 and K_2 , respectively, and let $\psi : F(V_1) \rightarrow F(V_2)$ be a homomorphism which maps atoms to atoms. Assume that $\dim V_1 \geq 3$. Then there exists a semilinear map $f : V_1 \rightarrow V_2$ which preserves linearly independent subsets of V_1 , such that $\langle x, y \rangle_1 = 0$ if and only if $\langle fx, fy \rangle_2 = 0$.*

THEOREM 2.7. *Let $(V_1, \langle \cdot, \cdot \rangle_1)$ and $(V_2, \langle \cdot, \cdot \rangle_2)$ be inner product spaces of dimension ≥ 3 over the same field $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $\psi : F(V_1) \rightarrow F(V_2)$ be a homomorphism which maps atoms to atoms. Then there exists a quasilinear isometry $f : V_1 \rightarrow V_2$, i.e., one has $\langle fx, fy \rangle_2 = \tau \langle x, y \rangle_1$ for all $x, y \in V_1$, where τ is the automorphism associated to f . In the complex case τ is either the identity or the conjugation.*

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