Yongjin Li

ISHIKAWA ITERATIVE SEQUENCE WITH ERRORS
FOR K-SUBACCETIVE OPERATORS
IN ARBITRARY BANACH SPACES

Abstract. In this paper, the iterative solution is studied for equation $x+Tx = f$ with a Lipschitz K-subaccretive operator in arbitrary Banach spaces, some previously results are generalized.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$ and E^* be the dual space of E . The duality mapping $J : E \rightarrow 2^{E^*}$ is defined by

$$Jx = \{f \in E^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|\}$$

where $\langle x, f \rangle$ denotes the value of the continuous linear functional $f \in E^*$ at $x \in E$.

An operator $T : D(T) \subset E \rightarrow E$ is said to be *k-subaccretive* if for all $x, y \in D(T)$ there exists $j \in J(x - y)$ and $k \in (-\infty, +\infty)$ such that

$$(1) \quad \langle Tx - Ty, j(x - y) \rangle \geq k\|x - y\|^2.$$

An operator $T : D(T) \subset E \rightarrow E$ is said to be *accretive* if $k = 0$ in (1) and T is said to be *strongly accretive* if $k > 0$ in (1).

An accretive operator T is said to be *m-accretive*, if $R(T + \lambda I) = E$ for some $\lambda > 0$ (hence for all $\lambda > 0$), where I stands for the identity operator.

For Hilbert space E Chidume [3], Dontson [9] and Bruck [2] considered the iterative process of the nonlinear operator equation $f = x + Tx$, $x \in D(T)$, $f \in R(I + T)$. For L_p some iterative techniques have been studied by

1991 *Mathematics Subject Classification*: 47H17, 47H10.

Key words and phrases: k-subaccretive operator; nonlinear equation; Ishikawa iterative sequences with errors.

This work is supported by grant 10471155 from NNSF of China, grant 031608 from the Foundation of the Guangdong province Natural Science Committee, and the Foundation of Sun Yat-sen University advanced Research Centre.

Chidume [4]. For an uniformly smooth Banach space E , Zhu [16] consider iterative techniques, when T is Lipschitzian (or locally Lipschitzian with constant $L \geq 1$) and m -accretive, or when is m -accretive and $R(T)$ is bounded and E is uniformly convex. Liu in [11] proved the following theorem:

THEOREM A ([11]). *Let E be uniformly smooth Banach space, $T : D(T) = E \rightarrow E$ be m -accretive Lipschitz operator (with Lipschitz constant $L \geq 1$), $\{u_n\}$, $\{v_n\}$ be sequences in E and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0, 1]$ such that*

- (1) $\alpha_n \rightarrow 0$, as $n \rightarrow \infty$. $\limsup \beta_n L(L + 1) < 1$;
- (2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (3) $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$.

Then, for any $x_0 \in E$, the Ishikawa iteration sequence $\{x_n\}$ with errors defined by

$$\begin{aligned} y_n &= (1 - \beta_n)x_n + \beta_n(f - Tx_n) + v_n \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n(f - Ty_n) + u_n \end{aligned}$$

converges strongly to the unique solution $x^ \in D(T)$ of the equation $x + Tx = f$ if $\{Ty_n\}$ is bounded.*

Our objective in this paper is to consider an iterative process which converges to a solution of $f = x + Tx$ in Banach space. Our results generalize and improve the all results of [11] and [16].

In order to prove the main results of the paper we need the following lemmas.

LEMMA 1 ([11]). *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be nonnegative sequence satisfying*

$$(2) \quad a_{n+1} \leq (1 - t_n)a_n + b_n + c_n.$$

With $\{t_n : n = 0, 1, 2, \dots\} \subset [0, 1]$, $\sum_{n=1}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=1}^{\infty} c_n < \infty$ then $\lim_{n \rightarrow \infty} a_n = 0$.

LEMMA 2 ([16]). *Let E be Banach space and $T : D(T) \subset E \rightarrow E$ be an m -accretive operator. Then, for any given $f \in E$, the equation $x + Tx = f$ has an unique solution in $D(T)$.*

LEMMA 3. *Let E be Banach space, $J : E \rightarrow 2^{E^*}$ be the duality mapping, then for any $x, y \in E$ and $j(x + y) \in J(x + y)$ we have*

$$\|x + y\|^2 \leq \|x\| \|x + y\| + \langle y, j(x + y) \rangle.$$

Proof. It follows from a sequel: for $x, y \in E$ and $j(x + y) \in J(x + y)$ one has

$$\begin{aligned} \|x + y\|^2 &= \langle x + y, j(x + y) \rangle = \langle x, j(x + y) \rangle + \langle y, j(x + y) \rangle \\ &\leq \|x\| \|x + y\| + \langle y, j(x + y) \rangle. \end{aligned}$$

2. Main results

Now we state and prove the following theorems.

THEOREM 1. *Let E be Banach space, $T : E \rightarrow E$ be Lipschitz operator (with Lipschitz constant $L \geq 1$), and T satisfy*

$$(3) \quad \langle Tx - Ty, J(x - y) \rangle \geq -\frac{1}{2} \|x - y\|^2.$$

Let $\{u_n\}, \{v_n\}$ be sequences in E and $\{\alpha_n\}, \{\beta_n\}$ be sequences in $[0, 1]$ such that

- (1) $\sum_{n=1}^{\infty} \|u_n\| < \infty, \sum_{n=1}^{\infty} \|v_n\| < \infty;$
- (2) $\sum_{n=1}^{\infty} \alpha_n = \infty;$
- (3) $\sup \beta_n < \frac{\lambda}{2L(L+1)}, \sup \alpha_n < \frac{1-\lambda}{2[L(L+1)(L+2)+2]},$ for some $\lambda \in (0, 1).$

Then, for any $x_0 \in E$, the Ishikawa iteration sequence $\{x_n\}$ with errors defined by

$$(4) \quad y_n = (1 - \beta_n)x_n + \beta_n(f - Tx_n) + v_n,$$

$$(5) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(f - Ty_n) + u_n$$

converges strongly to the unique solution x^ of the equation $x + Tx = f$.*

Proof. Since $\langle Tx - Ty, J(x - y) \rangle \geq -\frac{1}{2}\|x - y\|^2$ for any x, y , we have $\langle (I + 2T)x - (I + 2T)y, J(x - y) \rangle = \langle x - y, J(x - y) \rangle + 2\langle Tx - Ty, J(x - y) \rangle \geq \|x - y\|^2 - \|x - y\|^2 = 0$. Thus $I + 2T$ is an accretive operator. Since T is Lipschitz, we conclude that $I + 2T$ is m -accretive operator. By Lemma 2, the equation $x + (I + 2T)x = f$ has an unique solution x^* . So, the equation $2x + 2Tx = f$ has a unique solution. Thus the equation $x + Tx = f$ has an unique solution x^* . Let $Sx = f - Tx$. Since $Sx^* = f - Tx^* = x^*$ then x^* is a fixed point of S and there exists $j(x - y) \in J(x - y)$ such that

$$\langle Sx - Sy, j(x - y) \rangle = -\langle Tx - Ty, j(x - y) \rangle \leq \frac{1}{2}\|x - y\|^2.$$

Since $T : E \rightarrow E$ is Lipschitz operator, by

$$\|Sx - Sy\| = \|f - Tx - (f - Ty)\| \leq L\|x - y\|,$$

S and T has the same Lipschitz constant. Note that

$$(6) \quad y_n = (1 - \beta_n)x_n + \beta_n(Sx_n) + v_n,$$

$$(7) \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n(Sx_n) + u_n.$$

By using (7) we obtain

$$\begin{aligned}
 x_n &= x_{n+1} + \alpha_n x_n - \alpha_n S y_n - u_n \\
 &= x_{n+1} + \alpha_n (x_{n+1} + \alpha_n x_n - \alpha_n S y_n - u_n) - \alpha_n S y_n - u_n \\
 &= (1 + \alpha_n) x_{n+1} + \alpha_n^2 x_n - \alpha_n^2 S y_n - \alpha_n u_n - \alpha_n S y_n - u_n \\
 &= (1 + \alpha_n) x_{n+1} - \alpha_n S y_n + \alpha_n^2 (x_n - S y_n) - \alpha_n u_n - u_n \\
 &= (1 + \alpha_n) x_{n+1} - \alpha_n S x_{n+1} + \alpha_n^2 (x_n - S y_n) \\
 &\quad + \alpha_n (S x_{n+1} - S y_n - u_n) - u_n, \\
 x^* &= (1 + \alpha_n) x^* - \alpha_n S x^*.
 \end{aligned}$$

So

$$\begin{aligned}
 x_n - x^* &= (1 + \alpha_n)(x_{n+1} - x^*) - \alpha_n (S x_{n+1} - S x^*) \\
 &\quad + \alpha_n^2 (x_n - S y_n) + \alpha_n (S x_{n+1} - S y_n - u_n) - u_n \\
 &= (1 + \alpha_n) \left[\|x_{n+1} - x^* - \frac{\alpha_n}{1 + \alpha_n} (S x_{n+1} - S x^*)\| \right] \\
 &\quad + \alpha_n^2 (x_n - S y_n) + \alpha_n (S x_{n+1} - S y_n - u_n) - u_n
 \end{aligned}$$

and thus

$$\begin{aligned}
 \|x_{n+1} - x^*\|^2 &= \|x_{n+1} - x^* - \frac{\alpha_n}{1 + \alpha_n} (S x_{n+1} - S x^*) + \frac{\alpha_n}{1 + \alpha_n} (S x_{n+1} - S x^*)\|^2 \\
 &\leq \|x_{n+1} - x^* - \frac{\alpha_n}{1 + \alpha_n} (S x_{n+1} - S x^*)\| \|x_{n+1} - x^*\| \\
 &\quad + \frac{\alpha_n}{1 + \alpha_n} \langle S x_{n+1} - S x^*, j(x_{n+1} - x^*) \rangle \\
 &\leq \|x_{n+1} - x^* - \frac{\alpha_n}{1 + \alpha_n} (S x_{n+1} - S x^*)\| \|x_{n+1} - x^*\| \\
 &\quad + \frac{\alpha_n}{2(1 + \alpha_n)} \|x_{n+1} - x^*\|^2.
 \end{aligned}$$

So

$$\|x_{n+1} - x^* - \frac{\alpha_n}{1 + \alpha_n} (S x_{n+1} - S x^*)\| \geq \frac{2 + \alpha_n}{2(1 + \alpha_n)} \|x_{n+1} - x^*\|$$

and hence

$$\begin{aligned}
 \|x_n - x^*\| &\geq \left(1 + \frac{1}{2} \alpha_n\right) \|x_{n+1} - x^*\| - \alpha_n^2 \|x_n - S y_n\| \\
 &\quad - \alpha_n \|S x_{n+1} - S y_n - u_n\| - \|u_n\|.
 \end{aligned}$$

Therefore

$$\begin{aligned}
 &\left(1 + \frac{1}{2} \alpha_n\right) \|x_{n+1} - x^*\| \\
 &\leq \|x_n - x^*\| + \alpha_n^2 \|x_n - S y_n\| + \alpha_n \|S x_{n+1} - S y_n - u_n\| + \|u_n\|
 \end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \left(1 - \frac{1}{2}\alpha_n + \frac{1}{4}\alpha_n^2\right)\|x_n - x^*\| + \alpha_n^2\|x_n - Sy_n\| \\ &\quad + \alpha_n\|Sx_{n+1} - Sy_n - u_n\| + \|u_n\|.\end{aligned}$$

But

$$y_n = (1 - \beta_n)x_n + \beta_n Sx_n + v_n.$$

So we have

$$\begin{aligned}\|y_n - x^*\| &= \|(1 - \beta_n)x_n + \beta_n Sx_n + v_n - x^*\| \\ &= \|(1 - \beta_n)x_n - (1 - \beta_n)x^* + \beta_n(Sx_n - x^*) + v_n\| \\ &\leq (1 - \beta_n)\|x_n - x^*\| + \beta_n\|Sx_n - Sx^*\| + \|v_n\| \\ &= (1 - \beta_n)\|x_n - x^*\| + \beta_n L\|x_n - x^*\| + \|v_n\| \\ &= (1 - \beta_n + \beta_n L)\|x_n - x^*\| + \|v_n\|\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \left(1 - \frac{1}{2}\alpha_n + \frac{1}{4}\alpha_n^2\right)\|x_n - x^*\| \\ &\quad + \alpha_n^2\|x_n - Sy_n\| + \alpha_n\|Sx_{n+1} - Sy_n - u_n\| + \|u_n\|.\end{aligned}$$

But

$$\begin{aligned}\|x_n - Sy_n\| &\leq \|x_n - x^*\| + \|Sy_n - x^*\| \leq \|x_n - x^*\| + L\|y_n - x^*\| \\ &\leq \|x_n - x^*\| + L(1 - \beta_n + \beta_n L)\|x_n - x^*\| + L\|v_n\| \\ &= (1 + L - L\beta_n + L^2\beta_n)\|x_n - x^*\| + L\|v_n\|\end{aligned}$$

and

$$\begin{aligned}\|Sy_n - y_n\| &\leq \|Sy_n - x^*\| + \|y_n - x^*\| = \|Sy_n - Sx^*\| + \|y_n - x^*\| \\ &\leq L\|y_n - x^*\| + \|y_n - x^*\| \\ &= (L + 1)(1 - \beta_n + \beta_n L)\|x_n - x^*\| + (L + 1)\|v_n\|.\end{aligned}$$

So

$$\begin{aligned}\|Sx_{n+1} - Sy_n - u_n\| &\leq \|Sx_{n+1} - Sy_n\| + \|u_n\| \leq L\|x_{n+1} - y_n\| + \|u_n\| \\ &\leq L(1 - \alpha_n)\|\beta_n(x_n - Sy_n - v_n)\| + L\alpha_n\|Sy_n - y_n\| + (L + 1)\|u_n\| \\ &\leq L(1 - \alpha_n)(\|\beta_n(x_n - Sx_n)\| + \|v_n\|) \\ &\quad + L\alpha_n[(L + 1)(1 - \beta_n + \beta_n L)\|x_n - x^*\| + (L + 1)\|v_n\|] + (L + 1)\|u_n\|\end{aligned}$$

$$\begin{aligned}
&\leq L(1 - \alpha_n)(\beta_n(L + 1)\|x_n - x^*\| + \|v_n\|) \\
&\quad + L(L + 1)\alpha_n(1 - \beta_n + \beta_n L)\|x_n - x^*\| \\
&\quad + L(L + 1)\alpha_n\|v_n\| + (L + 1)\|u_n\| \\
&= L(L + 1)\beta_n(1 - \alpha_n)\|x_n - x^*\| + L(1 - \alpha_n)\|v_n\| \\
&\quad + L(L + 1)\alpha_n(1 - \beta_n + \beta_n L)\|x_n - x^*\| \\
&\quad + L(L + 1)\alpha_n\|v_n\| + (L + 1)\|u_n\|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&\leq \left(1 - \frac{1}{2}\alpha_n + \frac{1}{4}\alpha_n^2\right)\|x_n - x^*\| \\
&\quad + \alpha_n^2\|x_n - Sy_n\| + \alpha_n\|Sx_{n+1} - Sy_n - u_n\| + \|u_n\| \\
&\leq \left(1 - \frac{1}{2}\alpha_n + \frac{1}{4}\alpha_n^2\right)\|x_n - x^*\| + [\alpha_n^2(1 + L - L\beta_n \\
&\quad + L^2\beta_n)\|x_n - x^*\| + \alpha_n L\|y_n\|] + \alpha_n[L(L + 1)\beta_n(1 - \alpha_n)\|x_n - x^*\| \\
&\quad + L(1 - \alpha_n)\|v_n\| + L(L + 1)\alpha_n(1 - \beta_n + \beta_n L)\|x_n - x^*\| \\
&\quad + L(L + 1)\alpha_n\|v_n\| + (L + 1)\|u_n\|] + \|u_n\| \\
&= \left(1 - \frac{1}{2}\alpha_n\right)\|x_n - x^*\| + \alpha_n(1 - \alpha_n)L(L + 1)\beta_n\|x_n - x^*\| \\
&\quad + \alpha_n^2\left[\frac{1}{4}\|x_n - x^*\| + (1 + L - L\beta_n + L^2\beta_n)\|x_n - x^*\| \right. \\
&\quad \left. + L(L + 1)(1 - \beta_n + \beta_n L)\|x_n - x^*\| + (\|v_n\| - \alpha_n L\|v_n\| \right. \\
&\quad \left. + \alpha_n^2 L(L + 1)\|v_n\|) + (\alpha_n(L + 1)\|u_n\| + \|u_n\|)\right] \\
&\leq (1 - \frac{1}{2}\alpha_n)\|x_n - x^*\| + \alpha_n L(L + 1)\beta_n\|x_n - x^*\| + \alpha_n^2\left[\left(\frac{5}{4} + L - L\beta_n \right. \right. \\
&\quad \left. \left. + L^2\beta_n\right) + L(L + 1)(1 - \beta_n + \beta_n L)\right]\|x_n - x^*\| + M\|v_n\| + N\|u_n\| \\
&\leq \left(1 - \frac{1}{2}\alpha_n\right)\|x_n - x^*\| + \alpha_n L(L + 1)\beta_n\|x_n - x^*\| \\
&\quad + [L(L + 1)(L + 2) + 2]\alpha_n^2\|x_n - x^*\| + M\|v_n\| + N\|u_n\|.
\end{aligned}$$

Since

$$\sup \beta_n < \frac{\lambda}{2L(L + 1)}, \quad \sup \alpha_n < \frac{1 - \lambda}{2[L(L + 1)(L + 2) + 2]}$$

for some $\lambda \in (0, 1)$, there exists $N > 0$ such that for all $n > N$, we have

$$\tau = \frac{1}{2} - L(1 + L)\beta_n - [L(L + 1)(L + 2) + 2]\alpha_n > 0.$$

Thus we have

$$\|x_{n+1} - x^*\| \leq (1 - \tau\alpha_n)\|x_n - x^*\| + M\|v_n\| + N\|u_n\|.$$

Set

$$\begin{aligned} t_n &= \tau\alpha_n, \\ b_n &= 0, \\ c_n &= M\|v_n\| + N\|u_n\|, \\ a_n &= \|x_n - x^*\|. \end{aligned}$$

Then we have

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n.$$

According to above argument, it can be easily seen that

$$\sum_{n=1}^{\infty} t_n = \infty, \quad b_n = o(t_n), \quad \sum_{n=1}^{\infty} c_n \leq \infty$$

and so, by Lemma 1, $\lim a_n = \lim \|x_n - x^*\| = 0$. But the latter means that the sequence $\{x_n\}$ converges strongly to the unique solution x^* of equation $x + Tx = f$. ■

REMARK 1. Theorem 1 extends Theorem 2 [11] from uniformly smooth Banach space to general Banach spaces and extends the Ishikawa sequence in Corollary 12 in [16] to Ishikawa sequence with errors.

The author would like to thank the anonymous referee for his or her suggestions and corrections.

References

- [1] F. E. Browder, *Nonlinear mappings of nonexpansive and accretive type in Banach spaces*, Bull. Amer. Math. Soc. 73 (1967), 875–882.
- [2] R. E. Bruck, *The iterative solution of the equation $y = x + Tx$ for a monotone operator T in Hilbert space*, Bull. Amer. Math. Soc. 79 (1973), 1258–1262.
- [3] C. E. Chidume, *An approximation method for monotone operator*, J. Austral. Math. Soc. (series A) 41 (1986), 59–63.
- [4] C. E. Chidume, *The iterative solution of the equation $f = x + Tx$ for a monotone operator T in L^p space*, J. Math. Anal. Appl. 166 (1986), 531–537.
- [5] C. E. Chidume, H. Zegeye, *Approximation of the zeros of m -accretive operator*, Nonlinear Anal. 37 (1999), 81–96.
- [6] C. E. Chidume, M. O. Osilike, *Nonlinear accretive and pseudo-contractive operator equations in Banach spaces*, Nonlinear Anal. 31 (7) (1998), 779–789.

- [7] C. E. Chidume, M. O. Osilike, *Iterative solutions of nonlinear accretive operator equations in arbitrary Banach spaces*, Nonlinear Anal. 36 (1999), 863–872.
- [8] C. E. Chidume, H. Zegeye, *Iterative solution of $0 \in Ax$ for an m -accretive operator A in certain Banach spaces*, J. Math. Anal. Appl. 269 (2002), 421–430.
- [9] W. G. Dotson, *An iterative process for nonlinear monotonic nonexpansive operators in Hilbert space*, Math. Comp. 32 (1978), 223–225.
- [10] T. Kato, *Nonlinear semi-groups and evolutions equations*, J. Math. Soc. Japan. 19 (1964), 508–520.
- [11] Liu Lishan, *Ishikawa-type and Mann-type iterative process with errors for constructing solutions of nonlinear equations involving m -accretive operators in Banach spaces*, Nonlinear Anal. 34 (1998), 307–317.
- [12] R. H. Martin, Jr., *A global existence theorem for autonomous differential equations in Banach space*, Proc. Amer. Math. Soc. 26 (1970), 307–314.
- [13] S. Reich, *An iterative procedure for constructing zeros of accretive sets in Banach spaces*, Nonlinear Anal. 2 (1978), 85–92.
- [14] R. T. Rockafellar, *Local boundedness of nonlinear monotone operator*, Michigan Math J. 16 (1969), 397–407.
- [15] Xu Yuguang, *Ishikawa and Mann Iterative process with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl. 224 (1998), 91–101.
- [16] Zhu Liang, *Iterative solution of nonlinear equations involving m -accretive operators in Banach spaces*, J. Math. Anal. Appl. 188 (1994), 410–416.

DEPARTMENT OF MATHEMATICS
SUN YAT-SEN UNIVERSITY
GUANGZHOU 510275, P. R. CHINA
e-mail: stslyj@mail.sysu.edu.cn

Received August 3, 2004; revised version December 22, 2004.