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LIFTING OF VECTOR FIELDS TO 1-FORMS ON r -JET PROLONGATION OF THE BUNDLE OF TENSORS OF TYPE $(0, 2)$

Abstract. For natural numbers $n \geq 2$ and $r \geq 1$ all natural operators $T_{|\mathcal{M}f_n} \rightarrow T^*(J^r(\otimes^2 T^*))$ lifting vector fields from n -manifolds M into 1-forms on $J^r(\otimes^2 T^*)(M)$ are classified. It's proved that the set of all natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^r(\otimes^2 T^*))$ is a free $(4r+3)$ -dimensional $C^\infty(\mathbf{R}^{r+1})$ module and we construct explicitly the basis of this module.

0. Introduction

The present paper is devoted to natural operators in the sense of [2]. Namely, we study the problem how a vector field $X \in \mathcal{X}(M)$ induces a 1-form $A(X) \in \Omega^1(J^r(\otimes^2 T^*)(M))$, where $J^r(\otimes^2 T^*)(M)$ is the r -jet prolongation of the bundle of tensors of type $(0, 2)$ on M . This problem is a reflection on natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^r(\otimes^2 T^*))$.

The main result is that the set of all natural operators $A : T_{|\mathcal{M}f_n} \rightsquigarrow T^*(J^r(\otimes^2 T^*))$ is a free $(4r+3)$ -dimensional $C^\infty(\mathbf{R}^{r+1})$ module and we construct explicitly the basis of this module. The proof of the main result is a modification of a respective theorem in [5].

Similar problems of natural operators were studied in many papers, e.g. [1], [3], [4].

All manifolds are assumed to be finite dimensional and smooth, i.e. of class C^∞ . Mappings between manifolds are assumed to be smooth.

1. Natural bundle $J^r(\otimes^2 T^*)$ over n -manifolds

For every n -dimensional manifold M we have the vector bundle

$$J^r(\otimes^2 T^*)(M) = \{j_x^r \tau \mid \tau \text{ is a tensor field of type } (0, 2) \text{ over } M, x \in M\}.$$

Every local diffeomorphism $\varphi : M \rightarrow N$ between n -manifolds gives a vector bundle homomorphism $J^r(\otimes^2 T^*)(\varphi) : J^r(\otimes^2 T^*)(M) \rightarrow J^r(\otimes^2 T^*)(N)$,

$j_x^r \tau \rightarrow j_{\varphi(x)}^r(\varphi_* \tau)$, $j_x^r \tau \in (J^r(\otimes^2 T^*)(M))_x$, $x \in M$. The correspondence $J^r(\otimes^2 T^*) : \mathcal{M}f_n \rightarrow \mathcal{VB}$ is a vector natural bundle.

2. Definition of natural operator $B : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)} J^r(\otimes^2 T^*)$

A natural operator $B : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)} J^r(\otimes^2 T^*)$ is a system of operators

$$B : \mathcal{X}(M) \rightarrow C^\infty(J^r(\otimes^2 T^*)(M))$$

for every n -manifold M satisfying :

- a) (*Naturality*) If vector fields $X \in \mathcal{X}(M)$ and $Y \in \mathcal{X}(N)$ are φ -connected ($T\varphi \circ X = Y \circ \varphi$) by local diffeomorphism $\varphi : M \rightarrow N$ between n -manifold then functions $B(X)$ and $B(Y)$ are $J^r(\otimes^2 T^*)(\varphi)$ -connected ($B(X) = B(Y) \circ J^r(\otimes^2 T^*)(\varphi)$).
- b) (*Regularity*) If $(X_t)_{t \in \mathbf{R}^k} \in \mathcal{X}(M)$ is a smoothly parametrized family of vector fields, (i.e. $(t, y) \rightarrow X_t(y)$ is smooth) then $(B(X_t))_{t \in \mathbf{R}^k}$ is a smooth family of functions.

3. Example of natural operator $B : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)} J^r(\otimes^2 T^*)$

Let $p = 0, \dots, r$. For every vector field $X \in \mathcal{X}(M)$ on n -manifold M we have a function $B^{(p)}(X) : J^r(\otimes^2 T^*)(M) \rightarrow \mathbf{R}$ satisfying

$$B^{(p)}(X)(j_x^r \tau) = (X^{(p)} \tau(X, X))_x$$

where $j_x^r \tau \in (J^r(\otimes^2 T^*)(M))_x$, $x \in M$ and $X^{(p)} = X \circ \dots \circ X$ (p -times).

Family of maps $B^{(p)} : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)} J^r(\otimes^2 T^*)$ such that

$$B^{(p)} : \mathcal{X}(M) \rightarrow C^\infty(J^r(\otimes^2 T^*)(M)), X \rightarrow B^{(p)}(X),$$

is a natural operator.

4. Classification of natural operators $B : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)} J^r(\otimes^2 T^*)$

The first main result of the paper is the following theorem

THEOREM 1. *Let r, n be natural numbers. Then for every natural operator $B : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)} J^r(\otimes^2 T^*)$ there exists a unique smooth function $h : \mathbf{R}^{r+1} \rightarrow \mathbf{R}$ such that*

$$B(X) = h(B^{(0)}(X), \dots, B^{(r)}(X)), \quad X \in \mathcal{X}(M).$$

We say that a natural operator $B : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)} J^r(\otimes^2 T^*)$ is linear if $B : \mathcal{X}(M) \rightarrow C^\infty(J^r(\otimes^2 T^*)(M))$ is $\mathbf{R}$ -linear for every n -manifold M .

From Theorem 1 we have

COROLLARY 1. *Every linear natural operator $B : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)} J^r(\otimes^2 T^*)$ is equal to 0.*

5. Proof of Theorem 1

We start with the following lemma.

LEMMA 1. Let $B : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}J^r(\otimes^2 T^*)$ be a natural operator. Let $B(\partial_1)|(J^r(\otimes^2 T^*)(\mathbf{R}^n))_0 = 0$ be the canonical vector field on $\mathbf{R}^n$, where $\partial_1 = \frac{\partial}{\partial x^1}$. Then $B = 0$.

Proof. Let $X \in \mathcal{X}(M)$, $y \in (J^r(\otimes^2 T^*)(M))_x$, $x \in M$. We must prove that $B(X)_y = 0$.

Consider two cases:

(a) $X_x \neq 0$. There exist a map φ on M such that $\varphi(x) = 0$ and $\varphi_* X = \partial_1$ in the neighborhood of 0. Then

$$B(X)_y = TJ^r(\otimes^2 T^*)(\varphi^{-1}) \circ B(\partial_1)(J^r(\otimes^2 T^*)(\varphi)(y)) = 0$$

from the assumption and the naturality condition.

(b) $X_x = 0$. There exist $Y \in \mathcal{X}(M)$ such that $Y_x \neq 0$. Then $\tilde{X}_t = X + tY \in \mathcal{X}(M)$ and $(\tilde{X}_t)_x \neq 0$ for $t \neq 0$. But $\tilde{X}_t$ is smooth family of fields, hence by regularity condition and (a) we have

$$B(X)_y = B(\tilde{X}_0)_y = \lim_{t \rightarrow 0^+} B(\tilde{X}_t)(y) = 0.$$

The lemma is proved. ■

We continue the proof of Theorem 1.

Let $B : T|_{\mathcal{M}f_n} \rightsquigarrow T^{(0,0)}J^r(\otimes^2 T^*)$ be a natural operator.

Suppose $h \in C^\infty(\mathbf{R}^{r+1})$ satisfies

$$(1) \quad B(X) = h(B^{(0)}(X), \dots, B^{(r)}(X))$$

for all $X \in \mathcal{X}(M)$.

Then in particular

$$h(a_0, \dots, a_r) = h(B^{(0)}(\partial_1)_{y_a}, \dots, B^{(r)}(\partial_1)_{y_a}) = B(\partial_1)_{y_a}$$

where $y_a = j_0^r(\sum_{q=0}^r \frac{1}{q!} a_q (x^1)^q dx^1 \otimes dx^1) \in (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0$ for $a = (a_0, \dots, a_r) \in \mathbf{R}^{r+1}$.

Hence if h exists, it has to be unique. We will show that $h : \mathbf{R}^{r+1} \rightarrow \mathbf{R}$ given by

$$h(a) := B(\partial_1)_{y_a}, \quad a \in \mathbf{R}^{r+1}$$

meets the condition (1) for every $X \in \mathcal{X}(M)$.

By Lemma 1 it is enough to show that

$$B(\partial_1)_y = h(B^{(0)}(\partial_1)_y, \dots, B^{(r)}(\partial_1)_y)$$

for every $y \in (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0$.

So let $y = j_0^r(\sum_{\alpha, i, j} \frac{1}{\alpha!} a_{\alpha, i, j} x^\alpha dx^i \otimes dx^j) \in J^r(\otimes^2 T^*)(\mathbf{R}^n)_0$.

Using the invariance of $B(\partial_1), B^{(0)}(\partial_1), \dots, B^{(r)}(\partial_1)$ with respect to diffeomorphism $a_t = (x^1, tx^2, \dots, tx^n)$, $t \neq 0$, we can see that

$$B(\partial_1)_y = B(\partial_1)_{J_0^r(\otimes^2 T^*)(a_t)(y)}$$

and

$$\begin{aligned} h(B^{(0)}(\partial_1)_y, \dots, B^{(r)}(\partial_1)_y) \\ = h(B^{(0)}(\partial_1)_{J_0^r(\otimes^2 T^*)(a_t)(y)}, \dots, B^{(r)}(\partial_1)_{J_0^r(\otimes^2 T^*)(a_t)(y)}). \end{aligned}$$

Putting $t \rightarrow 0$ we get

$$B(\partial_1)_y = B(\partial_1)_{y_0}$$

and

$$h(B^{(0)}(\partial_1)_y, \dots, B^{(r)}(\partial_1)_y) = h(B^{(0)}(\partial_1)_{y_0}, \dots, B^{(r)}(\partial_1)_{y_0})$$

where

$$y_0 = \lim_{t \rightarrow 0} J_0^r(\otimes^2 T^*)(a_t)(y) = j_0^r \left(\sum_{q=0}^r \frac{1}{q!} a_{(q,0,\dots,0),1,1}(x^1)^q dx^1 \otimes dx^1 \right).$$

We have to explain that

$$B(\partial_1)_{y_0} = h(B^{(0)}(\partial_1)_{y_0}, \dots, B^{(r)}(\partial_1)_{y_0}).$$

From the definition of h it follows that

$$\begin{aligned} h(B^{(0)}(\partial_1)_{y_0}, \dots, B^{(r)}(\partial_1)_{y_0}) &= h(a_{(0,\dots,0),1,1}, \dots, a_{(r,0,\dots,0),1,1}) \\ &= B(\partial_1)_{y_0} \end{aligned}$$

because $B^{(q)}(\partial_1)_{y_0} = a_{(q,0,\dots,0),1,1}$ for $q = 0, \dots, r$.

Theorem 1 is complete. ■

6. Examples of natural operators $T|_{\mathcal{M}f_n} \rightsquigarrow T^* J^r(\otimes^2 T^*)$

A natural operator $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^* J^r(\otimes^2 T^*)$ is a system of operators

$$A : \mathcal{X}(M) \rightarrow \Omega^1(J^r(\otimes^2 T^*)(M))$$

for every n -manifold M satisfying:

- (*Naturality*) If vector fields $X \in \mathcal{X}(M)$ and $Y \in \mathcal{X}(N)$ are φ -connected ($T\varphi \circ X = Y \circ \varphi$) by local diffeomorphism $\varphi : M \rightarrow N$ between n -manifolds, then 1-form $A(X)$ and $A(Y)$ are $J^r(\otimes^2 T^*)(\varphi)$ -connected ($A(X) = A(Y) \circ T J^r(\otimes^2 T^*)(\varphi)$).
- (*Regularity*) If $(X_t)_{t \in \mathbf{R}^k} \in \mathcal{X}(M)$ is a smoothly parametrized family of vector fields then $(A(X_t))_{t \in \mathbf{R}^k}$ is a smooth family of 1-forms.

EXAMPLE. (a) Let $p = 0, \dots, r$. For every vector field $X \in \mathcal{X}(M)$ on n -manifold M we have 1-form $A^{(p)}(X) = d(B^{(p)}(X))$ on $J^r(\otimes^2 T^*)(M)$,

where $B^{(p)}(X) : J^r(\otimes^2 T^*)(M) \rightarrow \mathbf{R}$ is natural function from Section 3. The family of maps $A^{(p)} : T|_{\mathcal{M}f_n} \rightsquigarrow T^* J^r(\otimes^2 T^*)$ satisfying

$$A^{(p)} : \mathcal{X}(M) \rightarrow \Omega^1(J^r(\otimes^2 T^*)(M)),$$

$X \rightarrow A^{(p)}(X)$ is a natural operator.

EXAMPLE. (b) Let $q = 0, \dots, r-1$. For every vector field $X \in \mathcal{X}(M)$ on n -manifold M we have 1-form $A^{[q]}(X) : TJ^r(\otimes^2 T^*)(M) \rightarrow \mathbf{R}$ on $J^r(\otimes^2 T^*)(M)$ such that

$$A^{[q]}(X)(v) = \langle d_x(X^{(q)}\tau(X, X)), T\pi(v) \rangle,$$

$v \in T_u J^r(\otimes^2 T^*)(M)$, $u = j_x^r(\tau) \in (J^r(\otimes^2 T^*)(M))_x$, $x \in M$ where $\pi : J^r(\otimes^2 T^*)(M) \rightarrow M$ is projection of bundle $J^r(\otimes^2 T^*)(M)$. The family of maps $A^{[q]} : T|_{\mathcal{M}f_n} \rightsquigarrow T^* J^r(\otimes^2 T^*)$ such that

$$A^{[q]} : \mathcal{X}(M) \rightarrow \Omega^1(J^r(\otimes^2 T^*)(M)), X \rightarrow A^{[q]}(X),$$

is a natural operator.

EXAMPLE. (c) Let $p = 0, \dots, r$. For every vector field $X \in \mathcal{X}(M)$ on n -manifold M we have 1-form $A^{<p>}(X) : TJ^r(\otimes^2 T^*)(M) \rightarrow \mathbf{R}$ on $J^r(\otimes^2 T^*)(M)$ satisfying

$$A^{<p>}(X)(v) = ((L_X)^{(p)}\tau)_x(X_x, T\pi(v)),$$

$v \in T_u J^r(\otimes^2 T^*)(M)$, $u = j_x^r(\tau) \in (J^r(\otimes^2 T^*)(M))_x$, $x \in M$ where $\pi : J^r(\otimes^2 T^*)(M) \rightarrow M$ is projection of bundle $J^r(\otimes^2 T^*)(M)$ while $(L_X)^{(p)} = L_X \circ \dots \circ L_X$ (p times), L_X means Lie derivative with respect to X . The family of maps $A^{<p>} : T|_{\mathcal{M}f_n} \rightsquigarrow T^* J^r(\otimes^2 T^*)$ such that

$$A^{<p>} : \mathcal{X}(M) \rightarrow \Omega^1(J^r(\otimes^2 T^*)(M)), X \rightarrow A^{<p>}(X),$$

is a natural operator.

EXAMPLE. (d) Let $p = 0, \dots, r$. For every vector field $X \in \mathcal{X}(M)$ on n -manifold M we have 1-form $A^{<<p>>}(X) : TJ^r(\otimes^2 T^*)(M) \rightarrow \mathbf{R}$ on $J^r(\otimes^2 T^*)(M)$ satisfying

$$A^{<<p>>}(X)(v) = ((L_X)^{(p)}\tau)_x(T\pi(v), X_x),$$

$v \in T_u J^r(\otimes^2 T^*)(M)$, $u = j_x^r(\tau) \in (J^r(\otimes^2 T^*)(M))_x$, $x \in M$ where π , L_X , $(L_X)^{(p)}$ is like in (c). The family of maps $A^{<<p>>} : T|_{\mathcal{M}f_n} \rightsquigarrow T^* J^r(\otimes^2 T^*)$ satisfying

$$A^{<<p>>} : \mathcal{X}(M) \rightarrow \Omega^1(J^r(\otimes^2 T^*)(M)), X \rightarrow A^{<<p>>}(X),$$

is a natural operator.

7. Classification of natural operators $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^*J^r(\otimes^2 T^*)$

Let n, r be natural numbers. The set of all natural operators $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^*J^r(\otimes^2 T^*)$ forms a module over an algebra $C^\infty(\mathbf{R}^{r+1})$. Indeed for $h \in C^\infty(\mathbf{R}^{r+1})$ and $A_1, A_2 : T|_{\mathcal{M}f_n} \rightsquigarrow T^*J^r(\otimes^2 T^*)$ it's enough to define

$$(A_1 + A_2)(X) = A_1(X) + A_2(X), \quad X \in \mathcal{X}(M),$$

and

$$(hA_1)(X) = h(B^{(0)}(X), \dots, B^{(r)}(X))A_1(X),$$

where $B^{(0)}(X), \dots, B^{(r)}(X)$ are described in Section 3.

The second main result of the paper is the following theorem

THEOREM 2. *Let $n \geq 2$, $r \geq 1$ be natural numbers. Then $C^\infty(\mathbf{R}^{r+1})$ -module of natural operators $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^*J^r(\otimes^2 T^*)$ is free and $(4r + 3)$ -dimensional. Natural operators $A^{(p)}, A^{[q]}, A^{<p>}$ and $A^{<<p>>}$ for $p=0, \dots, r$, $q = 0, \dots, r-1$ from examples 6.a-6.d form a basis of this module over $C^\infty(\mathbf{R}^{r+1})$.*

8. Proof of Theorem 2

The proof is a modification of respective proof in [5]. First we will show two lemmas.

LEMMA 2. *Let $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^*J^r(\otimes^2 T^*)$ be a natural operator. Suppose $A(\partial_1)|_{(TJ^r(\otimes^2 T^*)(\mathbf{R}^n))_0} = 0$. Then $A = 0$.*

The proof of Lemma 2 is similar to the proof of Lemma 1.

We will study $A(\partial_1)|_{(TJ^r(\otimes^2 T^*)(\mathbf{R}^n))_0}$.

LEMMA 3. *Let $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^*J^r(\otimes^2 T^*)$ be a natural operator. There exist uniquely determined functions $f_0, \dots, f_r \in C^\infty(\mathbf{R}^{r+1})$ such that*

$$(A - \sum_{p=0}^r f_p A^{(p)})(\partial_1)|_{(VJ^r(\otimes^2 T^*)(\mathbf{R}^n))_0} = 0,$$

where $VJ^r(\otimes^2 T^*)(\mathbf{R}^n) \subset TJ^r(\otimes^2 T^*)(\mathbf{R}^n)$ denotes π -vertical subbundle.

Proof. We have an identification $(VJ^r(\otimes^2 T^*)(\mathbf{R}^n))_0 \cong (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0 \times (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0$,

$$\frac{d}{dt}|_{t=0} (u + tw) \cong (u, w), \quad u, w \in (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0.$$

Consider the natural operator $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^*J^r(\otimes^2 T^*)$. Suppose

$$(2) \quad (A - \sum_{p=0}^r f_p A^{(p)})(\partial_1)(j_0^r \tau, j_0^r \tilde{\tau}) = 0$$

for every

$$(j_0^r \tau, j_0^r \tilde{\tau}) \in (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0 \times (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0 \\ \cong (VJ^r(\otimes^2 T^*)(\mathbf{R}^n))_0.$$

In particular it's true for $j_0^r \tau = j_0^r(\tau_a)$ and $j_0^r \tilde{\tau} = j_0^r(\tau_p)$, where

$$j_0^r(\tau_a) := j_0^r \left(\sum_{i=0}^r \frac{1}{i!} a_i (x^1)^i dx^1 \otimes dx^1 \right), \quad a = (a_i) \in \mathbf{R}^{r+1}$$

and

$$j_0^r(\tau_p) := j_0^r \left(\frac{1}{p!} (x^1)^p dx^1 \otimes dx^1 \right), \quad p = 0, \dots, r.$$

But for $j_0^r(\tau_a)$ and $j_0^r(\tau_p)$ occurs

$$(f_{\bar{p}} A^{(\bar{p})})(\partial_1)(j_0^r(\tau_a), j_0^r(\tau_p)) \\ = f_{\bar{p}}(B^{(0)}(\partial_1)_{j_0^r(\tau_a)}, \dots, B^{(r)}(\partial_1)_{j_0^r(\tau_a)}) A^{(\bar{p})}(\partial_1)(j_0^r(\tau_a), j_0^r(\tau_p)) \\ = \begin{cases} f_p(a_0, \dots, a_r) & \text{for } \bar{p} = p \\ 0 & \text{for } \bar{p} \neq p. \end{cases}$$

Therefore from (2) we have

$$(3) \quad f_p(a_0, \dots, a_r) = A(\partial_1)(j_0^r(\tau_a), j_0^r(\tau_p))$$

for $a = (a_0, \dots, a_r) \in \mathbf{R}^{r+1}$, where $j_0^r(\tau_a)$ and $j_0^r(\tau_p)$ are defined above. Hence f_p is unique if it exists.

Define $f_p : \mathbf{R}^{r+1} \rightarrow \mathbf{R}$, $p = 0, \dots, r$, by (3). It remains to show (2) for every

$$(j_0^r \tau, j_0^r \tilde{\tau}) \in (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0 \times (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0 \cong (VJ^r(\otimes^2 T^*)(\mathbf{R}^n))_0.$$

Under the definition of f_p we have (2) for $j_0^r \tau = j_0^r(\tau_a)$ and $j_0^r \tilde{\tau} = j_0^r(\tau_b)$ for every $a = (a_0, \dots, a_r) \in \mathbf{R}^{r+1}$ and $b = (b_0, \dots, b_r) \in \mathbf{R}^{r+1}$.

Now take any $j_0^r \tau, j_0^r \tilde{\tau}$. From the invariance of $A - \sum_{p=0}^r f_p A^{(p)}$ with respect to diffeomorphism $a_t = (x^1, \frac{1}{t}x^2, \dots, \frac{1}{t}x^n)$, $t \neq 0$, we have

$$\left(A - \sum_{p=0}^r f_p A^{(p)} \right) (\partial_1)(j_0^r(\tau), j_0^r(\tilde{\tau})) \\ = \left(A - \sum_{p=0}^r f_p A^{(p)} \right) (\partial_1)(j_0^r((a_t)_* \tau), j_0^r((a_t)_* \tilde{\tau}))$$

for $t \neq 0$. Putting $t \rightarrow 0$ we get

$$\begin{aligned}
& (A - \sum_{p=0}^r f_p A^{(p)})(\partial_1)(j_0^r(\tau), j_0^r(\tilde{\tau})) \\
&= (A - \sum_{p=0}^r f_p A^{(p)})(\partial_1)(j_0^r((x^1, 0, \dots, 0)_* \tau), j_0^r((x^1, 0, \dots, 0)_* \tilde{\tau})) = 0,
\end{aligned}$$

for $j_0^r((x^1, 0, \dots, 0)_* \tau) = j_0^r(\tau_a)$ and $j_0^r((x^1, 0, \dots, 0)_* \tilde{\tau}) = j_0^r(\tau_b)$ for some $a \in \mathbf{R}^{r+1}$ and $b \in \mathbf{R}^{r+1}$.

The proof of the lemma is complete. ■

Let $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^* J^r(\otimes^2 T^*)$ be a natural operator. Replacing A by $A - \sum_{p=0}^r f_p A^{(p)}$, where $f_p \in C^\infty(\mathbf{R}^{r+1})$ are like in Lemma 3, we can assume that

$$(4) \quad A|_{VJ^r(\otimes^2 T^*)(\mathbf{R}^n)_0} = 0.$$

To prove Theorem 2 it remains to show that there exist uniquely determined functions $\tilde{f}_q, g_p, h_p \in C^\infty(\mathbf{R}^{r+1})$, $q = 0, \dots, r-1$, $p = 0, \dots, r$, such that

$$(5) \quad A = \sum_{q=0}^{r-1} \tilde{f}_q A^{[q]} + \sum_{p=0}^r g_p A^{<p>} + \sum_{p=0}^r h_p A^{<<p>>}.$$

LEMMA 4. *Functions $\tilde{f}_q, g_p, h_p$ are uniquely determined.*

Proof. For $a = (a_0, \dots, a_r) \in \mathbf{R}^{r+1}$ and $q = 0, \dots, r-1$ and $p = 0, \dots, r$ define

$$\begin{aligned}
j_0^r(\tau_{a,q}) &= j_0^r \left(\sum_{\iota=0}^r \frac{1}{\iota!} a_\iota (x^1)^\iota dx^1 \otimes dx^1 + \frac{1}{q!} (x^1)^q x^2 dx^1 \otimes dx^1 \right), \\
j_0^r(\mu_{a,p}) &= j_0^r \left(\sum_{\iota=0}^r \frac{1}{\iota!} a_\iota (x^1)^\iota dx^1 \otimes dx^1 + \frac{1}{p!} (x^1)^p dx^1 \otimes dx^2 \right), \\
j_0^r(\rho_{a,p}) &= j_0^r \left(\sum_{\iota=0}^r \frac{1}{\iota!} a_\iota (x^1)^\iota dx^1 \otimes dx^1 + \frac{1}{p!} (x^1)^p dx^2 \otimes dx^1 \right).
\end{aligned}$$

If $\tilde{f}_q, g_p, h_p \in C^\infty(\mathbf{R}^{r+1})$ satisfy (2) then in particular

$$(6) \quad A(\partial_1)(\partial_{2|j_0^r(\tau_{a,q})}^C) = \tilde{f}_q(a_0, \dots, a_r),$$

and similarly

$$(7) \quad A(\partial_1)(\partial_{2|j_0^r(\mu_{a,p})}^C) = g_p(a_0, \dots, a_r),$$

$$(8) \quad A(\partial_1)(\partial_{2|j_0^r(\rho_{a,p})}^C) = h_p(a_0, \dots, a_r),$$

where ∂_2^C denotes the complete lifting of vector field $\partial_2 = \partial/\partial x^2$ to $J^r(\otimes^2 T^*)(\mathbf{R}^n)$. We have

$$B^{(\bar{p})}(\partial_1)_{j_0^r(\tau_{a,q})} = a_{\bar{p}},$$

$$B^{(\bar{p})}(\partial_1)_{j_0^r(\mu_{a,p})} = a_{\bar{p}},$$

$$B^{(\bar{p})}(\partial_1)_{j_0^r(\rho_{a,p})} = a_{\bar{p}}$$

for $\bar{p} = 0, \dots, r$. Similarly

$$A^{[\bar{q}]}(\partial_1)(\partial_2^C|_{j_0^r(\tau_{a,q})}) = \delta_{\bar{q}}^q,$$

$$A^{[\bar{q}]}(\partial_1)(\partial_2^C|_{j_0^r(\mu_{a,p})}) = 0,$$

$$A^{[\bar{q}]}(\partial_1)(\partial_2^C|_{j_0^r(\rho_{a,p})}) = 0$$

for $\bar{q} = 0, \dots, r-1$; and

$$A^{<\bar{p}>}(\partial_1)(\partial_2^C|_{j_0^r(\tau_{a,q})}) = 0,$$

$$A^{<\bar{p}>}(\partial_1)(\partial_2^C|_{j_0^r(\mu_{a,p})}) = \delta_{\bar{p}}^p,$$

$$A^{<\bar{p}>}(\partial_1)(\partial_2^C|_{j_0^r(\rho_{a,p})}) = 0$$

for $\bar{p} = 0, \dots, r$; and

$$A^{<<\bar{p}>>}(\partial_1)(\partial_2^C|_{j_0^r(\tau_{a,q})}) = 0,$$

$$A^{<<\bar{p}>>}(\partial_1)(\partial_2^C|_{j_0^r(\mu_{a,p})}) = 0,$$

$$A^{<<\bar{p}>>}(\partial_1)(\partial_2^C|_{j_0^r(\rho_{a,p})}) = \delta_{\bar{p}}^p$$

for $\bar{p} = 0, \dots, r$.

Hence if $\tilde{f}_q, g_p, h_p \in C^\infty(\mathbf{R}^{r+1})$ meets (5) then they are uniquely defined by:

$$\begin{aligned} \tilde{f}_q(a_0, \dots, a_r) &= A(\partial_1)(\partial_2^C|_{j_0^r(\tau_{a,q})}), \\ g_p(a_0, \dots, a_r) &= A(\partial_1)(\partial_2^C|_{j_0^r(\mu_{a,p})}), \\ h_p(a_0, \dots, a_r) &= A(\partial_1)(\partial_2^C|_{j_0^r(\rho_{a,p})}) \end{aligned} \quad (9)$$

for $a = (a_0, \dots, a_r) \in \mathbf{R}^{r+1}$ where $j_0^r(\tau_{a,q}), j_0^r(\mu_{a,p}), j_0^r(\rho_{a,p})$ are defined in the beginning of the proof. The proof of Lemma 4 is complete. ■

9. Existence of functions $\tilde{f}_q, g_p, h_p$

LEMMA 5. Define $\tilde{f}_q, g_p, h_p \in C^\infty(\mathbf{R}^{r+1})$ by (9). Then (5) holds.

Proof. By assumption 4 and Lemma 2 it's enough to check that

$$A(\partial_1)(\partial_{j_0^r \tau}^C) = \tilde{A}(\partial_1)(\partial_{j_0^r \tau}^C)$$

for any constant vector field ∂ on $\mathbf{R}^n$ such that ∂_1 and ∂ are linear independent, ($n \geq 2$), and for any $j_0^r(\tau) \in (J^r(\otimes^2 T^*)(\mathbf{R}^n))_0$, where ∂^C is complete lift of vector field ∂ to $J^r(\otimes^2 T^*)(\mathbf{R}^n)$ and

$$\tilde{A} := \sum_{q=0}^{r-1} \tilde{f}_q A^{[q]} + \sum_{p=0}^r g_p A^{<p>} + \sum_{p=0}^r h_p A^{<<p>>}.$$

Using the invariance of A and $\tilde{A}$ with respect to linear isomorphism $\mathbf{R}^n \rightarrow \mathbf{R}^n$ transforming ∂_1 into ∂_1 and ∂ into ∂_2 we can assume that $\partial = \partial_2$.

So we have to show

$$(10) \quad A(\partial_1)(\partial_{2|j_0^r(\tau)}^C) = \tilde{A}(\partial_1)(\partial_{2|j_0^r(\tau)}^C).$$

Define $\Phi : \mathbf{R}^{K_{r,n}} \rightarrow \mathbf{R}$ by

$$\Phi(a_{\alpha,i,j}) = A(\partial_1)(\partial_{2|j_0^r(\sum a_{\alpha,i,j} x^\alpha dx^i \otimes dx^j)}^C)$$

and similarly $\tilde{\Phi} : \mathbf{R}^{K_{r,n}} \rightarrow \mathbf{R}$

$$\tilde{\Phi}(a_{\alpha,i,j}) = \tilde{A}(\partial_1)(\partial_{2|j_0^r(\sum a_{\alpha,i,j} x^\alpha dx^i \otimes dx^j)}^C)$$

where

$$K_{r,n} = \{(\alpha, i, j) \in (\mathbf{N} \cup \{0\})^n \times \mathbf{N} \times \mathbf{N} \mid \alpha = (\alpha_1, \dots, \alpha_n), \\ 0 \leq |\alpha| = \alpha_1 + \dots + \alpha_n \leq r, i, j = 1, \dots, n\}.$$

The invariance of A and $\tilde{A}$ with respect to

$$(t_1 x^1, t_2 x^2, t_3 x^3, \dots, t_n x^n), t_2, \dots, t_n \neq 0, t_1 = 1$$

gives the homogeneous conditions

$$\Phi(t_2^{\alpha_2} \dots t_n^{\alpha_n} t_i t_j a_{\alpha,i,j}) = t_2 \Phi(a_{\alpha,i,j})$$

and

$$\tilde{\Phi}(t_2^{\alpha_2} \dots t_n^{\alpha_n} t_i t_j a_{\alpha,i,j}) = t_2 \tilde{\Phi}(a_{\alpha,i,j}).$$

Then from the homogeneous function theorem, it follows that $\Phi(a_{\alpha,i,j})$ is the linear combination of variables $a_{(q,1,0,\dots,0),1,1}$, $a_{(p,0,\dots,0),1,2}$ and $a_{(p,0,\dots,0),2,1}$, $q = 0, \dots, r-1$, $p = 0, \dots, r$. Similarly for $\tilde{\Phi}(a_{\alpha,i,j})$.

Hence it's enough to check (10) only for

$$j_0^r(\tau) = j_0^r(\tau_{a,q}), j_0^r(\mu_{a,p}), j_0^r(\rho_{a,p}),$$

where $a = (a_0, \dots, a_r) \in \mathbf{R}^{r+1}$, $q = 0, \dots, r-1$, $p = 0, \dots, r$.

It's clear because of definitions of functions $\tilde{f}_q(a)$, $g_p(a)$, $h_p(a)$ and equalities (6), (7), (8).

The proof of Lemma 5 is complete. ■

The proof of Theorem 2 is complete. ■

10. Corollaries from Theorem 2

We say that a natural operator $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^*J^r(\otimes^2 T^*)$ is linear if $A : \mathcal{X}(M) \rightarrow \Omega^1(J^r(\otimes^2 T^*)(M))$ is $\mathbf{R}$ -linear for every n -manifold M . It's clear that operators $A^{<0>}$ and $A^{<<0>>}$ from Examples (c) and (d) for $p = 0$ are linear.

From Theorem 2 we have

COROLLARY 2. *Let $n \geq 2$, $r \geq 1$ be natural numbers. Let $A : T|_{\mathcal{M}f_n} \rightsquigarrow T^*J^r(\otimes^2 T^*)$ be a linear natural operator. Then there exist real numbers $a, b \in \mathbf{R}$ such that*

$$A = aA^{<0>} + bA^{<<0>>},$$

in particular a, b are uniquely determined.

A natural 1-form on $J^r(\otimes^2 T^*)$ is a family of 1-forms

$$\omega \in \Omega^1(J^r(\otimes^2 T^*)(M))$$

for every n -manifold M satisfying

$$\omega = \omega \circ TJ^r(\otimes^2 T^*)(\varphi)$$

for every linear diffeomorphism $\varphi : M \rightarrow N$ between n -manifolds.

COROLLARY 3. *Let $n \geq 2$, $r \geq 1$. Every natural 1-form on $J^r(\otimes^2 T^*)$ is equal to 0.*

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