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BASES FOR THE k-NORMALIZATIONS OF VARIETIES OF BANDS

Abstract. The usual depth measurement on terms of a fixed type type τ assigns to each term a non-negative integer called its depth. For $k \geq 1$, an identity $s \approx t$ of type τ is said to be k-normal (with respect to the depth measurement) if either $s = t$ or both s and t have depth $\geq k$. Taking $k = 1$ gives the well-known property of normality of identities. A variety is called k-normal (with respect to the depth measurement) if all its identities are k-normal. For any variety V, there is a least k-normal variety $N_k(V)$ containing V, the variety determined by the set of all k-normal identities of V. In this paper we produce for every subvariety V of the variety B of bands (idempotent semigroups) a finite equational basis for $N_k(V)$, for $k \geq 1$.

1. Introduction

Let $\tau = (n_i)_{i \in I}$ be any type of algebras, with an operation symbol f_i of arity n_i for each $i \in I$. Let $X = \{x_1, x_2, x_3, \ldots\}$ be a set of variable symbols, and let $W_\tau(X)$ be the set of all terms of type τ formed using variables from X. We will use the well-known Galois connection $Id - Mod$ between classes of algebras and sets of identities. For any class K of algebras of type τ and any set Σ of identities of type τ, we have

$$Mod\,\Sigma = \{\text{algebras } \mathcal{A} \text{ of type } \tau \mid \mathcal{A} \text{ satisfies all identities in } \Sigma\},$$

and

$$Id\,K = \{\text{identities } s \approx t \text{ of type } \tau \mid \text{ all algebras in } K \text{ satisfy } s \approx t\}.$$

For each $t \in W_\tau(X)$, we denote by $v(t)$ the depth of t, a parameter which is defined inductively by

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(i) $v(t) = 0$, if t is a variable;

(ii) $v(t) = 1 + max\{v(t_j) : 1 \leq j \leq n_i\}$, if t is a composite term $t = f_i(t_1, \ldots, t_{n_i})$, for some $i \in I$ and some terms $t_1, \ldots, t_{n_i}$.

When a term is portrayed by a tree diagram, with the nodes corresponding to operation symbols in the term and the leaves to variable symbols, the depth of the term t corresponds to the length of the longest path from root to leaves in the tree diagram for t. The depth parameter defines a valuation function v on the set of all terms of type τ (see [3]).

Let $k \geq 0$ be any natural number. An identity $s \approx t$ of type τ is called *k-normal* (with respect to the depth valuation) if either $s = t$ or $v(t)$, $v(s) \geq k$. We denote by $N_k(\tau)$ the set of all k-normal identities of type τ. It was proved in [3] that k-normality with respect to the depth of terms is a hereditary property of identities, meaning that the set $N_k(\tau)$ is closed under the usual five rules of deduction for identities. This is equivalent to the fact that $N_k(\tau)$ is an equational theory. For a variety V of type τ, we consider the set $Id_k\,V = N_k(\tau) \cap Id\,V$ of all k-normal identities satisfied by V. This is an equational theory, and the variety it determines, $N_k(V) = Mod\,Id_k\,V$, is called the *k-normalization of V*. It can happen that $N_k(V) = V$, when every identity satisfied by V is k-normal, and in this case we say that V is a *k-normal variety.* Otherwise V is a proper subvariety of $N_k(V)$, and $N_k(V)$ is the least k-normal variety to contain V. The case $k = 1$ gives the usual definitions of a normal identity or variety and the normalization of V; see for instance [7].

The variety $N_k(V)$ is defined equationally, by means of the k-normal identities of V. An algebraic characterization of the algebras in $N_k(V)$ was given by Denecke and Wismath in [2], using the concept of a k-choice algebra. They showed that any algebra in $N_k(V)$ is a homomorphic image of a k-choice algebra constructed from an algebra in V.

In this paper we return to the equational approach, to examine the question of when the variety $N_k(V)$ will have a finite basis. We consider the subvarieties of the type (2) variety B of bands (idempotent semigroups), and produce for each such subvariety V a finite equational basis for $N_k(V)$, for $k \geq 1$. Section 2 provides the necessary background: a description of the countably infinite collection of varieties of bands, and a summary of the finite basis (from [8]) for the variety $N_k(Sem)$, for $k \geq 1$, where Sem is the variety of all semigroups. We will refer to the identities in this basis as the *k-normal associative identities.* In Section 3, we add two additional *k-normal idempotent identities* to this basis, to produce a basis for $N_k(B)$. Finally in Section 4 we show that for any proper subvariety V of B, there is a finite equational basis for $N_k(V)$ consisting of the basis for $N_k(B)$ plus one additional k-normal identity (determined by V).

2. Background information

In this section we present some background information about the variety *Sem* of all semigroups and the variety B of bands and its subvarieties. All of these varieties are of type (2), and we shall follow the convention of using juxtaposition instead of the binary operation symbol f. Thus $Sem = Mod\{x(yz) \approx (xy)z\}$, the model class determined by the associative law, and $B = Mod\{x(yz) \approx (xy)z, x^2 \approx x\}$, the variety of idempotent semigroups or bands.

It is known that B has a countably infinite number of subvarieties, and the lattice they form has been completely described by Birjukov ([1]), Fennemore ([4]), Gerhard ([5]) and Gerhard and Petrich ([6]). For our purposes, the most significant feature of these varieties is that each proper subvariety of B is equationally defined by associativity, idempotence, and one additional identity.

We now describe the equational bases for the varieties $N_k(Sem)$, for $k \geq 1$, as produced in [8]. First, we observe that the associativity identity is actually k-normal for $k = 1, 2$. Thus for these k-values we have $N_k(Sem) = Sem$, and the set consisting of the associative identity is a basis. So we need a basis only for the case that $k \geq 3$.

DEFINITION 2.1. Let $k \geq 1$. A type (2) term will be called a *skeleton term (of depth k)* if it contains exactly k occurrences of f and exactly one occurrence of each of the variables $x_1, \ldots, x_{k+1}$, in that order from left to right in the term, and no other variables.

For $k = 1$, there is only one skeleton term, the term $f(x_1, x_2)$. The skeleton terms for $k = 3$ are given in Figure 1. In general, there are 2^{k-1} skeleton terms for $k \geq 2$.

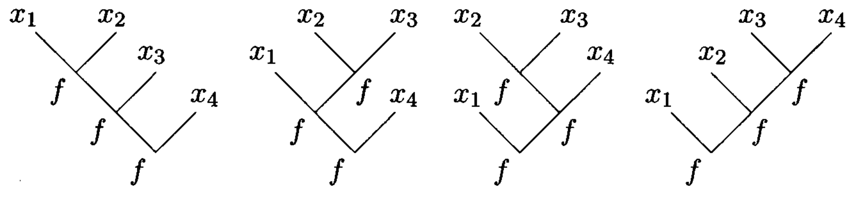

Figure 1

DEFINITION 2.2. A staircase term is any term of the form

$$f(f(\ldots(f(x_{i_1}, x_{i_2}), x_{i_3}), \ldots), x_{i_k})$$

for some variables $x_{i_1}, \ldots, x_{i_p}$. For any $k \geq 1$, we shall refer to the term $stair = stair(x_1, \ldots, x_{k+1}) = f(f(\ldots(f(x_1, x_2), x_3), \ldots), x_{k+1})$, shown in

Figure 2 below, as the *staircase term* of depth k. This is one of the skeleton terms of depth k.

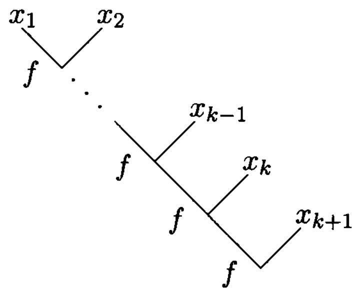

Figure 2

THEOREM 2.3 ([8]). *Let* $k \geq 3$. *The set*

$$\Sigma_{N_k(Sem)} = \{stair \approx w \mid w \text{ any depth } k \text{ non-stair skeleton term}\}$$

forms a finite equational basis for the variety $N_k(Sem)$.

We shall refer to the identities in the basis $\Sigma_{N_k(Sem)}$ as the *k-normal associativity* identities.

3. An equational basis for $N_k(B)$

The variety B of all bands is equationally defined by the two identities of associativity and idempotence. As noted above, associativity is k-normal for $k = 1, 2$ but not k-normal for $k \geq 3$. The idempotent identity $x \approx f(x, x)$ is not k-normal for any $k \geq 1$. That means that $N_k(B)$ no longer satisfies idempotence for $k \geq 1$, although it does still satisfy any k-normal consequences of idempotence. We shall consider first the case that $k \geq 3$, leaving till later the simpler cases of $k = 1, 2$.

THEOREM 3.1. *Let* $k \geq 3$. *A finite equational basis for the variety* $N_k(B)$ *is given by the set* $\Sigma_{N_k(B)}$ *of the following identities*:

k-Normal Associativity

(1) $stair(x_1, \ldots, x_{k+1}) \approx w(x_1, \ldots, x_{k+1})$

for all non-staircase skeleton terms w.

k-Normal Idempotence

(2a) $(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_1)x_2)x_3)\cdots)x_{k+1}$

(2b) $(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_2)x_3)\cdots)x_{k+1})x_{k+1}$.

Note that all of the identities in $\Sigma_{N_k(B)}$ are k-normal consequences of associativity or idempotence, and hence hold in $N_k(B)$. To prove Theorem 3.1, we will show that we can produce a deduction of any k-normal band identity $s \approx t$ from the set $\Sigma_{N_k(B)}$ using the five rules of deduction. Our proof will use the $(k+1)$-variable staircase term $stair = stair(x_1, x_2, \ldots, x_{k+1})$ from Section 2. Our strategy will be to deduce the three identities $s \approx stair(s, \ldots, s)$, $stair(s, \ldots, s) \approx stair(t, \ldots, t)$, and $stair(t, \ldots, t) \approx t$ from $\Sigma_{N_k(B)}$, allowing us to conclude $s \approx t$. We start by deducing four identities from $\Sigma_{N_k(B)}$ that we will use in subsequent proofs.

LEMMA 3.2. *The following identities are consequences of the identities in* $\Sigma_{N_k(B)}$:

(1*) $(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_1)x_1)\cdots)x_1)x_2)x_3)\cdots)x_{k+1}$,
with k or $k+1$ occurrences of x_1 on the right side of the equation.

(2*) $(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_2)x_3)\cdots)x_{k+1})\cdots)x_{k+1}$,
with k or $k+1$ occurrences of x_{k+1} on the right side of the equation.

(3*) $(\cdots(x_1x_2)x_3)\cdots)x_{k+2} \approx (\cdots(x_1x_2)x_3)\cdots)x_k)(x_{k+1}x_{k+2})$.

(4*) $(\cdots(x_1x_2)x_3)\cdots)x_k)(x_{k+1}x_{k+2})$
$\approx (\cdots(x_1x_2)x_3)\cdots)x_k)(x_{k+1}x_{k+2}))x_{k+2}$.

Proof. To deduce (1*), we first apply the substitution deduction rule on identity $(2a)$ from our basis, to replace variable x_j by x_{j-1} for $2 \le j \le k+1$. This gives us $(\cdots(x_1x_1)x_2)\cdots)x_k \approx (\cdots(x_1x_1)x_1)x_2)\cdots)x_k$. Then we use the compatibility deduction rule on this identity and $x_{k+1} \approx x_{k+1}$ to obtain $(\cdots(x_1x_1)x_2)\cdots)x_{k+1} \approx (\cdots(x_1x_1)x_1)x_2)\cdots)x_{k+1}$. By transitivity then we get $(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_1)x_1)x_2)\cdots)x_{k+1}$. Next we use the substitution rule on identity $(2a)$ again, this time replacing x_2 with x_1 and x_j with x_{j-2} for $3 \le j \le k+1$. This gives $(\cdots(x_1x_1)x_1)x_2)\cdots)x_{k-1} \approx (\cdots(x_1x_1)x_1)x_1)x_2)\cdots)x_{k-1}$. Multiplying this identity on the right by x_k and then again on the right by x_{k+1} (by the compatibility deduction rule) gives us $(\cdots(x_1x_1)x_1)x_2)\cdots)x_{k+1} \approx (\cdots(x_1x_1)x_1)x_1)x_2)\cdots)x_{k+1}$. We apply transitivity again to obtain

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_1)x_1)x_1)x_2)\cdots)x_{k+1}.$$

With repeated applications of substitution, compatibility and transitivity we obtain (1*).

The deduction of (2*) is similar. We first use compatibility on identity $(2b)$ and $x_{k+1} \approx x_{k+1}$ to obtain

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1})x_{k+1} \approx (\cdots(x_1x_2)x_3)\cdots)x_{k+1})x_{k+1})x_{k+1}.$$

Transitivity then gives

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_2)x_3)\cdots)x_{k+1})x_{k+1})x_{k+1}.$$

Then we alternate using compatibility on $x_{k+1} \approx x_{k+1}$ and the previous step in our deduction with transitivity on (2*b*) and the previous step in the deduction to obtain (2*).

For (3*), we start with the identity

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1(x_2x_3))x_4)\cdots)x_{k+1},$$

which is one of the k-normal associativity consequences from the set $\Sigma_{N_k(B)}$. Using the compatibility rule we can multiply both sides of the identity on the right by x_{k+2} to get

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1})x_{k+2} \approx (\cdots(x_1(x_2x_3))x_4)\cdots)x_{k+1})x_{k+2}.$$

Next we apply the substitution rule to the $\Sigma_{N_k(B)}$ identity

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx x_1[(\cdots(x_2x_3)x_4)\cdots)x_{k+1}]$$

to replace the variable x_j with x_{j+1}, for $3 \leq j \leq k+1$, and the variable x_2 with x_2x_3; this results in

$$(\cdots(x_1(x_2x_3))x_4)\cdots)x_{k+2} \approx x_1[(\cdots(x_2x_3)x_4)\cdots)x_{k+2}].$$

From transitivity we obtain the identity

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1})x_{k+2} \approx x_1[(\cdots(x_2x_3)x_4)\cdots)x_{k+2}],$$

which we shall denote by (1^+).

Now we replace x_j with x_{j+1}, for $1 \leq j \leq k+1$, in the $\Sigma_{N_k(B)}$ identity $(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx x_1(x_2(\cdots(x_{k-1}(x_kx_{k+1})\cdots)$. Applying the compatibility rule on the resulting identity and the identity $x_1 \approx x_1$ gives $x_1[(\cdots(x_2x_3)x_4)\cdots)x_{k+2}] \approx x_1(x_2(\cdots(x_k(x_{k+1}x_{k+2})\cdots)$, which we denote by (2^+).

Finally, we use $x_1(x_2(\cdots(x_{k-1}(x_kx_{k+1})\cdots) \approx (\cdots(x_1x_2)x_3)\cdots)x_{k+1}$ and replace x_{k+1} with $x_{k+1}x_{k+2}$. This gives the following identity (3^+):

$$x_1(x_2(\cdots(x_{k-1}(x_k(x_{k+1}x_{k+2})\cdots) \approx (\cdots(x_1x_2)x_3)\cdots)(x_{k+1}x_{k+2}).$$

Our identity (3*) then follows from (1^+), (2^+), and (3^+).

To deduce (4*), we apply the substitution rule on identity (2*b*), replacing x_j with x_{j+1} for $2 \leq j \leq k+1$, and x_1 with x_1x_2. This leads to

$$(1^\diamond) \qquad (\cdots(x_1x_2)x_3)\cdots)x_{k+2} \approx (\cdots(x_1x_2)x_3)\cdots)x_{k+2})x_{k+2}.$$

We then apply the compatibility rule on identity (3*) from above and $x_{k+2} \approx x_{k+2}$ to obtain

$$(2^\diamond) \qquad (\cdots(x_1x_2)x_3)\cdots)x_{k+1})x_{k+2})x_{k+2} \approx (\cdots(x_1x_2)x_3)\cdots)(x_{k+1}x_{k+2}))x_{k+2}.$$

From $(\cdots(x_1x_2)x_3)\cdots)x_k)(x_{k+1}x_{k+2}) \approx (\cdots(x_1x_2)x_3)\cdots)x_{k+2}$, (1°) and (2°) we obtain (4*). ∎

LEMMA 3.3. *For any term u of depth $\geq k$, the identity $u \approx stair(u,\ldots,u)$ can be deduced from $\Sigma_{N_k(B)}$.*

Proof. We first note that it is straightforward to deduce from $\Sigma_{N_k(B)}$ the identity

$$stair(x_1,\ldots,x_{k+1}) \approx stair(stair(x_1,\ldots,x_{k+1}),\ldots,stair(x_1,\ldots,x_{k+1})),$$

using the identities (2*), (1*) and (3*) from the previous proof and identity (2*b*).

Now let u be any term of depth $\geq k$. The depth restriction means that we can write $u = w(u_1,\ldots,u_{k+1})$, for some skeleton term w and some terms $u_1,\ldots,u_{k+1}$. Thus we have

$u = w(u_1,\ldots,u_{k+1})$
$\approx stair(u_1,\ldots,u_{k+1})$ from the k-associative identities
$\approx stair(stair(u_1,\ldots,u_{k+1}),\ldots,stair(u_1,\ldots,u_{k+1}))$ from above
$\approx stair(u,\ldots,u)$ since $w \approx stair$.

Therefore, $u \approx stair(u,\ldots,u)$ can be deduced from the available identities. ∎

We have shown so far that for any k-normal band identity $s \approx t$, we can deduce $s \approx stair(s,\ldots,s)$ and $stair(t,\ldots,t) \approx t$ from $\Sigma_{N_k(B)}$. All that remains for our proof is to show that $stair(s,\ldots,s) \approx stair(t,\ldots,t)$ can also be deduced from $\Sigma_{N_k(B)}$.

Since $s \approx t$ holds in B there is a deduction of it, using the five rules of deduction, from the usual basis $\Sigma_B = \{x(yz) \approx (xy)z, x^2 \approx x\}$ for the variety of bands. We shall refer to this deduction as the *given deduction*. Now, we produce a new list of identities called the *derived list* by replacing each step $u_j \approx w_j$ in the given deduction by $stair(u_j,\ldots,u_j) \approx stair(w_j,\ldots,w_j)$. We want to show that the derived list is a deduction of $stair(s,\ldots,s) \approx stair(t,\ldots,t)$ from $\Sigma_{N_k(B)}$ and some of its consequences. In particular, we want to be able to use in addition to $\Sigma_{N_k(B)}$ the identities in $stair(\Sigma_B) = \{stair(p,\ldots,p) \approx stair(q,\ldots,q) \mid p \approx q \in \Sigma_B\}$.

LEMMA 3.4. *The identities in the set $stair(\Sigma_B)$ can be deduced from $\Sigma_{N_k(B)}$.*

Proof. We need to consider the "staired" version of the two basis identities associativity and idempotence. First, $stair(x_1(x_2x_3),\ldots,x_1(x_2x_3)) \approx stair((x_1x_2)x_3,\ldots,(x_1x_2)x_3)$ can be deduced from $\Sigma_{N_k(B)}$ in a straightforward deduction using the identities

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1(x_2x_3))x_4)x_5)\cdots)x_{k+1},$$

(1*) and (2*),

$$x_1(x_2(\cdots(x_{k_2}((x_{k_1}x_k)x_{k+1}))\cdots)\approx x_1(x_2(\cdots(x_{k_1}(x_kx_{k+1})\cdots)$$

and

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1}\approx x_1(x_2(\cdots(x_{k_1}(x_kx_{k+1})\cdots).$$

The deduction of $stair(x_1x_1,\ldots,x_1x_1)\approx stair(x_1,\ldots,x_1)$ for idempotence is similarly straightforward, using the identities (2*a*), (1*), (2*) and (4*). ■

Now it suffices to show that the identity $stair(s,\ldots,s)\approx stair(t,\ldots,t)$ can be deduced from $\Sigma_{N_k(B)}\bigcup stair(\Sigma_B)$. To prove this, we will show that each step j in the derived list can be justified, by the same justification used for step j in the given deduction. We will use the following two lemmas to handle two of the cases.

LEMMA 3.5. *For any terms u, w, p and q, the identity*

$$stair((uw),\ldots,(uw))\approx stair((pq),\ldots,(pq))$$

can be deduced from the identity

$$stair(u,\ldots,u)stair(w,\ldots,w)\approx stair(p,\ldots,p)stair(q,\ldots,q)$$

and the identities in $\Sigma_{N_k(B)}$.

Proof. First we note that the two identities

$$stair((uw),\ldots,(uw))\approx stair(u,\ldots,u)stair(w,\ldots,w)$$

and

$$stair(p,\ldots,p)stair(q,\ldots,q)\approx stair((pq),\ldots,(pq))$$

can be deduced from the identities

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1}\approx x_1[(\cdots(x_2x_3)x_4)\cdots)x_{k+1}],$$

(1*),

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1}\approx(\cdots(x_1(x_2x_3))x_4)x_5)\cdots)x_{k+1},$$

(2*) and (4*). Then we use

$$stair((uw),\ldots,(uw))\approx stair(u,\ldots,u)stair(w,\ldots,w)$$

and

$$stair(u,\ldots,u)stair(w,\ldots,w)\approx stair(p,\ldots,p)stair(q,\ldots,q)$$

and

$$stair(p,\ldots,p),stair(q,\ldots,q)\approx stair((pq),\ldots,(pq))$$

to deduce

$$stair((uw),\ldots,(uw))\approx stair((pq),\ldots,(pq)).\ ■$$

We will denote by $Subs(u,x,w)$ the term obtained by replacing every occurrence of the variable x by the term u in the term w.

LEMMA 3.6. *For any terms u and w and any variable x, we have*

$$stair(Subs(u,x,w),\dots,stair(Subs(u,x,w)) = Subs(u,x,stair(w,\dots,w)).$$

Proof. Let u, w be any terms and let x be any variable. We have

$stair(Subs(u,x,w),\dots,Subs(u,x,w))$
$= [\cdots[Subs(u,x,w)Subs(u,x,w)]Subs(u,x,w)]\cdots Subs(u,x,w)$
$= Subs(u,x,[\cdots[ww]w]\cdots w)$
$= Subs(u,x,stair(w,\dots,w))$. ■

Now we are ready to prove that the derived list is indeed a deduction of our identity $stair(s,\dots,s) \approx stair(t,\dots,t)$ from the identities in $\Sigma_{N_k(B)} \bigcup stair(\Sigma_B)$, which will complete the proof of Theorem 3.1.

LEMMA 3.7. *The derived list is a deduction of* $stair(s,\dots,s)\approx stair(t,\dots,t)$ *from* $\Sigma_{N_k(B)} \bigcup stair(\Sigma_B)$.

Proof. We need to verify that the justification, the rule of deduction used on previous steps, is the same for each step j in the derived list as the justification for step j in the given deduction. Consider the identity $u_j \approx w_j$ at any step j in the given deduction. If step j was an instance of an identity from Σ_B, then step j in the derived list is an instance of the corresponding identity from $stair(\Sigma_B)$. If step j was an instance of the reflexive, symmetric, or transitive rules of deduction, then clearly step j in the derived list is an instance of the same rule.

If step j in the given deduction was an instance of the compatibility rule on two previous steps c and d, then step j involved deducing $u_c u_d \approx w_c w_d$ from $u_c \approx w_c$ and $u_d \approx w_d$. According to our construction of the derived list, step j in the derived list is $stair(u_c u_d,\dots,u_c u_d) \approx stair(w_c w_d,\dots,w_c w_d)$.

This is not what we obtain from the application of the compatibility rule to steps c and d in the derived list. Instead, we obtain the identity $stair(u_c,\dots,u_c)stair(u_d,\dots,u_d) \approx stair(w_c,\dots,w_c)stair(w_d,\dots,w_d)$. But by Lemma 3.5 we can produce a deduction of

$$stair(u_c u_d,\dots,u_c u_d) \approx stair(w_c w_d,\dots,w_c w_d)$$

from $\Sigma_{N_k(B)}$ and the identity

$$stair(u_c,\dots,u_c)stair(u_d,\dots,u_d) \approx stair(w_c,\dots,w_c)stair(w_d,\dots,w_d).$$

If step j in the given deduction was an instance of the substitution rule on a previous step e, then step j in the given deduction was $Subs(z,x,u_e) \approx Subs(z,x,w_e)$ and so step j in the derived list is

$$stair(Subs(z,x,u_e),\dots,Subs(z,x,u_e)) \approx stair(Subs(z,x,w_e),\dots,Subs(z,x,w_e)).$$

When we apply the substitution rule to step e in the derived list, we obtain

$$Subs(z, x, stair(u_e, \ldots, u_e)) \approx Subs(z, x, stair(w_e, \ldots, w_e)).$$

By Lemma 3.6,

$$stair(Subs(z, x, u_e), \ldots, Subs(z, x, u_e)) = Subs(z, x, stair(u_e, \ldots, u_e))$$

and

$$stair(Subs(z, x, w_e), \ldots, Subs(z, x, w_e)) = Subs(z, x, stair(w_e, \ldots, w_e));$$

hence step j in the derived list is an instance of the substitution rule applied to step e in the derived list. ∎

We conclude this section with a look at the special cases $k = 1, 2$. As noted above, the associative law still holds in $N_k(B)$ for these values of k, and so we have a simplified version of the basis identities from Theorem 1. In particular, we can use associativity to omit brackets from terms in the identities. For $k = 1, 2$, our basis $\Sigma_{N_k(B)}$ consists of:

(1) Associativity $\quad x_1(x_2x_3) \approx (x_1x_2)x_3.$

(2) k-Normal Idempotence $\quad x_1x_2x_3 \cdots x_{k+1} \approx x_1x_1x_2x_3 \cdots x_{k+1}$

$\quad x_1x_2x_3 \cdots x_{k+1} \approx x_1x_2x_3 \cdots x_{k+1}x_{k+1}.$

The proof that $\Sigma_{N_k(B)}$ forms a finite equational basis for $N_k(B)$ for $k = 1, 2$ is similar to the proof for $k \geq 3$, but much simpler because of the presence of associativity.

4. Equational bases for subvarieties of bands

Every subvariety V of the variety of all bands can be equationally defined by the associative and idempotent identities and one additional identity, which we will call the *defining identity* (with respect to B) of V. We shall show that for each such subvariety V, the identities of k-normal associativity and k-normal idempotence, plus one specific k-normal consequence of the defining identity of V form a finite equational basis for $N_k(V)$. We consider first the case that $k \geq 3$.

THEOREM 4.1. *Let* $V = Mod\{x(yz) \approx (xy)z, x^2 \approx x,\ p \approx q\}$ *be a variety of bands, with defining identity* $p \approx q$. *The set* $\Sigma_{N_k(V)}$ *of identities listed below forms a finite equational basis for* $N_k(V)$, *for* $k \geq 3$.

(1) k-Normal Associativity $\quad stair(x_1, \ldots, x_{k+1}) \approx w(x_1, \ldots, x_{k+1})$ *for all skeleton terms w other than the staircase term.*

(2) k-Normal Idempotence

$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_1)x_2)x_3)\cdots)x_{k+1}$$
$$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_2)x_3)\cdots)x_{k+1})x_{k+1}.$$

(3) k-normal Defining Identity $\quad stair(p,\ldots,p) \approx stair(q,\ldots,q)$.

Proof. We will use the same technique as in Section 3 to prove Theorem 4.1. Let $\Sigma_V = \{x(yz) \approx (xy)z, x^2 \approx x, p \approx q\}$ be a basis for V. The identities of $\Sigma_{N_k(V)}$ are all k-normal consequences of the identities in Σ_V. Let $s \approx t$ be any identity that holds in V such that s, t have depth $\geq k$. We want to show that $s \approx stair(s,\ldots,s)$, $stair(s,\ldots,s) \approx stair(t,\ldots,t)$, and $stair(t,\ldots,t) \approx t$ can each be deduced from $\Sigma_{N_k(V)}$, using the five rules of deduction.

First, by Lemma 3.3 the identities $s \approx stair(s,\ldots,s)$ and $stair(t,\ldots,t) \approx t$ can be deduced from $\Sigma_{N_k(B)}$; and since $\Sigma_{N_k(B)} \subseteq \Sigma_{N_k(V)}$, these two identities can certainly also be deduced from $\Sigma_{N_k(V)}$. As before, we need some additional identities: we set

$$stair(\Sigma_V) = \{stair(u,\ldots,u) \approx stair(w,\ldots,w) \mid u \approx w \in \Sigma_V\},$$

and show that this set of identities can be deduced from $\Sigma_{N_k(V)}$. The deductions for associativity and idempotence can be obtained exactly as in Lemma 3.4, using only $\Sigma_{N_k(B)}$, while the third identity $stair(p,\ldots,p) \approx stair(q,\ldots,q)$ is in fact part of our basis set $\Sigma_{N_k(V)}$.

Next we want to prove that $stair(s,\ldots,s) \approx stair(t,\ldots,t)$ can be deduced from $\Sigma_{N_k(V)}$. Defining the given deduction and the derived list as in Section 3, we show that the derived list is indeed a deduction of $stair(s,\ldots,s) \approx stair(t,\ldots,t)$ from $\Sigma_{N_k(V)} \bigcup stair(\Sigma_V)$. The proof of this is identical to the proof of Lemma 3.7. The only possible difficulty is the case where step j in the given deduction was an instance of the compatibility rule on two previous steps c and d. In this case, step j has $u_cu_d \approx w_cw_d$ deduced from $u_c \approx w_c$ and $u_d \approx w_d$. According to our construction of the derived list, step j in the derived list is $stair(u_cu_d,\ldots,u_cu_d) \approx stair(w_cw_d,\ldots,w_cw_d)$, but when we apply the compatibility rule to steps c and d in the derived list we obtain $stair(u_c,\ldots,u_c)stair(u_d,\ldots,u_d) \approx stair(w_c,\ldots,w_c)stair(w_d,\ldots,w_d)$ instead. However, by Lemma 3.5 we can produce a deduction of $stair(u_cu_d,\ldots,u_cu_d) \approx stair(w_cw_d,\ldots,w_cw_d)$ from $\Sigma_{N_k(B)}$ and the identity

$$stair(u_c,\ldots,u_c)stair(u_d,\ldots,u_d) \approx stair(w_c,\ldots,w_c)stair(w_d,\ldots,w_d).$$

But since $\Sigma_{N_k(B)} \subseteq \Sigma_{N_k(V)}$, we can still produce a deduction of

$$stair(u_cu_d,\ldots,u_cu_d) \approx stair(w_cw_d,\ldots,w_cw_d)$$

from $\Sigma_{N_k(V)}$ and the identity

$$stair(u_c, \ldots, u_c)stair(u_d, \ldots, u_d) \approx stair(w_c, \ldots, w_c)stair(w_d, \ldots, w_d).$$

This shows that $stair(s, \ldots, s) \approx stair(t, \ldots, t)$ can be deduced from $\Sigma_{N_k(V)} \bigcup stair(\Sigma_V)$ and hence from $\Sigma_{N_k(V)}$. Since $s \approx stair(s, \ldots, s)$ and $stair(t, \ldots, t) \approx t$ can also be deduced from $\Sigma_{N_k(V)}$, so can $s \approx t$. ∎

In the special cases that k equals 1 or 2, we can simplify our basis for $N_k(V)$. Associativity still holds, so we can simply use the associative identity instead of the k-normal associativity basis from Section 2; and the remaining identities in the basis can be simplified by the omission of brackets. This gives us the following basis for $N_k(V)$, when $k = 1, 2$ and V is the variety of bands determined by the defining identity $p \approx q$:

(1) Associativity $\quad x_1(x_2x_3) \approx (x_1x_2)x_3.$

(2) k-Normal Idempotence $\quad x_1x_2x_3 \cdots x_{k+1} \approx x_1x_1x_2x_3 \cdots x_{k+1}$

$x_1x_2x_3 \cdots x_{k+1} \approx x_1x_2x_3 \cdots x_{k+1}x_{k+1}.$

(3) k-normal Defining Identity $\quad p^{k+1} \approx q^{k+1}.$

The finite basis given for $N_k(V)$ in Theorem 4.1 is of course not unique. We now give a different basis for $N_k(V)$, for several well-known varieties V of bands. In each case the new basis consists of the k-normal associativity identities, the k-normal idempotence identities, and one additional identity; we list for each variety only the additional identity.

Subvariety	Additional Identity
Left Zero Bands	$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_1)x_2)x_3)\cdots)x_k.$
Right Zero Bands	$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_2x_3)x_4)\cdots)x_{k+1})x_{k+1}.$
Semilattices	$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_2x_1)x_3)x_4)\cdots)x_{k+1}$ $(\cdots(x_1x_2)x_3)\cdots)x_{k-1})x_k)x_{k+1} \approx (\cdots(x_1x_2)x_3)\cdots)x_{k-1})x_{k+1})x_k.$
Rectangular Bands	$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_1)x_3)x_4)\cdots)x_{k+1}.$
Normal Bands	$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_1x_3)x_2)x_4)x_5)\cdots)x_{k+1}.$
Left Normal Bands	$(\cdots(x_1x_2)x_3)\cdots)x_{k-1})x_k)x_{k+1} \approx (\cdots(x_1x_2)x_3)\cdots)x_{k-1})x_{k+1})x_k.$
Right Normal Bands	$(\cdots(x_1x_2)x_3)\cdots)x_{k+1} \approx (\cdots(x_2x_1)x_3)x_4)\cdots)x_{k+1}.$

The proofs for these bases are similar to the proof given for Theorem 4.1. For each choice of V, with defining identity $p \approx q$, we need to prove that $stair(p, \ldots, p) \approx stair(q, \ldots, q)$ can be deduced from the proposed new basis for $N_k(V)$. These deductions are straightforward and involve the use of the identities (2a), (2b), (1*), (2*), (3*), (4*), and the additional identities listed above.

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