DEMONSTRATIO MATHEMATICA
Vol. XL No 4 2007

M. S. N. Murty, G. Suresh Kumar

INITIAL AND BOUNDARY VALUE PROBLEMS FOR FUZZY DIFFERENTIAL EQUATIONS

Abstract. This paper deals with the study of existence and uniqueness criteria for initial and boundary value problems associated with fuzzy differential equations of first and second order with the help of a modified Lipschitz condition.

1. Introduction

The theory of fuzzy differential equations has attracted much attention in recent times [8]. Many authors ([3]–[10]) have studied initial and boundary value problems associated with first and second order Fuzzy differential equations on the metric space (E^n, D) of normal fuzzy convex sets with the distance D given by the supremum of the Hausdorff distance between the corresponding α - level sets.

Recently in [3] they have been obtained existence and uniqueness conditions of solutions for initial value problems associated with nonlinear second order and higher order fuzzy differential equations satisfying a Lipschitz condition. In this paper, we prove existence and uniqueness theorems for first and second order nonlinear fuzzy differential equations satisfying a modified Lipschitz condition.

Section 2 is concerned with notations and terminology relating to fuzzy sets and also deals with existence and uniqueness theorem for initial value problems associated with first order fuzzy differential equations. In section 3 we prove existence and uniqueness results for initial and boundary value problems for second order fuzzy differential equations with the help of Green's functions and contraction mapping theorem. Here, we use a modified Lipschitz condition that involves all the variables. The results obtained

2000 *Mathematics Subject Classification*: 34A10, 26E50, 34B15.
Key words and phrases: fuzzy differential equations, existence, uniqueness, initial and boundary value problems, contraction mapping, Green's function.

here, include more general class of problems than in ([3], [5] and [7]) obtained for first and second order fuzzy differential equations.

2. Preliminaries

Let $P_k(R^n)$ denotes the family of all nonempty compact convex subsets of R^n. Define the addition and scalar multiplication in $P_k(R^n)$ as usual. Radstrom [12] states that $P_k(R^n)$ is a commutative semigroup under addition, which satisfies the cancellation law. Moreover, if $\alpha, \beta \in R$ and $A, B \in P_k(R^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1 \cdot A = A$$

and if $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. The distance between A and B is defined by the Hausdorff metric

$$d(A,B) = \inf\{\epsilon : A \subset N(B,\epsilon), B \subset N(A,\epsilon)\},$$

where

$$N(A,\epsilon) = \{x \in R^n : ||x-y|| < \epsilon, \text{ for some } y \in A\}.$$

Let $I = [a,b] \subset R$ be a compact interval and let

$$E^n = \{u : R^n \to [0,1]/u \text{ satisfies (i)–(iv) below}\},$$

where

(i) u is normal, i.e. there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;
(ii) u is fuzzy convex, i.e. for $x, y \in R^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1-\lambda)y) \geq \min[u(x), u(y)];$$

(iii) u is upper semicontinuous;
(iv) $[u]^0 = cl\{x \in R^n/u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, the α-level set is denoted and defined by $[u]^\alpha = \{x \in R^n/u(x) \geq \alpha\}$. Then, from (i)-(iv) it follows that $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0,\infty)$ by the equation

$$D(u,v) = \sup\{d([u]^\alpha, [v]^\alpha)/\alpha \in [0,1]\},$$

where d is the Hausdorff metric defined in $P_k(R^n)$. It is easy to show that D is a metric in E^n and using results of [[2], [11]], we see that (E^n, D) is a complete metric space, but not locally compact. Moreover, the distance D verifies that

$$D(u+w, v+w) = D(u,v), \quad u, v, w \in E^n,$$
$$D(\lambda u, \lambda v) = |\lambda| D(u,v), \quad u, v \in E^n, \lambda \in R,$$
$$D(u+w, v+z) \leq D(u,v) + D(w,z), \quad u, v, w, z \in E^n.$$

We note that (E^n, D) is not a vector space. But it can be embedded isomorphically as a cone in a Banach space [12].

Regarding fundamentals of differentiability and integrabilitry of fuzzy functions we refer to O. Kaleva [5], Lakshmikantham and Mohapatra [8].

Let $I = [a, b] \subset R$ and $f : I \times E^n \to E^n$ be continuous. A mapping $\phi : I \to E^n$ is a solution of the initial value problem

$$y' = f(t, y), \quad y(a) = k, \tag{2.1}$$

where k is a real constant, if and only if ϕ is a solution of the integral equation

$$y(t) = k + \int_a^t f(s, y(s))ds. \tag{2.2}$$

It is easily seen that $C(I, E^n)$, the set of all continuous mappings from I to E^n is complete with the distance

$$H(u, v) = \sup_{t \in I} \left\{ D(u(t), v(t))e^{-\rho t} \right\},$$

where $u, v \in C(I, E^n)$ and $\rho(\geq 0) \in R$ is fixed.

THEOREM 2.1. *Let $f : I \times E^n \to E^n$ be a continuous map and suppose that there exists $M > 0$ such that*

$$D(f(t, u_1), f(t, u_2)) \leq MD(u_1, u_2)e^{-\rho t} \tag{2.3}$$

for all $t \in I, u_1, u_2 \in E^n$. Then the initial value problem (2.1) has a unique solution on the interval I.

Proof. For any $u \in C(I, E^n)$ define the operator $Q : C(I, E^n) \to C(I, E^n)$ given by

$$[Qu](t) = k + \int_a^t f(s, u(s))ds, \quad \forall t \in I.$$

Consider

$$\begin{aligned} H(Qu, Qv) &= \sup_{t \in I} \left\{ D([Qu](t), [Qv](t))e^{-\rho t} \right\} \\ &= \sup_{t \in I} \left\{ D\left(\int_a^t f(s, u(s))ds, \int_a^t f(s, v(s))ds\right)e^{-\rho t} \right\} \\ &\leq \sup_{t \in I} \left\{ \int_a^t D(f(s, u(s)), f(s, v(s)))ds \ \ e^{-\rho t} \right\} \\ &\leq \sup_{t \in I} \left\{ \int_a^t MD(u(s), v(s))dse^{-\rho s} \ \ e^{-\rho t} \right\} \\ &\leq MH(u, v) \sup_{t \in I} \left\{ (t - a) \ \ e^{-\rho t} \right\} \\ &\leq MH(u, v)(b - a)e^{-\rho a}. \end{aligned}$$

We can choose $\rho > 0$ such that $M(b-a)e^{-\rho a} < 1$ and Q is a contraction mapping. By using Contraction Mapping Principle, Q has a unique fixed point which is a unique solution of the initial value problem (2.1). ∎

EXAMPLE 2.1 Consider the initial value problem

$$y'(t) = qe^{-\rho t}y(t) + F(t), \quad y(1) = k, \quad t \in [1,2] \tag{2.4}$$

where $F \in C([1,2], E^n)$, and q, $k \in R$. Here

$$f(t,y) = qe^{-\rho t}y(t) + F(t).$$

Consider

$$\begin{aligned} D(f(t,y), f(t,z)) &\leq D(qe^{-\rho t}y + F(t), qe^{-\rho t}z + F(t)) \\ &= |q|e^{-\rho t}D(y,z) \\ &= MD(y,z)e^{-\rho t}, \end{aligned}$$

where $M = |q|$. Clearly, f satisfies (2.3). By taking $q = 1$, and from Theorem 2.1 the initial value problem (2.4) has a unique solution for all $\rho > 0$.

We denote by $C'(I, E^n)$ the set of all continuously differentiable mappings from I to E^n. For $u, v \in C'(I, E^n)$, we define the distance

$$H_1(u,v) = H(u,v) + H(u',v').$$

LEMMA 2.1 (Lemma 1 of [3]). *$(C'(I,E^n), H_1)$ is a complete metric space.*

3. Initial and boundary value problems

In this section we prove the existence and uniqueness results for initial and boundary value problems associated with second order fuzzy differential equations.

Consider the nonlinear fuzzy differential equation of second order

$$y'' + f(t,y,y') = 0 \tag{3.1}$$

satisfying

$$y(a) = m_1 \quad y'(a) = m_2 \tag{3.2}$$

where $f : I \times E^n \times E^n \to E^n$ is continuous, m_1 and m_2 are real constants.

If ϕ is a solution of (3.1) satisfying (3.2) if and only if ϕ is a solution of the integral equation

$$y(t) = m_1 + m_2(t-a) + \int_a^t (s-t)f(s,y(s),y'(s))ds. \tag{3.3}$$

Now we prove the existence and uniqueness theorem for initial value problem (3.1) satisfying (3.2) using the integral representation (3.3).

THEOREM 3.1. *Let $f : I \times E^n \times E^n \to E^n$ be a continuous map and suppose that there exist $M_1, M_2 > 0$ such that*

$$D(f(t,u_1,u_2), f(t,v_1,v_2)) \le [M_1 D(u_1,v_1) + M_2 D(u_2,v_2)]e^{-\rho t} \tag{3.4}$$

for all $t \in I = [a,b]$, $u_1, u_2, v_1, v_2 \in E^n$. Then the initial value problem (3.1) *satisfying* (3.2) *has a unique solution on the interval I.*

Proof. Consider the complete metric space $(C'(I,E^n), H_1)$. For any $u \in C'(I,E^n)$ define the operator $T : C'(I,E^n) \to C'(I,E^n)$ by

$$[Tu](t) = m_1 + m_2(t-a) + \int_a^t (s-t) f(s, y(s), y'(s))ds, \quad t \in I.$$

Using definitions of H_1, H, T, and (3.4) we have

$$H_1(Tu,Tv) = H(Tu,Tv) + H([Tu]', [Tv]')$$

$$= \sup_{t\in I}\left\{ D\Big(\int_a^t (s-t)f(s,u(s),u'(s))ds, \int_a^t (s-t)f(s,v(s),v'(s))ds\Big)e^{-\rho t}\right\}$$

$$+ \sup_{t\in I}\left\{ D\Big(\int_a^t f(s,u(s),u'(s))ds, \int_a^t f(s,v(s),v'(s))ds\Big)e^{-\rho t}\right\}$$

$$\le \sup_{t\in I}\left\{ \int_a^t |s-t|\; D(f(s,u(s),u'(s)), f(s,v(s),v'(s)))ds\, e^{-\rho t}\right\}$$

$$+ \sup_{t\in I}\left\{ \int_a^t D(f(s,u(s),u'(s)), f(s,v(s),v'(s)))ds \;\; e^{-\rho t}\right\}$$

$$\le \sup_{t\in I}\left\{ \int_a^t (t-s)[M_1 D(u(s),v(s)) + M_2 D(u'(s),v'(s))]e^{-\rho s}ds \;\; e^{-\rho t}\right\}$$

$$+ \sup_{t\in I}\left\{ \int_a^t [M_1 D(u(s),v(s)) + M_2 D(u'(s),v'(s))]e^{-\rho s}ds \;\; e^{-\rho t}\right\}$$

$$\le \sup_{t\in I}\left\{ \int_a^t (t-s)[M_1 H(u,v) + M_2 H(u',v')]ds \;\; e^{-\rho t}\right\}$$

$$+ \sup_{t\in I}\left\{ \int_a^t [M_1 H(u,v) + M_2 H(u',v')]ds \;\; e^{-\rho t}\right\}$$

$$= [M_1 H(u,v) + M_2 H(u',v')]\left(\sup_{t\in I}\left\{ \int_a^t (t-s)ds\, e^{-\rho t}\right\} + \sup_{t\in I}\left\{ \int_a^t ds\; e^{-\rho t}\right\}\right)$$

$$\leq \max\{M_1, M_2\} H_1(u,v) \left(\sup_{t \in I} \left\{ \frac{(t-a)^2}{2} e^{-\rho t} \right\} + \sup_{t \in I} \left\{ (t-a)\, e^{-\rho t} \right\} \right)$$

$$= H_1(u,v) \max\{M_1, M_2\} \left(\frac{(b-a)^2}{2} e^{-\rho a} + (b-a) e^{-\rho a} \right).$$

We can choose $\rho > 0$ such that

$$\max\{M_1, M_2\} \left(\frac{(b-a)^2}{2} e^{-\rho a} + (b-a) e^{-\rho a} \right) < 1.$$

It follows that T is a contraction mapping in the complete metric space $C'((I, E^n), H_1)$. By Contraction Mapping Principle, T has a unique fixed point u, which is a unique solution of the initial value problem (3.1) satisfying (3.2). ∎

Now consider the nonlinear fuzzy differential equation of second order

$$y'' + f(t, y, y') = 0 \tag{3.5}$$

satisfying the boundary condition

$$y(a) = k_1 \quad y(b) = k_2 \tag{3.6}$$

where $f : I \times E^n \times E^n \to E^n$ is continuous, k_1 and k_2 are real constants. We know that $C'((I, E^n), H_1)$ is a complete metric space. For any $\phi \in C'((I, E^n), H_1)$ define the operator $T\phi \in C'(I, E^n)$ by

$$[T\phi](t) = \int_a^b G(t,s) f(s, \phi(s), \phi'(s)) ds, \quad \forall\ t \in I,$$

where $G(t,s)$ is the Green's function for the homogeneous boundary value problem. Hence, $\phi \in C'(I, E^n)$ is a solution of (3.5) satisfying (3.6) if and only if ϕ is a fixed point of T.

THEOREM 3.2. *Let $f : I \times E^n \times E^n \to E^n$ be continuous and suppose that there exist $M_1, M_2 > 0$ such that (3.4) is satisfied. Then the two point fuzzy boundary value problem (3.5) satisfying (3.6) has a unique solution on the interval I.*

Proof. Consider the boundary value problem

$$y'' = 0 \tag{3.7}$$

satisfying

$$y(a) = 0, \quad y(b) = 0. \tag{3.8}$$

This problem has no nontrivial solution. Therefore, if h is any continuous function on $[a, b]$, the equation $y''(t) + h(t) = 0$ has a unique solution satisfying the boundary condition (3.8) given by

$$y(t) = \int_a^b G(t,s)h(s)ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{(b-t)(s-a)}{b-a} & a \le s \le t \le b \\ \dfrac{(b-s)(t-a)}{b-a} & a \le t \le s \le b. \end{cases}$$

Consider

$$\begin{aligned} \sup_{t\in I} \int_a^b |G(t,s)|ds &= \sup_{t\in I} \left\{ \int_a^t |G(t,s)|ds + \int_t^b |G(t,s)|ds \right\} \\ &= \sup_{t\in I} \left\{ \frac{b-t}{b-a} \int_a^t (s-a)ds + \frac{t-a}{b-a} \int_t^b (b-s)ds \right\} \\ &= \sup_{t\in I} \left\{ \frac{(b-t)(t-a)}{2} \right\}. \end{aligned}$$

This function attains its maximum value at $t = \frac{a+b}{2}$, and hence

$$\sup_{t\in I} \int_a^b |G(t,s)|ds \le \frac{(b-a)^2}{8}. \tag{3.9}$$

Again consider

$$\sup_{t\in I} \int_a^b |G_t(t,s)|ds = \sup_{t\in I} \left\{ \frac{(t-a)^2 + (t-b)^2}{2(b-a)} \right\}.$$

The maximum of this function is attained at a and b which yields

$$\sup_{t\in I} \int_a^b |G_t(t,s)|ds \le \frac{b-a}{2}. \tag{3.10}$$

We know that $C'((I,E^n),H_1)$ is complete metric space. For any $u \in C'(I,E^n)$ define the operator $F : C'(I,E^n) \to C'(I,E^n)$ by

$$[Fu](t) = \int_a^b G(t,s)f(s,u(s),u'(s))ds, \quad t \in I. \tag{3.11}$$

Using the bounds on G, G_t given by (3.9) and (3.10), definitions of H_1, H, F, and from (3.4) we have

$$
\begin{aligned}
H_1(Fu, Fv) &= H(Fu, Fv) + H([Fu]', [Fv]') \\
&= \sup_{t\in I}\left\{ D\Big(\int_a^b G(t,s) f(s,u(s),u'(s))ds, \int_a^b G(t,s) f(s,v(s),v'(s))ds\Big) e^{-\rho t}\right\} \\
&\quad + \sup_{t\in I}\left\{ D\Big(\int_a^b G_t(t,s) f(s,u(s),u'(s))ds, \int_a^b G_t(t,s) f(s,v(s),v'(s))ds\Big) e^{-\rho t}\right\} \\
&\le \sup_{t\in I}\left\{ \int_a^b |G(t,s)| D(f(s,u(s),u'(s)), f(s,v(s),v'(s)))ds \;\; e^{-\rho t}\right\} \\
&\quad + \sup_{t\in I}\left\{ \int_a^b |G_t(t,s)| D(f(s,u(s),u'(s)), f(s,v(s),v'(s)))ds \;\; e^{-\rho t}\right\} \\
&\le \sup_{t\in I}\left\{ \int_a^b |G(t,s)| [M_1 D(u(s),v(s)) + M_2 D(u'(s),v'(s))] e^{-\rho s} ds \;\; e^{-\rho t}\right\} \\
&\quad + \sup_{t\in I}\left\{ \int_a^b |G_t(t,s)| [M_1 D(u(s),v(s)) + M_2 D(u'(s),v'(s))] e^{-\rho s} ds \;\; e^{-\rho t}\right\} \\
&\le \sup_{t\in I}\left\{ \int_a^b |G(t,s)| [M_1 H(u,v) + M_2 H(u',v')] ds \;\; e^{-\rho t}\right\} \\
&\quad + \sup_{t\in I}\left\{ \int_a^b |G_t(t,s)| [M_1 H(u,v) + M_2 H(u',v')] ds \;\; e^{-\rho t}\right\} \\
&= [M_1 H(u,v) + M_2 H(u',v')] \\
&\quad \times \left(\sup_{t\in I}\left\{ \int_a^b |G(t,s)| ds \, e^{-\rho t}\right\} + \sup_{t\in I}\left\{ \int_a^b |G_t(t,s)| ds \, e^{-\rho t}\right\}\right) \\
&\le \max\{M_1, M_2\} H_1(u,v) \left(\sup_{t\in I}\left\{ \frac{(b-a)^2}{8} \, e^{-\rho t}\right\} + \sup_{t\in I}\left\{ \frac{(b-a)}{2} \, e^{-\rho t}\right\}\right) \\
&= \max\{M_1, M_2\} H_1(u,v) \left(\frac{(b-a)^2}{8} \;\; e^{-\rho a} + \frac{(b-a)}{2} \;\; e^{-\rho a}\right).
\end{aligned}
$$

We can choose $\rho > 0$ such that

$$
\max\{M_1, M_2\}\left(\frac{(b-a)^2}{8} \;\; e^{-\rho a} + \frac{(b-a)}{2} \;\; e^{-\rho a}\right) < 1
$$

and F is contraction mapping. By Contraction Mapping Principle, F has a unique fixed point, which is a unique solution of the boundary value problem (3.5) satisfying (3.8).

By applying the above procedure to the boundary value problem

$$y'' + f(t, y(t) + p(t), y'(t) + p'(t)) = 0$$
$$y(a) = 0, \quad y(b) = 0,$$

where p is a polynomial of first degree such that $p(a) = k_1$, $p(b) = k_2$ a unique solution $y_1(t)$ is constructed. Let $y(t) = y_1(t) + p(t)$. Then it is easily seen that y is a solution of the boundary value problem (3.5) satisfying (3.6). Hence the theorem. ∎

THEOREM 3.3. *Let $f : I \times E^n \times E^n \to E^n$ be continuous and suppose that there exist $M_1, M_2 > 0$ such that (3.4) is satisfied. Then the two point fuzzy boundary value problems of the second kind, (3.5) satisfying*

$$y'(a) = k_1 \quad y(b) = k_2 \tag{3.12}$$

and (3.5) satisfying

$$y(a) = k_1 \quad y'(b) = k_2 \tag{3.13}$$

have unique solutions on the interval I.

Proof. First, we consider the boundary value problem

$$y'' = 0 \tag{3.14}$$

satisfying

$$y(a) = 0 \qquad y'(b) = 0. \tag{3.15}$$

This problem has no nontrivial solution. Therefore, if g is any continuous function on $[a, b]$, the equation

$$y''(t) + g(t) = 0 \tag{3.16}$$

satisfying the boundary condition (3.15) has a unique solution, given by

$$y(t) = \int_a^b K(t, s) g(s) ds,$$

where

$$K(t, s) = \begin{cases} s - a & a \le s \le t \le b \\ t - a & a \le t \le s \le b. \end{cases}$$

Consider

$$\sup_{t \in I} \int_a^b |K(t, s)| ds = \sup_{t \in I} \left\{ \int_a^t (s - a) ds + (t - a) \int_t^b ds \right\}$$
$$= \sup_{t \in I} \left\{ \frac{(t - a)^2}{2} + (t - a)(b - t) \right\}.$$

This function attains its maximum value at $t = \frac{a+b}{2}$ and hence

$$\sup_{t\in I}\int_a^b |K(t,s)|ds \le \frac{(b-a)^2}{2}. \tag{3.17}$$

Again consider

$$\sup_{t\in I}\int_a^b |K_t(t,s)|ds = \sup_{t\in I}\{b-t\} \le b-a. \tag{3.18}$$

We know that $C'((I,E^n),H_1)$ is complete metric space. For any $v \in C'(I,E^n)$ define the operator $\Phi : C'(I,E^n) \to C'(I,E^n)$ by

$$[\Phi v](t) = \int_a^b K(t,s)f(s,v(s),v'(s))ds \quad t \in I. \tag{3.19}$$

Similarly using the bounds on K, K_t given by (3.17) and (3.18), definitions of H_1, H, Φ, (3.4), and following the procedure as in Theorem 3.2, we have

$$\begin{aligned} &H_1(\Phi v, \Phi w) \\ &\quad\le [M_1 H(v,w) + M_2 H(v',w')] \\ &\qquad\times \left(\sup_{t\in I}\left\{\int_a^b |K(t,s)|ds\, e^{-\rho t}\right\} + \sup_{t\in I}\left\{\int_a^b |K_t(t,s)|ds\, e^{-\rho t}\right\}\right) \\ &\quad\le \max\{M_1,M_2\}H_1(v,w)\left(\sup_{t\in I}\left\{\frac{(b-a)^2}{2}e^{-\rho t}\right\} + \sup_{t\in I}\left\{(b-a)e^{-\rho t}\right\}\right) \\ &\quad= \max\{M_1,M_2\}H_1(v,w)\left(\frac{(b-a)^2}{2}e^{-\rho a} + (b-a)\,e^{-\rho a}\right). \end{aligned}$$

We can choose $\rho > 0$ such that

$$\max\{M_1,M_2\}\left(\frac{(b-a)^2}{2} + (b-a)\right) \quad e^{-\rho a} < 1$$

and Φ is a contraction mapping. By Contraction Mapping Principle, Φ has a unique fixed point, which is a unique solution of the boundary value problem (3.5) satisfying (3.15).

By applying the above procedure to the boundary value problem

$$\begin{gathered} y'' + f(t, y(t) + q(t), y'(t) + q'(t)) = 0, \\ y(a) = 0, \qquad y'(b) = 0, \end{gathered}$$

where q is a polynomial of first degree such that $q(a) = k_1$, $q'(b) = k_2$ a unique solution $\widetilde{y}(t)$ is constructed. Let $y(t) = \widetilde{y}(t) + q(t)$. Then it is easily seen that $y(t)$ is a solution of the boundary value problem (3.5) satisfying (3.13).

Similarly we can prove that the boundary value problem (3.5) satisfying (3.12) has a unique solution on I. ∎

EXAMPLE 3.1. Consider the two point boundary value problem

$$y''(t) = q_1 e^{-\rho t} y(t) + q_2 e^{-\rho t} y'(t) + \phi(t), \quad t \in [1,2] \tag{3.20}$$

$$y(1) = k_1, \qquad y(2) = k_2, \tag{3.21}$$

where $\phi \in C([1,2], E^n)$, q_1, q_2, k_1, $k_2 \in R$ and $\rho(\geq 0) \in R$ is fixed. Here

$$f(t, y_1, y_2) = q_1 e^{-\rho t} y_1(t) + q_2 e^{-\rho t} y_2(t) + \phi(t) \tag{3.22}$$

Consider

$$\begin{aligned}
D(f(t,y_1,y_2), f(t,z_1,z_2)) \\
&\leq D(q_1 e^{-\rho t} y_1(t) + q_2 e^{-\rho t} y_2(t) + \phi(t), q_1 e^{-\rho t} z_1(t) \\
&\quad + q_2 e^{-\rho t} z_2(t) + \phi(t)) \\
&= D(q_1 e^{-\rho t} y_1(t) + q_2 e^{-\rho t} y_2(t), q_1 e^{-\rho t} z_1(t) + q_2 e^{-\rho t} z_2(t)) \\
&\leq |q_1| e^{-\rho t} D(y_1, z_1) + |q_2| e^{-\rho t} D(y_2, z_2) \\
&= [M_1 D(y_1, z_1) + M_1 D(y_1, z_1)] e^{-\rho t},
\end{aligned}$$

where $M_1 = |q_1|$ and $M_2 = |q_2|$. Therefore f satisfies the modified Lipschitz condition.

In particular, if we take $q_1 = 2$, $q_2 = 4$, then the two point boundary value problem (3.20) satisfying (3.21) has a unique solution for all values of $\rho \geq 1$.

EXAMPLE 3.2. Consider Example 3.1 with boundary conditions (3.21) replaced by

$$y(1) = k_1, \qquad y'(2) = k_2. \tag{3.23}$$

By taking $q_1 = 1$ and $q_2 = \frac{3}{2}$, and from Theorem 3.3 the boundary value problem (3.20) satisfying (3.23) has a unique solution for all values of $\rho \geq 1$.

Acknowledgement. The authors are grateful to the learned referee for a very careful reading of the manuscript with many useful suggestions for improvement.

References

[1] P. B. Bailey, L.F. Shampine and P. E. Waltman, *Nonlinear two point boundary value problems*, Academic Press (1968), New York.

[2] G. Debreu, *Integration of correspondence*, in: Proc. Fifth Berkeley Symp. Math. Statist. Probab., Vol. 2, Part 1 (Univ. California Press, Berkeley, CA, 1967), 351–372.

[3] D. N. Georgious, Juan J. Nieto and Rosana Rodrigue-Lopez, *Initial value problems for higher-order fuzzy differential equations*, Nonlinear Anal. 63 (2005), 587–600.
[4] D. N. Georgious, I. E. Kougias, *On Cauchy problems for fuzzy differential equations*, Internat. J. Math. Math. Sci. 15 (2004), 799–805.
[5] O. Kaleva, *Fuzzy differential equations*, Fuzzy Sets and Systems 24 (1987), 301–317.
[6] O. Kaleva, *The Cauchy problem for fuzzy differential equations*, Fuzzy Sets and Systems 35 (1990), 389–396.
[7] V. Lakshmikantham, K. N. Murty and J. Turner, *Two-point boundary value problems associated with nonlinear fuzzy differential equations*, Mathematical Inequalities and Applications 4, No. 4 (2001), 527–533.
[8] V. Lakshmikantham and R. Mohapatra, *Theory of Fuzzy Differential Equations and Inclusions*, Taylor and Francis, London (2003).
[9] M. S. N. Murty and G. Suresh Kumar, *Three point boundary value problems for Fuzzy differential equations*, Journal of the ChungCheong Mathematical Society 19, No. 1 (2006), 101–110.
[10] J. J. Nieto, *The Cauchy problem for continuous fuzzy differential equations*, Fuzzy Sets and Systems 2500 (1997), 001–006.
[11] M. L. Puri and D. A. Ralescu, *Fuzzy random variables*, J. Math. Anal. Appl. 114 (1986), 409–422.
[12] H. Radstrom, *An embedding theorem for space of convex sets*, Proc. Amer. Math. Soc. 3 (1952), 165–169.

DEPARTMENT OF APPLIED MATHEMATICS
ACHARYA NAGARJUNA UNIVERSITY, POST GRADUATE CENTRE
NUZVID, ANDRA PRADESH, INDIA
E-mail: drmsn2002@yahoo.com

Received September 14, 2006; revised version January 6, 2007.