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ON SOME REGULAR
MULTI-VEBLEN CONFIGURATIONS,
THE GEOMETRY OF
COMBINATORIAL QUASI GRASSMANNIANS

Abstract. Multi-Veblen configurations which can be embedded into Desarguesian projective spaces were characterized in [8]: besides the class of combinatorial Grassmannians, only one further class of multi-Veblen configurations shares this property, namely the class of combinatorial quasi Grassmannians introduced in [8]. In this note we discuss relationships between combinatorial Grassmannians and combinatorial quasi Grassmannians, characterize automorphisms of combinatorial quasi Grassmannians, and present some visualizations of them.

Introduction

There is a great deal of a literature on Steiner triple systems, and the literature on partial (called also *incomplete*) Steiner triple systems is also growing rapidly. The subject of this note is the geometry of some particular partial Steiner triple systems; why does this particular class deserve our interest?

In our opinion, we can quote two reasons. First, the structures considered in the paper are multi-Veblen configurations. The class of multi-Veblen configurations was introduced in [10]. These configurations arise naturally as combinatorial schemes generalizing two classical configurations considered in geometry, namely Kantor's 10_3G configuration, and the Desargues configuration, presented as some systems of Veblen configurations. Secondly, the structures considered in the paper are *projectively embeddable* i.e. they can be embedded into Desarguesian projective spaces. It turns out (cf. [8] for details) that there are exactly two types of multi-Veblen configurations that can be projectively embedded. Structures of the first type are so called

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combinatorial Grassmannians $\mathbf{G}_2(n)$, n – an integer, $n \geq 4$, which directly generalize the Desargues configuration considered as a perspective of two simplices. In the paper [7] we have presented basic information on their geometry. The second class of projectively embeddable multi-Veblen configurations consists of *quasi Grassmannians* $\mathfrak{R}_n$, $n \geq 2$ (defined in [8]), which are studied in this note.

There are two main topics of our paper. First, we determine the collineation group $\text{Aut}(\mathfrak{R}_n)$ of a quasi Grassmannian $\mathfrak{R}_n$ (cf. Thm. 2.8). Secondly, we discuss the geometry of quasi Grassmannians in some detail. We present a series of representations which allow us to visualize the internal structure of $\mathfrak{R}_n$ as some “abstract” perspectives (2.4, 2.5, 2.6) and as a “fan” of Veblen (or Kantor’s 10_3G) configurations (Thm. 3.2). In closing remarks we show how $\mathfrak{R}_n$ looks like as a projective configuration (Prop. 3.6, 3.7, 3.8).

1. Definitions

Let X be a set, we write $\wp_k(X)$ for the family of k -element subsets of X . Generally, a *simple undirected loopless graph* (in the paper called simply a *graph*, see [3]) on a set X is a structure $\langle X, \mathcal{P} \rangle$, where $\mathcal{P} \subset \wp_2(X)$. In what follows we shall frequently identify a set $\mathcal{P} \subset \wp_2(X)$ with the graph $\langle X, \mathcal{P} \rangle$. If $\mathcal{P} \subset \wp_2(X)$ and $Y \subset X$ then the graph $\mathcal{P} \restriction Y := \mathcal{P} \cap \wp_2(Y)$ is the edge-induced subgraph of Y in $\langle X, \mathcal{P} \rangle$. K_X and N_X denote the complete graph on X and its complement, respectively.

A partial Steiner triple system is a partial linear (or *semilinear*) space whose all the lines have size 3 (cf. [13]). The structure

$$\mathbf{G}_2(X) := \langle \wp_2(X), \wp_3(X), \subset \rangle \cong \langle \wp_2(X), \{ \wp_2(Y) : Y \in \wp_3(X) \}, \in \rangle$$

is referred to as a *combinatorial Grassmannian* (cf. [7], [10]; see also [2], [1]). We write $\mathbf{G}_2(n) = \mathbf{G}_2(X)$ if $|X| = n$. Note that $\mathbf{G}_2(n)$ is a $\left(\binom{n}{2}_{n-2} \binom{n}{3}_3 \right)$ -configuration, thus it is a partial Steiner triple system. In particular, $\mathbf{G}_2(4)$ is the *Veblen configuration* and $\mathbf{G}_2(5)$ is the *Desargues configuration*.

Let us first recall the definitions of a multi-Veblen configuration and a combinatorial quasi Grassmannian $\mathfrak{R}_n$ (cf. [10] and [8]):

DEFINITION 1.1. Let $\mathfrak{H} = \langle \wp_2(X), \mathcal{L} \rangle$ be a partial Steiner triple system and $\mathcal{P}$ be a graph defined on X . We take any two distinct elements $p_1, p_2 \notin X$ and put $p = \{p_1, p_2\}$, $X' = X \cup p$. Consider the following families of blocks:

$$\mathcal{L}_1 = \left\{ \{ \{p_1, p_2\}, \{p_1, i\}, \{p_2, i\} \} : i \in X \right\},$$

$$\mathcal{L}_2 = \left\{ \{ \{i, j\}, \{p_1, i\}, \{p_2, j\} \} : i, j \in X, i \neq j, \{i, j\} \notin \mathcal{P} \right\},$$

$$\mathcal{L}_3 = \left\{ \{ \{i, j\}, \{p_1, i\}, \{p_1, j\} \}, \{ \{i, j\}, \{p_2, i\}, \{p_2, j\} \} : i, j \in X, \{i, j\} \in \mathcal{P} \right\}.$$

Then the structure $\langle \wp_2(X'), \mathcal{L} \cup \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3 \rangle$ will be denoted by $\mathbb{M}_{X \triangleright_{\mathcal{P}}}^p \mathfrak{H}$ and it will be called *the multi-Veblen configuration with center p , consistency graph $\mathcal{P}$ defined on X , and axial configuration $\mathfrak{H}$* . A multi-Veblen configuration is *simple* if it has a combinatorial Grassmannian as its axial configuration.

Recall (cf. [10]) that $\mathbb{M}_{X \triangleright_{K_X}}^p \mathbf{G}_2(X) \cong \mathbf{G}_2(X \cup p)$. Let us denote $\mathbf{IB}(X) := \mathbb{M}_{X \triangleright_{N_X}}^p \mathbf{G}_2(X)$. We write $\mathbf{IB}(n) := \mathbf{IB}(X)$, where $|X| = n$, for short. ■

FACT 1.2. *If $\mathfrak{H}$ is a $\left(\binom{n}{2}_{n-2} \binom{n}{3}_3\right)$ -configuration then $\mathbb{M}_{X \triangleright_{\mathcal{P}}}^p \mathfrak{H}$ is a $\left(\binom{n+2}{2}_n \binom{n+2}{3}_3\right)$ -configuration.*

When we deal with the structures of the form $\mathfrak{M} = \mathbb{M}_{X \triangleright_{\mathcal{P}}}^p \mathfrak{H}$ we frequently use the following, auxiliary, labeling of the points of $\mathfrak{M}$:

CONSTRUCTION 1.3.

- * Let p be a point, *a center*.
- * We treat p as consisting of two symbols, a , and b , and we write $s_i = \{s, i\}$ when $i \in X$ and $s \in \{a, b\}$. Thus the points p, a_i, b_i with $i \in X$ yield a line for every $i \in X$. These are the lines of $\mathfrak{M}$ through p .
- * If $i_1, i_2 \in X$ are distinct then the point $\{i_1, i_2\}$ of $\mathfrak{M}$ is written as $c_{\{i_1, i_2\}} = \{i_1, i_2\}$. The point $c_{\{i_1, i_2\}}$ is the intersection point of the lines $\overline{a_{i_1}, a_{i_2}}$ and $\overline{b_{i_1}, b_{i_2}}$ when $\{i_1, i_2\} \in \mathcal{P}$, and it is the intersection point of the lines $\overline{a_{i_1}, b_{i_2}}$ and $\overline{b_{i_1}, a_{i_2}}$ when $\{i_1, i_2\} \notin \mathcal{P}$.
- * The points c_u, c_v, c_z with $u, v, z \in \wp_2(X)$ are joined in $\mathfrak{M}$ in the way in which u, v, z are joined in $\mathfrak{H}$. ■

This labeling can be considered, in fact, as *a definition* of the structure $\mathfrak{M}$ (cf. [10, Construction 3]). Note, however, that the above labeling corresponds to the way in which the structure $\mathfrak{M}$ is constructed, but it is not determined by the geometry of $\mathfrak{M}$. This means that in many structures of this form we can choose another point p' of $\mathfrak{M}$ as a center and, with possibly another graph $\mathcal{P}'$ and configuration $\mathfrak{H}'$, we obtain the same structure, up to an isomorphism.

A reiterated multi-Veblen configuration is any structure of the form

$$(1) \quad \mathfrak{M} = \mathbb{M}_{X_{k-1} \triangleright_{\mathcal{P}_{k-1}}}^{p^k} \cdots (\mathbb{M}_{X_1 \triangleright_{\mathcal{P}_1}}^{p^2} (\mathbb{M}_{X_0 \triangleright_{\mathcal{P}_0}}^{p^1} \mathbf{G}_2(X_0))),$$

where p^j are two element sets, the set X_0 and the p^j are pairwise disjoint, $X_j = X_{j-1} \cup p^j$ for $j = 1, \dots, k$, and $\mathcal{P}_j$ is a graph defined on X_j for $j = 0, \dots, k-1$.

By definition, if $k = 1$ then the structure $\mathfrak{M}$ of the form (1) is a simple multi-Veblen configuration (note that the converse is not true).

Let $\mathcal{P}', \mathcal{P}''$ be two graphs on a set X . We write $\mathcal{P}' \approx \mathcal{P}''$ iff there is a sequence $\mu_{x_1}, \dots, \mu_{x_l}$ of maps ($x_1, \dots, x_l \in X$) such that the composition

$\mu_{x_1} \dots \mu_{x_l}$ maps $\mathcal{P}'$ onto $\mathcal{P}''$ and every μ_{x_j} switches connections with the vertex x_j : x_j, y are connected in $\mu_{x_j}(\mathcal{P})$ iff they are not connected in $\mathcal{P}$, for arbitrary graphs $\mathcal{P}$ on X and $y \in X$.

FACT 1.4. (cf. [10, Prop. 9]) *If $\mathfrak{H}$ is any partial Steiner triple system defined on the set $\wp_2(X)$, p is a two-element set disjoint with X , and $\mathcal{P}', \mathcal{P}''$ are graphs defined on X then $\mathcal{P}' \approx \mathcal{P}''$ yields $\mathbb{M}_{X \cup p}^p \triangleright_{\mathcal{P}'} \mathfrak{H} \cong \mathbb{M}_{X \cup p}^p \triangleright_{\mathcal{P}''} \mathfrak{H}$.*

A multi-Veblen configuration $\mathfrak{M}$ is *regular* iff it can be represented in the form (1), where $\mathcal{P}_{i-1} = \mathcal{P}_i \wedge X_{i-1}$ for $i = 1, \dots, k-1$.

CONSTRUCTION 1.5. Let $n \geq 2$ be an integer. We define the regular multi-Veblen configuration $\mathfrak{R}_n$ as follows (the construction below differs inessentially from that originally presented in [8]). If $n = 2k$ for some integer k then we take $m = 2$, if $n = 2k + 1$ we set $m = 3$; in both cases we obtain $n = m + 2(k-1)$. Let $p^0 = \{1, 2\}$; we put $X = p^0$ when $m = 2$, and $X = p^0 \cup \{0\}$ when $m = 3$. Next we take $p^j = \{1 + 2j, 2 + 2j\}$ for $j = 1, \dots, k$. Let $\mathcal{P} = \wp_2(X \cup p^1 \cup \dots \cup p^{k-1}) \setminus \{p^0\}$. We put $\mathcal{P}_0 := \mathcal{P} \wedge X$ and $\mathcal{P}_j := \mathcal{P} \wedge (X \cup p^1 \cup \dots \cup p^j)$ for $0 < j < k$. If $m = 2$ then $\mathcal{P}_0 = N_X$; if $m = 3$ then $\mathcal{P}_0 \approx N_X$. We set

$$\mathfrak{R}_n := \mathbb{M}_{X \cup p^1 \cup \dots \cup p^{k-1}}^{p^k} \triangleright_{\mathcal{P}_{k-1}} \left(\dots \left(\mathbb{M}_{X \cup p^1}^{p^2} \triangleright_{\mathcal{P}_1} \left(\mathbb{M}_X^{p^1} \triangleright_{\mathcal{P}_0} \mathbf{G}_2(X) \right) \dots \right) \right).$$

The configuration $\mathfrak{R}_n$ will be called a *combinatorial quasi Grassmannian*. ■

In particular, $\mathfrak{R}_2$ is simply the *Veblen configuration* $\mathbf{B}(p^0) = \mathbf{B}(2)$ (see Figure 1), $\mathfrak{R}_3 \cong \mathbf{B}(3)$ is the *Veronese configuration* (in the terminology of [11]) i.e. Kantor's 10_3G configuration (cf. [5], see Figure 2). Then $\mathfrak{R}_4$ is the least *not simple* multi-Veblen configuration embeddable into a projective space (see Figure 3).

The configuration $\mathbf{G}_2(4)$ The configuration $\mathfrak{R}_2 = \mathbf{B}(2)$

Fig. 1. Two presentations of the Veblen configuration

In general, $\mathfrak{R}_n$ is a $\left(\binom{n+2}{2}_n \binom{n+2}{3}_3 \right)$ -configuration. Intuitively, we can write $\mathfrak{R}_{n+2} = \mathbb{M}_{U \cup p}^p \triangleright_{\mathcal{P}} \mathfrak{R}_n$ for a suitable graph $\mathcal{P}$, point p , and set U .

PROPOSITION 1.6. ([8, Thm. 2.17]) *A regular multi-Veblen configuration $\mathfrak{M}$ with the points of degree $n \geq 2$ can be embedded into a Desarguesian projective space iff $\mathfrak{M} \cong \mathbf{G}_2(n+2)$ or $\mathfrak{M} \cong \mathfrak{R}_n$.*

Fig. 2. The configuration $\mathfrak{R}_3 \cong \mathbb{B}(3)$

Comparing the definition of a combinatorial Grassmannian and the definition of a combinatorial quasi Grassmannian we obtain immediately the following observation which links structures of these two types and which plays a very important though technical role in our further investigations.

PROPOSITION 1.7. (cf. [8, Rem. 2.14]) *Let $n \geq 2$ be an integer. If n is odd we take $X = \{0, 1, 2, \dots, 2k + 2\}$ and write $n = 2k + 1$; if n is even we take $X = \{1, 2, \dots, 2k + 2\}$ and $n = 2k$. If one replaces in $\mathbf{G}_2(X) = \mathbf{G}_2(n + 2)$ the class of all its lines passing through $\{1, 2\}$ (i.e. the family of sets of the form $\{\{1, 2\}, \{1, 2j + 2\}, \{2, 2j + 2\}\}$, $\{\{1, 2\}, \{2, 2j + 1\}, \{1, 2j + 1\}\}$ with $1 \leq j \leq k$) by the family of sets of the form*

$$\{\{1, 2\}, \{1, 2j + 2\}, \{2, 2j + 1\}\}, \quad \{\{1, 2\}, \{2, 2j + 2\}, \{1, 2j + 1\}\}$$

with $1 \leq j \leq k$, leaving all the remaining lines of $\mathbf{G}_2(X)$ unchanged then $\mathfrak{R}_n$ is obtained.

2. The collineation group of $\mathfrak{R}_n$

Let $n > 2$. Recall that $\text{Aut}(\mathbf{G}_2(n)) \cong S_n$ (cf. [7, Prop. 1.5]). Let W be an arbitrary set and let S_W be the group of permutations of W , we define the (standard) action of S_W on $S_{\wp_2(W)}$ by the formula

$$S_W \ni \alpha \mapsto \alpha^\# \in S_{\wp_2(W)},$$

$$\text{where } \alpha^\#(A) = \{\alpha(x) : x \in A\} \text{ for } A \subset W \text{ and } \alpha \in S_W.$$

The above map yields an isomorphism of S_X and the group $\text{Aut}(\mathbf{G}_2(X))$.

At first, we show some immediate automorphisms of the structure $\mathfrak{R}_n$.

Adopt notation of 1.5. In particular, $m = |X|$ and $n = m + 2(k - 1)$. Let us set $Z_j := p^1 \cup \dots \cup p^j$ for $0 < j \leq k$. Next, we set $Y_j := X \cup Z_j$ for $j = 1, \dots, k$ and $Z := Z_k$, $Y := Y_k$. Then $\mathfrak{R}_n$ is defined on $\wp_2(Y)$.

Note that $\mathfrak{R}_m \cong \mathbf{IB}(m)$ for $m \leq 3$ i.e it is the Veblen configuration for $m = 2$ and the Veronese configuration for $m = 3$. In any case we can write

$$\mathfrak{R}_n = \mathbf{M}_{Y_{k-1}}^{p^k} \triangleright (\dots \mathbf{M}_{Y_1}^{p^2} \triangleright \mathfrak{R}_m).$$

The following notation will be also useful:

$$\mathcal{C} := \{p^1, \dots, p^k\}, \quad \text{and} \quad p^0 := \{1, 2\}.$$

First, note the following

PROPOSITION 2.1. *Let $\varphi \in S_Z$. Assume that φ leaves the decomposition of the set Z into the p^j invariant i.e. assume that there is a permutation α of the set $\{1, \dots, k\}$ such that $\varphi^\#(p^j) = p^{\alpha(j)}$ for $j = 1, \dots, k$, and set $\psi = \varphi \cup \text{id}_X$. Then the map $\psi^\#$ is an automorphism of $\mathfrak{R}_n$.*

Proof. First, let us take an arbitrary $\alpha \in S_k$ and define

$$\varphi(1 + 2j) = 1 + 2\alpha(j) \quad \text{and} \quad \varphi(2 + 2j) = 2 + 2\alpha(j) \quad \text{for } j = 1, \dots, k.$$

Then, clearly, $\varphi \in S_Z$ and $\varphi^\#(p^j) = p^{\alpha(j)}$. Put $\psi = \varphi \cup \text{id}_Z$. Evidently, $\psi^\#$ preserves the 3-subsets of $\wp_2(Y_k)$ that are lines of $\mathbf{G}_2(Y_k)$. Note two relations:

$$\begin{aligned} \psi^\#(\{\{1, 2\}, \{1, 2 + 2j\}, \{2, 1 + 2j\}\}) &= \{\{1, 2\}, \{1, 2 + 2\alpha(j)\}, \{2, 1 + 2\alpha(j)\}\}; \\ \psi^\#(\{\{1, 2\}, \{2, 2 + 2j\}, \{1, 1 + 2j\}\}) &= \{\{1, 2\}, \{2, 2 + 2\alpha(j)\}, \{1, 1 + 2\alpha(j)\}\}, \end{aligned}$$

for $1 \leq j \leq k$. In view of 1.7 this proves that $\psi^\#$ is an automorphism of $\mathfrak{R}_n$.

Next, let us consider an arbitrary transposition $\varphi = (1 + 2i, 2 + 2i)$ for fixed i , $1 \leq i \leq k$, as a permutation of the set Z . Then clearly, $\varphi^\#(p^j) = p^j$ for every j . We set $\psi = \varphi \cup \text{id}_X$; as above we prove that ψ yields an automorphism of $\mathfrak{R}_n$. This closes the proof. ■

Informally, 2.1 states that in the configuration $\mathfrak{R}_n$ the “perspective center” p^j can be chosen arbitrarily in $\mathcal{C}$; the configuration has k such centers.

PROPOSITION 2.2. *Let $\varphi = (1, 2) \in S_X$ be the transposition of $1, 2 \in X$ and $\psi = \varphi \cup \text{id}_{Z_k}$. Then $\psi^\#$ is an automorphism of $\mathfrak{R}_n$.*

Proof. In view of 1.7 it suffices to determine the images under the map $\psi^\#$ of the “modified” lines through $p^0 = \{1, 2\}$. We have

$$\begin{aligned} \psi^\#(\{\{1, 2\}, \{1, 2 + 2j\}, \{2, 1 + 2j\}\}) &= \{\{1, 2\}, \{2, 2 + 2j\}, \{1, 1 + 2j\}\}; \\ \psi^\#(\{\{1, 2\}, \{2, 2 + 2j\}, \{1, 1 + 2j\}\}) &= \{\{1, 2\}, \{1, 2 + 2j\}, \{1, 1 + 2j\}\}, \end{aligned}$$

which proves our claim. ■

We are going to prove that every automorphism of $\mathfrak{R}_n$ is a composition of maps given in 2.1 and 2.2. First, we prove that an automorphism $F \in \text{Aut}(\mathfrak{R}_n)$ leaves the set $\mathcal{C}$ invariant. To this aim we recall, first, that the

antineighborhood $\mathcal{N}^-(u)$ of a point u consists of all the points *not collinear* with u (cf. [10]). After that we determine the set

$$\mathcal{M} = \{u: u \text{ is a point of } \mathfrak{R}_n, \mathcal{N}^-(u) \text{ is a subspace of } \mathfrak{R}_n^{(1)}\}.$$

Clearly, the set $\mathcal{M}$ is invariant under the automorphisms of $\mathfrak{R}_n$.

PROPOSITION 2.3. $\mathcal{M} = \mathcal{C} \cup \wp_2(X)$.

If $q \in \mathcal{C}$ then $\mathcal{N}^-(q) \cong \mathfrak{R}_{n-2}$, and if $q \in \wp_2(X)$ then $\mathcal{N}^-(q) \cong \mathbf{G}_2(Y_k \setminus q) \cong \mathbf{G}_2(n)$. Consequently, every automorphism of $\mathfrak{R}_n$ leaves the sets $\mathcal{C}$ and $\wp_2(X)$ invariant.

Proof. It is evident that $p^k \in \mathcal{M}$ and that $\mathcal{N}^-(p^k) = \wp_2(Y_{k-1})$, thus $\mathcal{N}^-(p^k) \cong \mathfrak{R}_{n-2}$. By 2.1 every p^j belongs to $\mathcal{M}$ as well and thus $\mathcal{C} \subset \mathcal{M}$. Moreover, $\mathcal{N}^-(q) \cong \mathfrak{R}_{n-2}$ for every $q \in \mathcal{C}$.

From the (very technical) Lemma 8 in [10] we prove that no point on a line through p^k distinct from p^k belongs to $\mathcal{M}$ (in accordance with Lemma 8 in [10] it suffices to note that $\mathcal{P}_k \wedge \{i, 1, 2\} \approx N_3$ and thus the point a_i on a line through p^k is not in $\mathcal{M}$). Consequently, from 2.1 we infer that for any point v on a line through p^j with $1 \leq j \leq k$ we have $v \notin \mathcal{M}$.

The only points that remain to be examined are $q = \{1, 2\}$ and, when $m = 3$, $q = \{0, 1\}$ and $q = \{0, 2\}$. Here, we use [10, Lemma 7]. First, we directly verify that q and $\{i, j\}$ are collinear iff $q \cap \{i, j\} \neq \emptyset$ for $q = \{1, 2\}, \{0, 1\}, \{0, 2\}$.

Secondly, we note that if $q \in \wp_2(X)$ then the set $T := \wp_2(Y_{k-1} \setminus q)$ does not contain $\{1, 2\}$ and it does not contain any pair of points of the form $(\{1, 2 + 2j\}, \{2, 1 + 2j\})$ or $(\{2, 2 + 2j\}, \{1, 1 + 2j\})$; in view of 1.7 this suffices to state that T is a subspace of $\mathfrak{R}_{n-2}$ isomorphic to $\mathbf{G}_2(Y_{k-1} \setminus q)$. From Lemma 7 of [10] we get that $\mathcal{N}^-(q)$ is a subspace of $\mathfrak{R}_n$; clearly it is isomorphic to $\mathbf{G}_2(Y_k \setminus q)$. ■

The neighborhood $\mathcal{N}^+(u)$ of a point u consists of all the points collinear with u . In the subsequent representations 2.4, 2.5, and 2.6 we shall consider the neighborhood $\mathcal{N}^+(q)$ of a point $q \in \mathcal{M}$. The obtained information will allow us to say more on action of an automorphism of $\mathfrak{R}_n$ on $\mathcal{M}$, but also it will give an insight into the geometry of $\mathfrak{R}_n$.

REPRESENTATION 2.4. Let us adopt $q = p^0$, let $H = Y_k \setminus q$ and $\mathcal{H} = \wp_2(H)$. The set $\mathcal{N}^+(q)$ is the union of two n -simplices $\mathcal{S}_l := \{u \in \wp_2(Y_k): l \in u \neq q\}$, where $l = 1, 2$.

The fact that $\mathcal{S}_l$ is an n -simplex i.e. that any two its points are collinear in $\mathfrak{R}_n$, is trivial. Proposition 1.7 implies that every line through q crosses both $\mathcal{S}_1$ and $\mathcal{S}_2$. Consequently, for any point $u \in \mathcal{S}_1$ there is a line $\overline{q, u}$ of

⁽¹⁾A set $\mathcal{Z}$ of points of $\mathfrak{R}_n$ is a *subspace* if for any distinct collinear points $v, w \in \mathcal{Z}$ the line through v, w is contained in $\mathcal{Z}$.

$\mathfrak{R}_n$ and this line crosses $\mathcal{S}_2$, and conversely, with 1, 2 interchanged. Indeed, if $u = \{1, 2i+1\} \in \mathcal{S}_1$ then $\{2, 2i+2\} \in \mathcal{S}_2 \cap \overline{q, u}$, and if $u = \{1, 2i+2\} \in \mathcal{S}_1$ then $\{2, 2i+1\} \in \mathcal{S}_2 \cap \overline{q, u}$. Finally, if n is odd, for $u := \{0, 1\} \in \mathcal{S}_1$ we have $\{0, 2\} \in \mathcal{S}_2 \cap \overline{q, u}$. This means that q is the center of a perspective ξ of the simplices $\mathcal{S}_1$ and $\mathcal{S}_2$.

A perspective establishes a correspondence between points, but also a correspondence between lines, and in the usual case lines that correspond to each other intersect. But this is not the case here. Indeed, take $u = \{1, 2i+1\}$ and $v = \{1, 2j+1\}$ for $i \neq j$, then $u, v \in \mathcal{S}_1$ and $w := \{2i+1, 2j+1\} \in \overline{u, v}$. We have $\xi(u) = \{2, 2i+2\}$ and $\xi(v) = \{2, 2j+2\}$; clearly $\xi(u), \xi(v) \in \mathcal{S}_2$. Note, however, that the third point on $\xi(\overline{u, v}) = \xi(u), \xi(v)$ is $\{2i+2, 2j+2\} \neq w$ and thus $\overline{u, v}$ and $\xi(u), \xi(v)$ miss each other. A line L that joins two points in $\mathcal{S}_1$ such that the two lines $L, \xi(L)$ intersect has the form $L = \overline{\{1, 2i+1\}, \{1, 2i+2\}}$, and the obtained intersection points $L \cap \xi(L)$ are all the points $\{2i+1, 2i+2\}$ i.e. these are exactly the points in $\mathcal{C}$. This observation allows us to consider $\mathfrak{R}_n$ as a *twisted perspective* of some type (cf. [9]).

Let us write $\mathcal{L}_l$ for the set of lines of $\mathfrak{R}_n$ which join points in $\mathcal{S}_l$ for $l = 1, 2$. Note that, in contrast to the above, *for every $L \in \mathcal{L}_1$ there is (exactly one) $L' =: \xi'(L) \in \mathcal{L}_2$ such that $L \cap L' \neq \emptyset$* . The correspondence ξ' between the lines in $\mathcal{L}_1$ and $\mathcal{L}_2$ – a suitable *line-perspective* – maps concurrent lines onto concurrent lines and thus it determines a correspondence $\xi': \mathcal{S}_1 \rightarrow \mathcal{S}_2$ with $\xi'(\overline{u, v}) = \overline{\xi'(u), \xi'(v)}$ for $u, v \in \mathcal{S}_1$. It is seen that $\xi'(\{1, i\}) = \{2, i\}$ for $i \in Z$ (for $i \in Z \cup \{0\}$, if n is odd) and therefore u and $\xi'(u)$ are not collinear for any $u \in \mathcal{S}_1, u \neq \{1, 0\}$.

The points in $\mathcal{H}$ i.e. the points not in $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \{q\}$ are joined in accordance with the rules of $\mathbf{G}_2(H)$ (let us say: as in an ordinary perspective, cf. [7]). ■

REPRESENTATION 2.5. Assume that $|X| = 3$ i.e. $X = \{0, 1, 2\}$. Let us take $q = \{0, 1\}$ and $H = Y_k \setminus q, \mathcal{H} = \wp_2(H)$. The set $\mathcal{N}^+(q)$ is the union of two n -simplices $\mathcal{S}_l := \{u \in \wp_2(Y_k) : l \in u \neq q\}$, with $l = 0, 1$.

Note that through every $\{0, i\} \in \mathcal{S}_0$ there passes a line which joins q with $\{1, i\} \in \mathcal{S}_1$. This means that q is the center of a perspective ξ , which maps $\mathcal{S}_0$ onto $\mathcal{S}_1$.

Let us determine lines L that join points in $\mathcal{S}_0$ (say: sides of $\mathcal{S}_0$) and satisfy $L \cap \xi(L) \neq \emptyset$. Set $p_2^0 = \{0, 2\}$. Let, first, L miss p_2^0 i.e. $L = \overline{u, v}$ where $u = \{0, i\}, v = \{0, j\}$ with $i, j \neq 2$. Then $\xi(u) = \{1, i\}, \xi(v) = \{1, j\}$, and $\{i, j\}$ is the common point of L and $\xi(L)$. Next, let $u = p_2^0, v = \{0, 2i+1\}$, and $L = \overline{u, v}$. Then $\xi(u) = p_2^0$ and $\xi(v) = \{1, 2i+1\}$. The third point on L is $\{2, 2i+1\}$, while the third point of $\mathfrak{R}_n$ on $\xi(L)$ is $\{2, 2i+2\}$ and

thus L and $\xi(L)$ miss each other. The same relation holds when $v = \{0, 2i + 2\}$. This proves that L and $\xi(L)$ intersect iff L misses one fixed line q, p_2^0 through q . ■

REPRESENTATION 2.6. Finally, let us take $q \in \mathcal{C}$. In view of 2.1 to try to determine a perspective ξ of two n -simplices S' and S'' with the center q it suffices to adopt $q = p^k$ and use the definition of $\mathfrak{R}_n$. The set $\mathcal{N}^+(q)$ consists of the points a_i and b_i , $0 \leq i \leq n$ ($1 \leq i \leq n$, resp.), cf. 1.3. Note, however, that here a procedure applied in 2.4 and 2.5 fails. Indeed, the points a_i with $i \neq 1, 2$ yield one clique and the points b_i yield another clique. And an evident perspective ξ with center p^k maps one of these families onto the other. But while a_1 and a_2 are collinear with all the remaining a_i , b_1 and b_2 are collinear with all the remaining b_i , and $\xi(a_1) = b_1$, $\xi(a_2) = b_2$, the points in pairs a_1, a_2 and b_1, b_2 are not collinear. Thus *neither all the a_i nor all the b_i yield a simplex*. ■

As an immediate consequence of 2.4 and 2.5 we obtain

PROPOSITION 2.7. *Let $n > 3$. Then every automorphism F of $\mathfrak{R}_n$ fixes the point $p^0 \in \mathcal{M}$.*

Note that the automorphisms of $\mathfrak{R}_2$ and $\mathfrak{R}_3$ are known:

$$\text{Aut}(\mathfrak{R}_2) \cong \text{Aut}(\mathbf{B}(2)) \cong \text{Aut}(\mathbf{G}_2(Y)) \cong S_4$$

$$\text{Aut}(\mathfrak{R}_3) \cong \text{Aut}(\mathbf{B}(3)) \cong S_{p^1} \oplus S_X \cong C_2 \oplus S_3 \cong D_6$$

(cf. [11, Fact 2.7], see also [6]). Therefore, in the sequel we can assume that $n > 3$.

THEOREM 2.8. *Let $n \geq 4$. The following conditions are equivalent:*

- (i) F is an automorphism of $\mathfrak{R}_n$.
- (ii) There is a permutation $\Psi \in S_Y$ such that
  - (a) Ψ maps X and its subset p^0 onto itself,
  - (b) Ψ maps Z onto itself and preserves the decomposition $\mathcal{C}$ of $\wp_2(Z)$, and $F = \Psi^\#$.

Proof. The implication (ii) $\Rightarrow$ (i) directly follows by 2.1 and 2.2, so it only remains to prove (i) $\Rightarrow$ (ii).

Let $F \in \text{Aut}(\mathfrak{R}_n)$. Set $I := \{1, \dots, k\}$. By 2.3 there exists $\alpha \in S_k$ such that

$$F(p^j) = p^{\alpha(j)} \text{ for every } j \in I.$$

There exists a permutation $\varphi \in S_Z$ such that $\varphi^\#(p^j) = p^{\alpha(j)}$ for $j \in I$; then

$$G := (\psi^{-1})^\# \circ F \in \text{Aut}(\mathfrak{R}_n) \quad \text{and} \quad G(p^j) = p^j \quad \text{for every } j \in I,$$

$$\text{where } \psi = \varphi \cup \text{id}_X \in S_Y.$$

Moreover, from 2.7, G fixes $p^0 = \{1, 2\} \in \wp_2(X)$. If $m = |X| = 3$, then G permutes the points $\{0, 1\}$ and $\{0, 2\}$. By 2.2 there is a permutation $\psi' \in S_Y$ of the form $\psi' = \text{id}_Z \cup \varphi'$ with $\varphi' \in S_X$ which fixes all the p^j and acts on $\wp_2(X)$ as G does. Let $T = (\psi'^{\#})^{-1} \circ G$; then $T \in \text{Aut}(\mathfrak{R}_n)$. If $m = 2$ then we take $T := G$. In both cases $T(q) = q$ for every $q \in \mathcal{M}$. Considering again the antineighborhoods of the points in $\mathcal{M}$ we get that T leaves $\mathfrak{R}_m$ invariant, as it is spanned suitably by p^1 and $\wp_2(X)$; next, T leaves $\mathfrak{R}_{m+2}$ invariant, as spanned by p^2 and $\mathfrak{R}_m$, and so on. Finally, T leaves the whole sequence $\mathfrak{R}_n \supset \mathfrak{R}_{n-2} \dots \supset \mathfrak{R}_{m+2} \supset \mathfrak{R}_m$ invariant.

Then we proceed as follows. Consider the substructure $\wp_2(Y_1)$ of $\mathfrak{R}_n$, isomorphic to $\mathfrak{R}_m$. Since T fixes $\wp_2(X)$ point-wise, if $m = 3$ we have that T leaves every line through p^1 invariant; set $\lambda^0 := \text{id}_X$ and then $T' := (\lambda_0 \cup \text{id}_Z)^{\#} \circ T = T$. If $m = 2$ then composing T with $(\text{id}_Z \cup \lambda^0)^{\#}$ for a suitable permutation λ^0 of X we can assume that this composition T' leaves every of the (two) lines through p^1 invariant. Consider a line through p^1 contained in $\wp_2(Y_1)$. If T' interchanges any two points on such a line we take as λ^1 the transposition of the elements of p^1 ; otherwise we set $\lambda^1 = \text{id}_{p^1}$. Since T' fixes $\mathbf{G}_2(X)$ point-wise from [10, Lem. 3] we infer that T' restricted to the set $\wp_2(Y_1)$ – the set of points of $\mathfrak{R}_m$ – coincides with $(\lambda^0 \cup \lambda^1)^{\#}$. Let $\sigma_1 := \lambda^0 \cup \lambda^1$ and $T^1 := (\sigma_1^{-1} \cup \text{id}_{p^2 \cup \dots \cup p^k})^{\#} \circ T'$.

Next, we consider the lines through p^2 which, together with the points in $\mathfrak{R}_m$, span $\mathfrak{R}_{m+2}$; since T^1 fixes $\mathfrak{R}_m$ point-wise we get that T^1 leaves every such line invariant. If T^1 interchanges any two points on any of these lines we take as λ^2 the transposition of the elements of p^2 and otherwise we take $\lambda^2 := \text{id}_{p^2}$. Set $\sigma_2 = \sigma_1 \cup \lambda^2$; as above we get that T acts on $\wp_2(Y_2)$ as $(\sigma_2)^{\#}$ acts. We set $T^2 := (\sigma_2^{-1} \cup \text{id}_{p^3 \cup \dots \cup p^k})^{\#} \circ T^1$. Continuing in this fashion we come to $\sigma_k = \bigcup_{j=0}^k \lambda^j \in S_Y$ such that $T = \sigma_k^{\#}$. This proves our claim; it suffices to take $\Psi = \psi \circ \psi' \circ \sigma_k$. ■

COROLLARY 2.9. *If $n > 3$ is odd then through a point p^j ($j = 0, 1, \dots, k$) there passes a line invariant under every automorphism of $\mathfrak{R}_n$ which fixes p^j .*

COROLLARY 2.10. *If $n \geq 4$ then the group $\text{Aut}(\mathfrak{R}_n)$ contains $k! 2^{k+1}$ elements.*

3. A visualization of $\mathfrak{R}_n$

The Veronese configuration $\mathfrak{R}_3$ is essentially known. Recall a further interpretation: $\mathfrak{R}_3$ is obtained from a hexagon inscribed into three lines with a common point p – its *center* – by completing it by three points of intersection of opposite sides of the hexagon, so called *focuses*, and a line – *an axis* – which joins these three focuses (cf. e.g. [11]).

CONSTRUCTION 3.1. Let $n = 2k$ or $n = 2k + 1$. We define the partial Steiner triple system $\mathfrak{M}_n$ as follows. Let p^0, p^j, a_i^j, b_i^j for $i = 1, 2, j = 1, \dots, k$ be points of $\mathfrak{M}_n$ for even n ; if n is odd, we add the points $p_i^0, i = 1, 2$, and $a_0^j, b_0^j, j = 1, \dots, k$.

* Let n be odd. The set $L_0 = \{p^0, p_1^0, p_2^0\}$ is a line of $\mathfrak{M}_n$.

Let $\mathcal{D}^j = \{a_0^j, a_1^j, a_2^j, b_0^j, b_1^j, b_2^j\}$. For each $0 < j \leq k$ we define on the set $L_0 \cup \{p^j\} \cup \mathcal{D}^j$ the Veronese configuration $\mathcal{V}^j$ with center p^j and axis L_0 so that the map $p^j \mapsto p, a_i^j \mapsto a_i, b_i^j \mapsto b_i$ extends to an isomorphism of $\mathcal{V}^j$ and $\mathfrak{R}_3 = \mathbf{MP}_{\{0,1,2\} \triangleright_{\mathcal{P}}} \mathbf{G}_2(\{0, 1, 2\})$, where $\mathcal{P} = \{\{0, 1\}, \{0, 2\}\}$ (cf. 1.3).

* Let n be even. Set $\mathcal{Q}^j = \{a_1^j, a_2^j, b_1^j, b_2^j\}$. For each $0 < j \leq k$ we define on the set $\{p^j, p^0\} \cup \mathcal{Q}^j$ the Veblen configuration $\mathcal{V}^j$ which is in an analogous way isomorphic to $\mathfrak{R}_2 = \mathbf{MP}_{\{1,2\} \triangleright_{N_{\{1,2\}}}} \mathbf{G}_2(\{1, 2\})$.

* Next, we introduce the points $a'_{\{j,l\}}, b'_{\{j,l\}}, \mathcal{d}_{j,l}$ by the requirements

$$(2) \quad a'_{\{j,l\}} \in \overline{a_i^j, a_i^l}, \quad b'_{\{j,l\}} \in \overline{b_i^j, b_i^l}, \quad \mathcal{d}_{j,l} \in \overline{a_i^j, b_i^l} \quad \text{for } i = 1, 2$$

and for any distinct $0 < j, l \leq k$ ⁽²⁾. If n is odd, we require that (2) holds also for $i = 0$.

* After that we define the class of lines of the form

$$(3) \quad \{p^j, \mathcal{d}_{l,j}, a'_{\{l,j\}}\}, \quad \{p^j, \mathcal{d}_{j,l}, b'_{\{j,l\}}\} \quad \text{for distinct } 0 < j, l \leq k,$$

so the points $p^j, p^l, a'_{\{j,l\}}, \mathcal{d}_{j,l}, \mathcal{d}_{l,j}, b'_{\{j,l\}}$ with $j \neq l, 0 < j, l \leq k$ yield in $\mathfrak{M}_n$ a Veblen figure. Finally, we introduce the following classes of lines:

$$(4) \quad \{a'_{\{i,j\}}, a'_{\{j,l\}}, a'_{\{i,l\}}\}, \quad \{b'_{\{i,j\}}, b'_{\{j,l\}}, b'_{\{i,l\}}\}, \quad \{\mathcal{d}_{i,l}, \mathcal{d}_{j,l}, a'_{\{i,j\}}\}, \\ \{\mathcal{d}_{l,i}, \mathcal{d}_{l,j}, b'_{\{i,j\}}\}$$

for pairwise distinct $i, j, l, 1 \leq i, j, l \leq k$.

The points of $\mathfrak{M}_n$ are p^0, p^j, a_i^j, b_i^j (and p_i^0 , if need be) and $a'_{\{j,l\}}, b'_{\{j,l\}}, \mathcal{d}_{j,l}$. ■

THEOREM 3.2. *The structures $\mathfrak{R}_n$ and $\mathfrak{M}_n$ are isomorphic.*

Proof. It suffices to introduce the following labelling of the points of $\mathfrak{R}_n$: $p^0 = \{1, 2\}$, $p^j = \{2j + 1, 2j + 2\}$, $a_i^j := \{i, 2j + 1\}$, $b_i^j := \{i, 2j + 2\}$, for $0 < j \leq k$ and $i = 1, 2$ and, if n is odd, $p_i^0 = \{0, i\}$, $a_0^j := \{0, 2j + 1\}$, $b_0^j := \{0, 2j + 2\}$; $a'_{\{i,j\}} := \{2i + 1, 2j + 1\}$, $\mathcal{d}_{i,j} := \{2i + 1, 2j + 2\}$, and $b'_{\{i,j\}} := \{2i + 2, 2j + 2\}$. Direct verification of the required collinearities is based on 1.7. ■

⁽²⁾The symbol $\mathcal{d}$ denoting a join of lines of the form $\overline{a, b}$, should be read as $a-b$.

Comparing Constructions 3.1 and 1.3 we note that $a_i^j = c_{\{i,2j+1\}}$, $b_i^j = c_{\{i,2j+2\}}$ for $0 < j < k$, and $a_i^k = a_i$, $b_i^k = b_i$, for $i = 1, 2$. Note that 3.1 can be used as a starting point for further generalizations consisting in modifications of relations (3) and (4).

Next, let us assume that $\mathfrak{R}_n$ is embedded into a Desarguesian projective space $\mathfrak{P}$. Now, the symbols $\bar{u}, \bar{v}$ and L_0 refer to the corresponding projective lines. Then some further incidences not mentioned in 3.1 appear. The following is evident.

LEMMA 3.3. *For every $0 < j \leq k$ in $\mathfrak{P}$ there is a plane Π^j that contains $\mathcal{D}^j$ ($\mathcal{Q}^j$, resp.).*

LEMMA 3.4. *Let $0 < j, l \leq k$ be distinct. In $\mathfrak{P}$ the points $a'_{\{j,l\}}$, $\bar{d}_{j,l}$, $\bar{d}_{l,j}$, $b'_{\{j,l\}}$ yield a complete projective quadrangle with the diagonal points p^0, p^i, p^j .*

Proof. In view of 3.1, 3.2 it only remains to verify that p^0 lies on $\overline{a'_{\{j,l\}}}, \overline{b'_{\{j,l\}}}$ and on $\overline{\bar{d}_{l,j}}, \overline{\bar{d}_{j,l}}$. On the other hand this condition follows from the criterion $p^0 = \overline{a_{2i+1}}, \overline{b_{2i+2}} \cap \overline{b_{2i+1}}, \overline{a_{2i+2}}$ ($1 \leq i \leq k$) of projective realizability of $\mathfrak{R}_n$ (cf. [8, Prop. 2.10 and Rem. 2.14]) and 1.7. ■

LEMMA 3.5. *Let us consider the following projective points:*

$$(5) \quad q^j := \overline{a_1^j}, \overline{a_2^j} \cap \overline{b_1^j}, \overline{b_2^j}, \quad j = 1, \dots, k.$$

Then $q^j = q^k$ for every $0 < j \leq k$, so we can write simply $q^0 = q^k$.

If n is even we set $L_0 := \overline{p^0}, \overline{q^0}$; the planes Π^j have the line L_0 in common.

In any case the points $a_1^j, a_2^j, b_1^j, b_2^j$ yield a complete projective quadrangle with diagonal points p^j, p^0, q^0 contained in Π^j .

Proof. The only non evident relation is $q^j = q^k$. As we noted, $a_i = a_i^k$ and $b_i = b_i^k$. Let $c_{\{i,j\}}^*$ be a projective point such that the points a_i, b_i, a_j, b_j yield a complete projective quadrangle with the diagonal points $p^k, c_{\{i,j\}}^*, c_{\{i,j\}}^*$. Let $\mathbf{L}$ denote the ternary relation of collinearity of points in $\mathfrak{P}$. Note that $c_{\{1,2\}}^* = q^k$. By the Desargues Axiom applied to the triangles a_1, a_2, a_{2j+1} and b_1, b_2, b_{2j+1} we obtain $\mathbf{L}(c_{\{1,2j+1\}}, c_{\{2,2j+1\}}, c_{\{1,2\}}^*)$; since $c_{\{1,2j+1\}} = a_1^j$ and $c_{\{2,2j+1\}} = a_2^j$ this relation can be written as $\mathbf{L}(a_1^j, a_2^j, c_{\{1,2\}}^*)$. Analogously we get $\mathbf{L}(c_{\{1,2j+2\}}, c_{\{2,2j+2\}}, c_{\{1,2\}}^*)$, i.e. $\mathbf{L}(b_1^j, b_2^j, c_{\{1,2\}}^*)$. This gives us now $c_{\{1,2\}}^* = q^j$, i.e. $q^j = q^k$. ■

This series of statements allow us to obtain several results which state that $\mathfrak{R}_n$ is "projectively spanned" by some its subconfigurations.

PROPOSITION 3.6. *Let $n = 2k$ and let $\mathfrak{R}_n$ be embedded into a Desarguesian projective space $\mathfrak{P}$. Assume that $\Pi^j \neq \Pi^l$ for $j \neq l$. Let us complete*

the Veblen configuration $\mathcal{Q}^k$ to a Veronese configuration contained in Π^k by adding a new line through p^k and two points a_0^k, b_0^k on it ⁽³⁾ After that we find in $\mathfrak{P}$ the common points of the lines a_0, a_i, b_0, b_i with $i = 3, \dots, 2k$, corresponding points are labelled as $\{0, i\} = c_{\{0, i\}}$. The obtained points yield the configuration $\mathfrak{R}_{2k+1}$ in $\mathfrak{P}$.

Proof. In accordance with Definition 1.5 we must verify that the new points and the points on the lines through $p^1, p^2, \dots, p^k$ with appropriate indices yield $\mathfrak{R}_{2k+1}$, with the collinearity relation $\mathbf{L}$ inherited from the underlying projective space.

It suffices to take arbitrary p^j with $j < k$ and prove that a Veronese configuration $\mathcal{V}^j$ is obtained. The collinearities that must be verified are those which refer to new points, i.e. those with indices 0. By the way of the example we shall prove explicitly one of them: *the triple $\{p^j, a_0^j, b_0^j\}$ is collinear*. By the Desargues Axiom applied to the triangles a_0, a_{2j+1}, a_{2j+2} and b_0, b_{2j+1}, b_{2j+2} we infer $\mathbf{L}(c_{\{2j+1, 2j+2\}}, c_{\{0, 2j+1\}}, c_{\{0, 2j+2\}})$, which is exactly the desired claim. Another required collinearities are proved in the same way. ■

PROPOSITION 3.7. *Let a family $\{\mathcal{Q}^j: j = 1, \dots, k\}$ of complete projective quadrangles with common two diagonal points p^0, q^0 be contained in a Desarguesian projective space $\mathfrak{P}$. Let their points be named as in 3.1 and 3.5. Finally, let the points $a'_{\{j, l\}}, \mathcal{C}_{l, j}, \mathcal{C}_{j, l}, b'_{\{j, l\}}$ satisfy (2) (their existence follows from the Veblen condition). The points of the given complete quadrangles (with q^0 excluded) together with the points $a'_{\{j, l\}}, \mathcal{C}_{l, j}, \mathcal{C}_{j, l}, b'_{\{j, l\}}, j \neq l, 0 < j, l \leq k$ yield the configuration $\mathfrak{R}_{2k}$ in $\mathfrak{P}$.*

Proof. First, we must prove that the relations (3) hold. To this aim it suffices to consider suitable perspective triangles, e.g. considering the triangles (a_1^l, a_1^j, b_1^l) and (a_2^l, a_2^j, b_2^l) we obtain $\mathbf{L}(p^l, a'_{\{j, l\}}, \mathcal{C}_{j, l})$.

To complete the proof it suffices to verify that the triples given in (4) are collinear in $\mathfrak{P}$ for pairwise distinct $i, j, l = 1, \dots, k$. The line through $a'_{\{i, j\}}, a'_{\{j, l\}}, a'_{\{i, l\}}$ is the perspective axis of the two triangles (a_1^i, a_1^j, a_1^l) and (a_2^i, a_2^j, a_2^l) with the perspective center q^0 . Analogously, the line through $b'_{\{i, j\}}, b'_{\{j, l\}}, b'_{\{i, l\}}$ is the perspective axis of the triangles (b_1^i, b_1^j, b_1^l) and (b_2^i, b_2^j, b_2^l) . Finally, the line through $a'_{\{i, j\}}, \mathcal{C}_{l, j}, \mathcal{C}_{j, l}$ is the perspective axis of (a_1^i, a_1^j, b_1^l) and (a_2^i, a_2^j, b_2^l) , and the line through $b'_{\{i, j\}}, \mathcal{C}_{l, j}, \mathcal{C}_{j, l}$ is the perspective axis of (b_1^i, a_1^l, b_1^j) and (b_2^i, a_2^l, b_2^j) . ■

⁽³⁾Necessary and sufficient conditions which state precisely how to construct the points a_0^k, b_0^k are given in [11, Facts 5.1, 5.2, Prop. 5.3]; they are inessential here, as we only need to know that our construction yields a required Veronese configuration.

As an immediate consequence we get

PROPOSITION 3.8. *Let a family $\{\mathcal{V}^j: j = 1, \dots, k\}$ of Veronese configurations with points named as in 3.1 and with common focuses on a line L_0 be contained in a Desarguesian projective space $\mathfrak{P}$ such that $q^j = q^k$ for $0 < j \leq k$ (cf. (5)). Let the points $a'_{\{j,l\}}, \mathfrak{d}_{l,j}, \mathfrak{d}_{j,l}, b'_{\{j,l\}}$ satisfy (2). The points of the given Veronese configurations together with the points $a'_{\{j,l\}}, \mathfrak{d}_{l,j}, \mathfrak{d}_{j,l}, b'_{\{j,l\}}, j \neq l, 0 < j, l \leq k$ yield the configuration $\mathfrak{R}_{2k+1}$ in $\mathfrak{P}$.*

Proof. In view of 3.7 it remains to prove (2) with $i = 0$. The required collinearities all follow directly from the Desargues Axiom. One example: considering the pair of perspective triangles $(a_1^i, a_1^j, p_1^0), (a_2^i, a_2^j, p_2^0)$ we obtain $\mathbf{L}(a'_{\{i,j\}}, a_0^i, a_0^j)$. ■

Finally, note an interesting observation. From Proposition 5.3 of [11] it follows that if n is odd and one triple of pairwise noncollinear points of $\mathcal{D}^j$ lie on a line, then the remaining three points of $\mathcal{D}^j$ are collinear as well, i.e. there is a pair of lines C_1^j, C_2^j such that $a_1^j, a_2^j, b_0^j \in C_1^j$ and $b_1^j, b_2^j, a_0^j \in C_2^j$. Consequently, if $\mathfrak{P}$ is Pappian, $\mathcal{D}^j$ is contained in a (possibly degenerate) conic C^j in Π^j . Using elementary techniques of projective geometry one can prove the following.

LEMMA 3.9. *Let $\mathfrak{R}_{2k+1}$ be embedded into a projective space $\mathfrak{P}$ such that the planes Π^i, Π^j are distinct for $i \neq j$. Assume that $\mathcal{D}^j$ is inscribed into a pair of lines for some j . Then $\mathcal{D}^i$ cannot be inscribed into a pair of lines for $i \neq j$.*

Therefore, in view of 3.9, if $\mathfrak{P}$ is Pappian and $\mathfrak{R}_{2k+1}$ is contained in $\mathfrak{P}$ as required in 3.9 then every, except possibly exactly one, Veronese configuration $\mathcal{V}^j$ contained in $\mathfrak{R}_{2k+1}$ is associated with a nondegenerate conic. Therefore, it seems natural to make the following

CONJECTURE 3.10. *Let n be odd and $\mathfrak{R}_n$ be embedded into a Pappian projective space $\mathfrak{P}$. In $\mathfrak{P}$ there is a quadric which contains all the sets C^j with $j = 1, \dots, k$.*

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