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A SHORT PROOF OF THE SEPARABLE REDUCTION THEOREM

Abstract. We present a simple proof of the separable reduction theorem, a crucial result of nonsmooth analysis which allows to extend to Asplund spaces the results known for separable spaces dealing with Fréchet subdifferentials. It relies on elementary results in convex analysis and avoids certain technicalities.

The separable reduction theorem is an important result of nonsmooth analysis. It enables to pass from fuzzy sum rules in spaces with smooth norms to fuzzy sum rules in general Asplund spaces. Asplund spaces form the appropriate setting for such approximate rules and for extremal principles ([3], [5], [4], [6], [7], [8], [9], [13], [14], [15]). The proof of that result is rather sophisticated and long (see [1, pp. 243–258], [14, pp. 183–221]). It is the purpose of this note to present a short proof. While the core of the proof uses arguments similar to the ones in the original proofs, the simplification stems from elementary results about convexification. Such results of independent interest are gathered in Section 1. The characterization of nonemptiness of the Fréchet subdifferential of a function is recalled in Section 2. The proof of the separable reduction theorem is the object of Section 3. The last section is devoted to known consequences of the theorem which may motivate the study and give an idea of its usefulness.

1. Useful facts from convex analysis

We assume the reader has a basic knowledge of convex analysis. In particular, we assume some familiarity with the *Fenchel-Moreau subdifferential*: for a function $f : X \rightarrow \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ on a normed vector space X , finite

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at $\bar{x} \in X$, it is defined by

$$(1) \quad \partial f(\bar{x}) := \partial_{FM} f(\bar{x}) := \{\bar{x}^* \in X^* : \forall x \in X \ f(x) \geq f(\bar{x}) + \langle \bar{x}^*, x - \bar{x} \rangle\}.$$

We recall that when f is convex, finite at $\bar{x}$ and $g : X \rightarrow \mathbb{R}$ is convex continuous, one has

$$(2) \quad \partial(f + g)(\bar{x}) = \partial f(\bar{x}) + \partial g(\bar{x}).$$

The following extension result is also a consequence of the Hahn-Banach theorem, as observed to us by M. Fabian. We derive it from (2). In the sequel, given a function f on X and a subspace W of X , we denote by $f|_W$ the restriction of f to W .

LEMMA 1. *Let W be a linear subspace of a normed vector space X and let $g : X \rightarrow \mathbb{R}$ be a convex continuous function. Given $z \in W$ and $z_W^* \in \partial(g|_W)(z)$ there exists some $z^* \in \partial g(z)$ such that $z^*|_W = z_W^*$.*

Proof. By the Hahn-Banach theorem, there exists some $y^* \in X^*$ which extends z_W^* . Let ι_W be the indicator function of W : it is defined by $\iota_W(w) := 0$ for $w \in W$, $\iota_W(x) := +\infty$ for $x \in X \setminus W$. Then, we clearly have $y^* \in \partial(g + \iota_W)(z)$. Now, definition (1) shows that $\partial \iota_W(z) = W^\perp := \{x^* \in X^* : x^*|_W = 0\}$ and since g is convex continuous, relation (2) ensures that

$$\partial(g + \iota_W)(z) = \partial g(z) + \partial \iota_W(z) = \partial g(z) + W^\perp.$$

Thus we can find $z^* \in \partial g(z)$ satisfying $y^* - z^* \in W^\perp$, hence $z^*|_W = z_W^*$. ■

Let us recall the concept of calmness. A function $f : X \rightarrow \mathbb{R}_\infty$ finite at $\bar{x} \in X$ is said to be *calm at $\bar{x}$* if there exist $c \in \mathbb{R}_+$ and a neighborhood V of $\bar{x}$ such that $f(x) - f(\bar{x}) \geq -c\|x - \bar{x}\|$ for all $x \in V$. If one can take $V = X$, one says that f is *globally calm at $\bar{x}$* . When f is convex, the two notions coincide. The *calmness rate* of f at $\bar{x}$ is the infimum $\gamma_f(\bar{x})$ of the constants $c > 0$ for which the preceding inequality is satisfied for some neighborhood of $\bar{x}$. The *remoteness* of a nonempty subset S of a nvs is the number $\rho(S) := \inf\{\|s\| : s \in S\}$. Such a simple notion enables us to give a quantitative version of a useful characterization of the subdifferentiability of a convex function g at some $\bar{x}$, i.e. of the nonemptiness of $\partial g(\bar{x})$.

LEMMA 2. *A convex function $g : X \rightarrow \mathbb{R}_\infty$ finite at some $\bar{x} \in X$ is subdifferentiable at $\bar{x}$ iff it is globally calm at $\bar{x}$, iff it is calm at $\bar{x}$. Moreover, the calmness rate of g at $\bar{x}$ is equal to the remoteness $\rho(\partial_F g(\bar{x})) := \inf\{\|x^*\| : x^* \in \partial_F g(\bar{x})\}$ of $\partial_F g(\bar{x})$.*

Proof. If $\partial g(\bar{x})$ is nonempty, for any element $\bar{x}^* \in \partial g(\bar{x})$ one can take $c = \|\bar{x}^*\|$ to get global calmness. Conversely, if one can find $c \in \mathbb{R}_+$ such that $g(x) + c\|x - \bar{x}\| \geq g(\bar{x})$ for any $x \in X$, then the sum rule yields $0 \in \partial g(\bar{x}) + c\partial\|\cdot\|(0)$, so that $\partial g(\bar{x})$ is nonempty and in fact contains an element of

$cB_{X^*} = c\partial\|\cdot\|(0)$. If g is calm at $\bar{x}$, the convex function $x \mapsto g(x) + c\|x - \bar{x}\|$ attains a local minimum at $\bar{x}$, hence a global minimum at $\bar{x}$. ■

Let us make clear some facts about convexification. Let us recall that the *convex hull* $\text{conv}(E)$ of a subset E of a nvs Z is the smallest convex subset C of Z containing E . Introducing the set $\mathbb{N}_m$ of the m first positive integers and the canonical simplex Δ_m by

$$\mathbb{N}_m := \{i \in \mathbb{N} : 1 \leq i \leq m\}, \quad \Delta_m = \left\{ (t_1, \dots, t_m) \in \mathbb{R}_+^m, \sum_{j=1}^m t_j = 1 \right\},$$

the convex hull of E can be described explicitly as

$$C := \text{conv}(E) \\ = \{x = t_1x_1 + \dots + t_mx_m : m \geq 1, t := (t_1, \dots, t_m) \in \Delta_m, x_i \in E\},$$

as is well known and easy to check. The convex hull $\text{conv}(h)$ of a function $h : Z \rightarrow \mathbb{R}_\infty$ is the greatest convex function g majorized by h . Its epigraph is almost the convex hull of the epigraph E of h . In fact, it is the vertical closure of $\text{conv}(E)$: one has $\text{epi}_s g \subset \text{conv}(E) \subset \text{epi } g$, where $\text{epi}_s g := \{(x, r) \in X \times \mathbb{R} : r > g(x)\}$ as easily seen. The fact that these sets are distinct has been pointed out to us by C. Zălinescu who provided the example of $h : \mathbb{R} \rightarrow \mathbb{R}$ given by $h(0) := 1$, $h(x) := |x|$ for $x \in \mathbb{R} \setminus \{0\}$. Thus

$$g(x) := \inf \left\{ \sum_{i=1}^m t_i h(x_i) : m \geq 1, t := (t_1, \dots, t_m) \in \Delta_m, \right. \\ \left. x_i \in X, t_1x_1 + \dots + t_mx_m = x \right\}.$$

LEMMA 3. (a) *Given a sequence (E_n) of nonempty subsets of a vector space Z , the convex hull C of the union E of the E_n 's is the union over $p \in \mathbb{N} \setminus \{0\}$ of the convex hulls C_p of $E_1 \cup \dots \cup E_p$:*

$$(3) \quad C := \text{conv}(E) = \bigcup_p C_p \quad \text{where} \quad C_p := \text{conv} \left(\bigcup_{n \leq p} E_n \right).$$

Denoting by $J_{m,p}$ the set of maps $j : \mathbb{N}_m \rightarrow \mathbb{N}_p$, for $m, p \in \mathbb{N} \setminus \{0\}$, the set C_p is given by

$$(4) \quad C_p := \bigcup_{m \geq 1} \bigcup_{j \in J_{m,p}} \left\{ \sum_{i=1}^m t_i x_i : t := (t_1, \dots, t_m) \in \Delta_m, x_i \in E_{j(i)} \right\}.$$

(b) *Given a sequence (h_n) of functions on a vector space X , the convex hull k of the function $h := \inf_n h_n$ is the infimum over $p \in \mathbb{N} \setminus \{0\}$ of the convex hulls $k_p := \text{conv}(h_1, \dots, h_p)$ of the functions $h_1, \dots, h_p$. The function k_p is*

given by

$$(5) \quad k_p(x) := \inf_{m \geq 1} \inf_{j \in J_{m,p}} \inf \left\{ \sum_{i=1}^m t_i h_{j(i)}(x_i) : (t_1, \dots, t_m) \in \Delta_m, \right. \\ \left. x_i \in X, \sum_{i=1}^m t_i x_i = x \right\}.$$

Notice that in general, the union of a family (C_p) of convex subsets is no more convex; but when the sequence (C_p) is increasing (with respect to inclusion), the union is convex. Similarly, the infimum of a countable family (k_p) of convex functions is convex when the sequence (k_p) is decreasing; but that is not the case if the sequence (k_p) does not satisfies this property.

Proof. (a) In fact, any element of C can be written as a convex combination of a finite family of elements of E , hence is an element of C_p for some p . The reverse containment is obvious since $C_p \subset C$ for all p .

The right-hand side of (4) is clearly contained in C_p . Using the fact that the concatenation of an element j of $J_{m,p}$ with an element j' of $J_{n,p}$ is an element of $J_{m+n,p}$, it is easily seen that this set is convex and contains all E_n for $n \in \mathbb{N}_p$, it coincides with C_p .

(b) Now, when E_n is the epigraph of a function h_n , the vertical closure of C_p is the epigraph of $k_p := \text{conv}(h_1, \dots, h_p)$, the greatest convex function majorized by $h_1, \dots, h_p$.

The right-hand side of (5) defines a function which is clearly bounded below by k_p . Since it is easily seen that it is convex and bounded above by h_n for all $n \in \mathbb{N}_p$, it coincides with k_p . One can also derive this formula from (4) by using epigraphs. ■

2. A characterization of Fréchet subdifferentiability

We devote the present section to a characterization of the nonemptiness of the Fréchet subdifferential $\partial_F f(\bar{x})$ of a (nonconvex, nonsmooth) function f at some point $\bar{x}$ of its domain, $\text{dom} f$. Recall that

$$\partial_F f(\bar{x}) := \{\bar{x}^* \in X^* : \liminf_{x \rightarrow 0, x \neq 0} \frac{f(\bar{x} + x) - f(\bar{x}) - \langle \bar{x}^*, x \rangle}{\|x\|} \geq 0\}.$$

Clearly, f is Fréchet differentiable at $\bar{x}$ if, and only if, $\partial_F f(\bar{x}) \cap (-\partial_F(-f)(\bar{x}))$ is nonempty. Simple examples show that the nonemptiness of $\partial_F f(\bar{x})$ may frequently occur when X is a nice space; but here X is an arbitrary Banach space. We will use the following notion of approximate Fréchet subdifferential. Given an arbitrary function $f : X \rightarrow \overline{\mathbb{R}}$ and $\varepsilon > 0$ the ε -Fréchet subdifferential $\partial_F^\varepsilon f(\bar{x})$ of f at $\bar{x} \in \text{dom} f$ is the set of all $\bar{x}^* \in X$ such that

there exists $\rho > 0$ for which

$$(6) \quad f(\bar{x} + x) - f(\bar{x}) - \langle \bar{x}^*, x \rangle \geq -\varepsilon \|x\| \quad \forall x \in \rho B_X.$$

When f is convex one has $\partial_F^\varepsilon f(\bar{x}) = \partial(f + \varepsilon \|\cdot - \bar{x}\|)(\bar{x})$. Clearly one has

$$\partial_F f(\bar{x}) = \bigcap_{\varepsilon > 0} \partial_F^\varepsilon f(\bar{x}).$$

One can give a characterization of the nonemptiness of the sets $\partial_F^\varepsilon f(\bar{x})$ and $\partial_F f(\bar{x})$ which parallels the one we gave in the convex case. For such a purpose, given $\bar{x} \in X$, a function $f : X \rightarrow \overline{\mathbb{R}}$ finite at $\bar{x}$ and $\varepsilon, \rho > 0$, we introduce the function $f^{\varepsilon, \rho}$ given by

$$\begin{aligned} f^{\varepsilon, \rho}(x) &:= f(\bar{x} + x) - f(\bar{x}) + \varepsilon \|x\| \quad \text{for } x \in \rho B, \\ f^{\varepsilon, \rho}(x) &:= +\infty \quad \text{for } x \in X \setminus \rho B. \end{aligned}$$

Here $B := B_X$ is the closed unit ball of X .

LEMMA 4. *Given an arbitrary function $f : X \rightarrow \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ and $\varepsilon > 0$, the ε -Fréchet subdifferential $\partial_F^\varepsilon f(\bar{x})$ of f at $\bar{x} \in \text{dom } f$ contains some element of norm at most $c \in \mathbb{R}_+$ if and only if there exists $\rho > 0$ such that $\text{conv}(f^{\varepsilon, \rho})(\cdot) \geq -c \|\cdot\|$.*

The last relation can be rephrased more explicitly as follows: for all $m \geq 1$, $(t_1, \dots, t_m) \in \Delta_m$, $x_1, \dots, x_m \in \rho B$

$$\sum_{i=1}^m t_i (f(\bar{x} + x_i) + \varepsilon \|x_i\|) \geq f(\bar{x}) - c \left\| \sum_{i=1}^m t_i x_i \right\|.$$

Proof. Let $\bar{x}^* \in \partial_F^\varepsilon f(\bar{x})$ and let $\rho > 0$ be as in the definition of this set. Then, for $m \geq 1$ and $(t_1, \dots, t_m) \in \Delta_m$, $x_1, \dots, x_m \in \rho B$ we have

$$f(\bar{x} + x_i) + \varepsilon \|x_i\| \geq f(\bar{x}) + \langle \bar{x}^*, x_i \rangle \quad i = 1, \dots, m.$$

Multiplying both sides by t_i and summing, we obtain the expected relation by taking $c := \|\bar{x}^*\|$.

Conversely, suppose $\text{conv}(f^{\varepsilon, \rho})(\cdot) \geq -c \|\cdot\|$ for some $\rho > 0$, $c \geq 0$. Then, since $\text{conv}(f^{\varepsilon, \rho})(0) \leq f^{\varepsilon, \rho}(0) = 0$, we have $\text{conv}(f^{\varepsilon, \rho})(x) \geq -c \|x\| \geq \text{conv}(f^{\varepsilon, \rho})(0) - c \|x\|$, hence $\text{conv}(f^{\varepsilon, \rho})(0) = 0$ and Lemma 2 yields some $\bar{x}^* \in \partial(\text{conv}(f^{\varepsilon, \rho}))(0) \cap c B_{X^*}$. Then, for $x \in \rho B$, we get

$$f^{\varepsilon, \rho}(x) \geq \text{conv}(f^{\varepsilon, \rho})(x) \geq \text{conv}(f^{\varepsilon, \rho})(0) + \langle \bar{x}^*, x \rangle = \langle \bar{x}^*, x \rangle.$$

That ensures that $\bar{x}^* \in \partial_F^\varepsilon f(\bar{x})$. ■

Now let us characterize the nonemptiness of the set $\partial_F f(\bar{x})$. In the sequel (ε_n) is a fixed sequence of positive numbers with limit 0.

LEMMA 5. *Given $c \in \mathbb{R}_+$, $(\varepsilon_n) \rightarrow 0_+$, and an arbitrary function $f : X \rightarrow \overline{\mathbb{R}}$ finite at $\bar{x}$, one has $\partial_F f(\bar{x}) \cap c B_{X^*} \neq \emptyset$ if, and only if, there exists a sequence*

(ρ_n) of positive numbers such that, for $f^{\varepsilon, \rho}$ given as in the preceding lemma, one has

$$(7) \quad \forall x \in X \quad \text{conv}(\inf_n f^{\varepsilon_n, \rho_n})(x) \geq -c\|x\|.$$

It could be added that $\bar{x}^* \in \partial_F f(\bar{x}) \cap cB_{X^*}$ if, and only if, there exists a sequence (ρ_n) of positive numbers such that $\text{conv}(\inf_n f^{\varepsilon_n, \rho_n})(0) = 0$ and $\bar{x}^* \in \partial_F \text{conv}(\inf_n f^{\varepsilon_n, \rho_n})(0)$. However, our aim is to obtain a condition which just involves an estimate about this function and no element of the dual space.

Proof. Let $\bar{x}^* \in \partial_F f(\bar{x}) \cap cB_{X^*}$. Let $\rho_n > 0$ be such that

$$f(\bar{x} + x) - f(\bar{x}) - \langle \bar{x}^*, x \rangle \geq -\varepsilon_n \|x\| \quad \forall x \in \rho_n B$$

and let $g := \text{conv}(\inf_n f^{\varepsilon_n, \rho_n})$. Then, for all $x \in X$ and n , one has $f^{\varepsilon_n, \rho_n}(x) \geq \langle \bar{x}^*, x \rangle$ hence $g(x) \geq \langle \bar{x}^*, x \rangle$, in particular $g(0) \geq 0$ and in fact $g(0) = 0$ since $f^{\varepsilon_n, \rho_n}(0) = 0$ for all n . Thus $\bar{x}^* \in \partial g(0)$. Using Lemma 2, we get (7). Conversely, this last relation ensures that $\partial g(0) \cap cB_{X^*} \neq \emptyset$. Now, for every $\bar{x}^* \in \partial g(0)$ we have $\bar{x}^* \in \partial_F f(\bar{x})$ since for each $n \in \mathbb{N}$ and each $x \in X$ we have

$$f^{\varepsilon_n, \rho_n}(x) \geq g(x) \geq \langle \bar{x}^*, x \rangle,$$

hence $\bar{x}^* \in \partial_F^{\varepsilon_n} f(\bar{x})$ (since $f^{\varepsilon_n, \rho_n}(0) = 0$). ■

Applying Lemmas 2 and 3 to $h_n := f^{\varepsilon_n, \rho_n}$, we see that relation (7) holds if, and only if, for all $p \in \mathbb{N} \setminus \{0\}$ one has

$$(8) \quad \forall x \in X \quad \text{conv}(h_1, \dots, h_p)(x) \geq -c\|x\|.$$

3. Simple separable reduction

The following striking result has an independent interest, but it is not our final aim.

THEOREM 6. *Let $f : X \rightarrow \mathbb{R}_\infty$ be a lower semicontinuous function finite at $\bar{x}$ and let $c \in \mathbb{R}_+$. Then, for every separable subspace W_0 of X there exists a separable subspace W containing W_0 such that for any $w \in W$ the relation $\partial_F f(w) \cap cB_{X^*} \neq \emptyset$ holds if, and only if, $\partial_F(f|_W)(w) \cap cB_{W^*} \neq \emptyset$ holds.*

Proof. Without loss of generality, we may assume that $\varepsilon_n \leq 1$ for all n and, changing f into $c^{-1}f$ if necessary, $c \leq 1$. We construct a sequence $((W_k, A_k))_k$ of pairs such that W_k is a separable subspace of X and A_k is a dense countable subset of W_k . We start with W_0 and take for A_0 any dense countable subset of W_0 .

Assume we have constructed (W_n, A_n) for $n = 0, \dots, k$. Since f is lsc, for all $x \in X$ we have

$$\eta(x) := \sup\{r \in \mathbb{R}_+ : \inf f(B(x, r)) > -\infty\} > 0.$$

Given $a \in X$, $k, m, p \in \mathbb{N} \setminus \{0\}$, $q > 0$, $\rho := (\rho_1, \dots, \rho_p) \in (0, \eta(a)/2)^p$, $j \in J_{m,p} = (\mathbb{N}_p)^m$ (so that j includes the data of m and p), $t := (t_1, \dots, t_m) \in \Delta_m$, let

$$(9) \quad X(a, \rho, j, q, t) := \{(x_1, \dots, x_m) \in X^m : x_i \in 2\rho_{j(i)}B, \\ \|t_1x_1 + \dots + t_mx_m\| < q\}$$

and let $(w_1, \dots, w_m) := w(a, \rho, j, k, q, t) \in X(a, \rho, j, q, t)$ be such that

$$\inf \left\{ \sum_{i=1}^m t_i(f(a + x_i) + \varepsilon_{j(i)} \|x_i\|) : (x_1, \dots, x_m) \in X(a, \rho, j, q, t) \right\} \\ \geq \sum_{i=1}^m t_i(f(a + w_i) + \varepsilon_{j(i)} \|w_i\|) - 1/k.$$

Let W_{k+1} be the closed linear subspace generated by W_k and the vectors $w_i(a, \rho, j, k, q, t)$ for $a \in A_k$, $m, p \in \mathbb{N} \setminus \{0\}$, $i \in \mathbb{N}_m$, $j \in J_{m,p} := \{j : \mathbb{N}_m \rightarrow \mathbb{N}_p\} = (\mathbb{N}_p)^m$, $q \in \mathbb{Q}$, $\rho \in (0, \eta(a)/2)^m \cap \mathbb{Q}^m$, $t := (t_1, \dots, t_m) \in \Delta_m \cap \mathbb{Q}^m$. Clearly W_{k+1} is separable, so that we can take a dense countable subset A_{k+1} of W_{k+1} containing A_k .

Let W be the closure of the union of the W_k 's. Let $w \in W$ be such that $\partial_F(f|_W)(w) \cap cB_{W^*} \neq \emptyset$. Thus, there exists some decreasing sequence $(\rho_n)_n$ of positive rational numbers in $(0, \eta(w)/4) \cap (0, 1)$ such that, setting

$$f_{n,w}^W(v) := f(w + v) - f(w) + \varepsilon_n \|v\| + \iota_{3\rho_n B}(v), \quad v \in W \\ f_{n,w}(x) := f(w + x) - f(w) + \varepsilon_n \|x\| + \iota_{\rho_n B}(x), \quad x \in X$$

where ι_{rB} denotes the indicator function of the ball rB , one has

$$(10) \quad \forall p \geq 1, \forall v \in W \quad \text{conv}(f_{1,w}^W, \dots, f_{p,w}^W)(v) \geq -c\|v\|.$$

Let us prove that, for all $\alpha > 0$, we have

$$(11) \quad \forall p \geq 1, \forall x \in X \quad \text{conv}(f_{1,w}, \dots, f_{p,w})(x) \geq -c\|x\| - \alpha.$$

This relation means that for all $m, p \in \mathbb{N} \setminus \{0\}$, all $j : \mathbb{N}_m \rightarrow \mathbb{N}_p$, all $t := (t_1, \dots, t_m) \in \Delta_m$ and all $x_1, \dots, x_m \in X$ with $x := t_1x_1 + \dots + t_mx_m$, we have

$$(12) \quad t_1f_{j(1),w}(x_1) + \dots + t_mf_{j(m),w}(x_m) \geq -c\|x\| - \alpha.$$

We may suppose that $t_1, \dots, t_m$ are rational numbers. It also suffices to consider the case in which $x_i \in \rho_{j(i)}B$ for all $i \in \mathbb{N}_m$. Let q be a rational number satisfying $\|x\| < q < \|x\| + \alpha/5$, and let $\beta \in (0, \alpha/5)$ be such that $\beta < \eta(w)/2$, $\|x\| + \beta < q$. We take $k \geq \beta^{-1}$ with $d(w, W_k) < \min(\beta, \rho_p)$ and $a \in A_k$ such that $\|a - w\| \leq \min(\beta, \rho_p)$. Then $B(a, \eta(w)/2) \subset B(w, \eta(w))$ so that $\eta(a) \geq \eta(w)/2$. For $i \in \mathbb{N}_m$, setting $v_i := x_i + w - a$, we note that we have $\|x_i - v_i\| \leq \beta$, $\|v_i\| \leq \rho_{j(i)} + \rho_p \leq 2\rho_{j(i)}$, $\rho_i < \eta(w)/4 \leq \eta(a)/2$ and

$$\|t_1 v_1 + \cdots + t_m v_m\| \leq \|t_1 x_1 + \cdots + t_m x_m\| + \max_i \|v_i - x_i\| \leq \|x\| + \beta < q.$$

Thus $v_i \in X(a, \rho, j, q, t)$. The relation $w + x_i = a + v_i$ and the triangular inequality justify the first of the following string of inequalities, while the second one stems from the definition of w_i ; the relations $a + w_i = w + z_i$, for $z_i := a - w + w_i$, explain the next ones:

$$\begin{aligned} \sum_{i=1}^m t_i (f(w + x_i) + \varepsilon_{j(i)} \|x_i\|) &\geq \sum_{i=1}^m t_i (f(a + v_i) + \varepsilon_{j(i)} \|v_i\| - \varepsilon_{j(i)} \|x_i - v_i\|) \\ &\geq \sum_{i=1}^m t_i (f(a + w_i) + \varepsilon_{j(i)} \|w_i\|) - 1/k - \beta \\ &\geq \sum_{i=1}^m t_i (f(w + z_i) + \varepsilon_{j(i)} \|a - w + w_i\| - \varepsilon_{j(i)} \|a - w\|) - 2\beta \\ &\geq \sum_{i=1}^m t_i (f(w + z_i) + \varepsilon_{j(i)} \|z_i\|) - 3\beta. \end{aligned}$$

Since $z_i := a - w + w_i \in W \cap 3\rho_{j(i)}B$, for $z := t_1 z_1 + \cdots + t_m z_m$, from (10), we get

$$\begin{aligned} \sum_{i=1}^m t_i f_{j(i), w}(z_i) &\geq \sum_{i=1}^m t_i (f(w + z_i) - f(w) + \varepsilon_{j(i)} \|z_i\|) - 3\beta \\ &= \sum_{i=1}^m t_i f_{j(i), w}^W(z_i) - 3\beta \\ &\geq -c \|z\| - 3\beta. \end{aligned}$$

Now, since $z_i - w_i = a - w$ and $t_1 + \cdots + t_m = 1$, by (9) we have the estimate

$$\|z\| \leq \|t_1 w_i + \cdots + t_m w_m\| + \|a - w\| \leq q + \beta \leq \|x\| + \alpha/5 + \beta.$$

Since $c \leq 1$ and $\beta \leq \alpha/5$, relation (12) is established. $\square$

4. Application to fuzzy calculus

Using the previous results, one can prove the following crucial separable reduction property for sums.

THEOREM 7. *Let W_0 be a separable subspace of a Banach space X , and let $f : X \rightarrow \mathbb{R}_\infty$ be a lsc function and $g : X \rightarrow \mathbb{R}$ be a convex continuous function. Then, there exists a separable subspace W of X containing W_0*

such that for any $w, z \in W$ the relation $(\partial_F f(w) + \partial g(z)) \cap cB_{X^*} \neq \emptyset$ holds if, and only if, $(\partial_F(f|_W)(w) + \partial(g|_W)(z)) \cap cB_{W^*} \neq \emptyset$ holds.

Proof. Let us first consider the case g is a continuous linear form. Then, since $\partial_F(f+g)(w) = \partial_F f(w) + g$, with a similar relation for the restrictions to W , the result follows from Theorem 6 applied to $f+g$. An examination of its proof shows that the construction of W for f remains valid for $f+g$ as the same sequence (ρ_n) can be used for $f+g$ and

$$(f+g)_{n,w}^W = f_{n,w}^W + g|_W, \quad (f+g)_{n,w} = f_{n,w} + g, \\ \text{conv}(f_{1,w} + g, \dots, f_{p,w} + g) = \text{conv}(f_{1,w}, \dots, f_{p,w}) + g$$

and a similar relation with $\text{conv}(f_{1,w}^W + g|_W, \dots, f_{p,w}^W + g|_W)$.

Now let us consider the general case. Given f and a separable subspace W_0 of X , let W be a separable subspace containing W_0 associated with f . Let $w, z \in W$ and let $w_W^* \in \partial_F(f|_W)(w)$, $z_W^* \in \partial(g|_W)(z)$ be such that $\|w_W^* + z_W^*\| \leq c$. Using Lemma 1, we can find some $z^* \in \partial g(z)$ such that $z^*|_W = z_W^*$. Then, by the preceding special case, we can find $w^* \in \partial f(w)$ such that $\|w^* + z^*\| \leq c$, i.e. $(w^* + z^*) \in (\partial_F f(w) + z^*) \cap cB_{X^*} \subset (\partial_F f(w) + \partial g(z)) \cap cB_{X^*}$. ■

In the next statement, following [10], [11] and subsequent papers, we say that X is a *subdifferentiability space* for the Fréchet subdifferential ∂_F if for any lower semicontinuous (lsc) function on X the set G of points $(w, f(w))$ such that $\partial_F f(w)$ is nonempty is dense in the graph of f . We say that X is a *reliable space* for the Fréchet subdifferential ∂_F if whenever $f : X \rightarrow \mathbb{R}_\infty$ is a lsc function, $g : X \rightarrow \mathbb{R}$ is a convex continuous function and $f+g$ attains its minimum on X at $\bar{x} \in X$, for every $\varepsilon > 0$ there exist $w, z \in B(\bar{x}, \varepsilon)$ with $|f(w) - f(\bar{x})| \leq \varepsilon$ such that the relation $(\partial_F f(w) + \partial g(z)) \cap \varepsilon B_{X^*} \neq \emptyset$ holds. The space X is said to be a *trustworthy space* for the Fréchet subdifferential ∂_F if the same holds when g is a Lipschitzian function and $\partial g(z)$ is replaced by $\partial_F g(z)$.

THEOREM 8. Any Asplund space X is a Fréchet subdifferentiability space. In fact, for a Banach space X the following properties are equivalent:

- (a) X is an Asplund space;
- (b) X is a reliable space for the Fréchet subdifferential ∂_F ;
- (c) X is a trustworthy space for the Fréchet subdifferential ∂_F ;
- (d) X is a subdifferentiability space for the Fréchet subdifferential ∂_F .

Proof. (a) $\Rightarrow$ (b) Let $\varepsilon > 0$, a lsc function $f : X \rightarrow \mathbb{R}_\infty$ and a convex continuous function $g : X \rightarrow \mathbb{R}$ be given such that $f+g$ attains its minimum on X at $\bar{x} \in X$. Let $W_0 := \mathbb{R}\bar{x}$ and let W be a separable subspace of X containing W_0 associated to f as in Theorem 7. Since X is Asplund, the

separable subspace W has a separable dual, hence a Fréchet smooth bump function. A classical penalization procedure then shows that there exist $w, z \in B(\bar{x}, \varepsilon)$ such that $|f(w) - f(\bar{x})| \leq \varepsilon$ and $(\partial_F(f|_W)(w) + \partial(g|_W)(z)) \cap \varepsilon B_{W^*} \neq \emptyset$. Then, we conclude from Theorem 7 that $(\partial_F f(w) + \partial g(z)) \cap \varepsilon B_{X^*} \neq \emptyset$.

(b) $\Rightarrow$ (c) This is a general implication, valid for any subdifferential; see [13], [15].

(c) $\Rightarrow$ (d) Again, this is a general (and easy) fact.

(d) $\Rightarrow$ (a) Let $f : X \rightarrow \mathbb{R}$ be a continuous convex function and let $\bar{x} \in X$, $\varepsilon > 0$ be given. Let $g := -f$. Since X is a subdifferentiability space for ∂_F there exists some $x \in B(\bar{x}, \varepsilon)$ such that $\partial_F g(x)$ is nonempty. Then f is Fréchet differentiable at x . Thus f is densely Fréchet differentiable and X is an Asplund space. ■

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