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SUBDIRECTLY IRREDUCIBLE FIBERED AUTOMATA

Abstract. A permutational automaton is a fibered automaton with one-element event set or an empty state space. This paper characterizes all subdirectly irreducible permutational automata.

1. Introduction

Fibered automata were introduced by J. D. H. Smith [6] as certain two-sorted algebras. A motivation came from a construction that Ivo Rosenberg used in the proof of his theorem: each clone on a set may be defined as the set of all operations preserving a single relation. This construction gave a basis to formulate the specification of the variety of fibered automata. Also the Division Algorithm and the Continuous Fraction Algorithm can both be modelled by a fibered automaton.

General properties of fibered automata and of the variety of fibered automata were studied in [6]. The lattice of varieties of fibered automata was constructed in [4]. Permutational automata form a subvariety of the variety of fibered automata. The aim of this paper is to describe all subdirectly irreducible permutational automata.

Every variety of algebras is generated by its subdirectly irreducible members. Moreover, every algebra of the variety can be represented as a subdirect product of these subdirectly irreducible algebras. This means that any permutational automaton can be represented as a subdirect product of subdirectly irreducible permutational automata.

Basic properties and a graph representation of fibered automata, presented in [5], are recalled in Section 2. In Section 3, we describe some

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congruences of fibered automata. Subdirect embedding and subdirectly irreducible many-sorted algebras are defined similarly, as in the case of one-sorted algebras. Next, in Section 4, we describe all subdirectly irreducible permutational automata. Finally we prove the following (Theorem 4.4):

THEOREM. *The only subdirectly irreducible permutational automata with a singleton event set are those with graphs:*

- (a) $2C_1$;
- (b) C_{q^k} for a prime power q^k ;
- (c) $C_{q^k} + C_1$ for a prime power q^k ;

and fibered automata with an empty state space and a two-element event set.

2. Fibered automata

In this section we recall basic definitions and facts concerning fibered automata.

DEFINITION 2.1. ([6]) A *fibered automaton* is a two-sorted algebra $(S, E, \mu, \delta, \varepsilon)$ (or briefly (S, E)) with a unary *maternal operation*

$$\delta : S \rightarrow S,$$

a unary *paternal operation*

$$\varepsilon : S \rightarrow E,$$

and a binary operation

$$\mu : S \times E \rightarrow S; (s, e) \mapsto s \cdot e,$$

such that the operation μ and the *parent map*

$$(\delta, \varepsilon) : S \rightarrow S \times E; s \mapsto (s\delta, s\varepsilon)$$

are mutual inverses. The set S is called the *state space* of the automaton, and its elements are called *states*. The set E is called the *event set* of the automaton, and its elements are called *elementary events*. For a state s , the state $s\delta$ is called its *mother*, and the event $s\varepsilon$ its *father*.

THEOREM 2.2. ([6]) *The mutual inverse relation between the operation μ and the parent map is equivalent to satisfaction of the following identities*

$$(2.3) \quad s = s\delta \cdot s\varepsilon,$$

$$(2.4) \quad (s \cdot e)\delta = s,$$

$$(2.5) \quad (s \cdot e)\varepsilon = e,$$

for all $s \in S$ and all $e \in E$.

COROLLARY 2.6. ([6]) *The class $\mathcal{A}$ of fibered automata forms a variety of two-sorted algebras.*

The subvarieties of the variety $\mathcal{A}$ of fibered automata can be briefly described as follows. (Here $x, y, \dots$ denote variables of the state sort, and $a, b, \dots$ of the event sort.) The lattice $L(\mathcal{A})$ of varieties of fibered automata contains a unique dual atom, the variety $\mathcal{P}$ of permutational fibered automata defined by the identity $x\varepsilon = y\varepsilon$ [6]. The event set of a permutational fibered automaton has exactly one element or the state space is empty. The variety $\mathcal{P}$ covers the variety $\mathcal{P}_0$ defined by the identity $a = b$. (See [4]). For $n \in \mathbb{Z}^+$, let us denote by $\mathcal{A}_n$ the variety of fibered automata defined by the identity $x\delta^n = x$, and by $\mathcal{A}_n^0$ the variety of fibered automata defined by the identities $x\delta^n = x$ and $a = b$. Then $\mathcal{A}_0$ is the variety defined by the identity $x = y$ and $\mathcal{A}_0^0$ is the variety of trivial fibered automata, defined by the identities $x = y$ and $a = b$. All varieties $\mathcal{A}_n^0$ are contained in $\mathcal{P}_0$, and each $\mathcal{A}_n^0$ is covered by $\mathcal{A}_n$. The lattice of subvarieties of the variety of $\mathcal{A}$ is fully described in [4].

DEFINITION 2.7. The *transition diagram* of a fibered automaton (S, E) is a directed graph on the vertex set S , whose edges are labelled by the elementary events. For each $s \in S$ and $e \in E$ there exists an edge $s \xrightarrow{e} s \cdot e$, and each vertex has indegree one. The unlabelled transition diagram of a fibered automaton is called its *graph*.

THEOREM 2.8. ([5]) *Let (S, E) be a fibered automaton with $|E| = 1$. Then weakly connected components of its graph are directed cycles or directed chains.*

In what follows, a directed cycle of length n will be denoted by C_n , an infinite directed chain by C_∞ .

DEFINITION 2.9. A fibered automaton whose graph is weakly connected is called a *weakly connected fibered automaton*.

LEMMA 2.10. ([4]) *For all $n \geq 1$ and $k, l \geq 0$, the following identities hold in the variety $\mathcal{A}$:*

- (1) $x\delta^n \cdot x\delta^{n-1}\varepsilon = x\delta^{n-1}$;
- (2) $(x\delta^k \cdot x\delta^l\varepsilon)\delta = x\delta^k$;
- (3) $(x\delta^k \cdot x\delta^l\varepsilon)\varepsilon = x\delta^l\varepsilon$;
- (4) $(xa_1 \dots a_r)\delta^n = \begin{cases} x\delta^{n-r}, & \text{if } n > r; \\ xa_1 \dots a_{r-n}, & \text{if } n < r; \\ x, & \text{if } n = r, \end{cases}$
- (5) $(xa_1 \dots a_r)\delta^n\varepsilon = \begin{cases} x\delta^{n-r}\varepsilon, & \text{if } n > r; \\ a_{r-n}, & \text{if } n < r; \\ x\varepsilon, & \text{if } n = r. \end{cases}$

THEOREM 2.11. ([6]) *Let s and t be states of a fibered automaton (S, E) . The following conditions are equivalent:*

- (a) $\exists n, m \in \mathbb{N}, s\delta^n = t\delta^m$.
- (b) *The vertices s and t are in one weakly connected component of the graph of (S, E) .*

Let (S, E) be a non-empty permutational fibered automaton. Let $e \in E$ be the unique elementary event. For $s \in S$ and $n \in \mathbb{Z}^+$, define the mapping $\delta^{-n}: S \rightarrow S; s \mapsto se^n$.

COROLLARY 2.12. *Let s and t be states of a permutational fibered automaton (S, E) . The following conditions are equivalent:*

- (a) $\exists n \in \mathbb{Z}, s\delta^n = t$.
- (b) *The vertices s and t are in one weakly connected component of the graph of (S, E) .*

LEMMA 2.13. *If a fibered automaton has at most one elementary event, then it satisfies the following equations:*

$$x\delta^m a^n = \begin{cases} x\delta^{m-n}, & \text{if } m > n; \\ xa^{n-m}, & \text{if } m < n; \\ x, & \text{if } m = n. \end{cases}$$

Proof. If a fibered automaton has an empty state space then it satisfies these equations trivially.

Let (S, E) be a fibered automaton with a non-empty state space and a one-element event set. Then the following equations are satisfied

$$a = x\varepsilon = x\delta\varepsilon = \dots = x\delta^{m-1}\varepsilon.$$

We apply $\min(m, n)$ times Lemma 2.10 (1), obtaining

$$\begin{aligned} x\delta^m a^n &= (x\delta^m)aa^{n-1} = (x\delta^m)(x\delta^{m-1}\varepsilon)a^{n-1} = x\delta^{m-1}a^{n-1} = \dots \\ &= \begin{cases} x\delta^{m-n}, & \text{if } m > n; \\ xa^{n-m}, & \text{if } m < n; \\ x, & \text{if } m = n. \end{cases} \blacksquare \end{aligned}$$

LEMMA 2.14. *Let (S, E) be a fibered automaton and let $s \in S$. Let n be a positive number. The following conditions are equivalent:*

- (a) *The state s is a vertex of a directed cycle of length n .*
- (b) $s\delta^n = s$ and if $s\delta^m = s$, for some positive m , then $n|m$.

THEOREM 2.15. ([5]) *A fibered automaton lies in $\mathcal{A}_n^0$ iff its graph is a disjoint union of directed cycles of length dividing n .*

DEFINITION 2.16. A subalgebra A of a direct product $\prod_{\gamma \in \Gamma} A_\gamma$ of algebras A_γ of the same type is called a *subdirect product*, if for all projections π_γ , for $\gamma \in \Gamma$, we have $A\pi_\gamma = A_\gamma$. An embedding $e: A \rightarrow \prod_{\gamma \in \Gamma} A_\gamma$ is called *subdirect*, if $e(A)$ is a subdirect product of algebras A_γ .

DEFINITION 2.17. An algebra A is called *subdirectly irreducible*, if for any subdirect embedding $e: A \rightarrow \prod_{\gamma \in \Gamma} A_\gamma$ to a direct product of algebras A_γ of the same type, there exists $\gamma \in \Gamma$ such that

$$e\pi_\gamma: A \rightarrow \prod_{\gamma \in \Gamma} A_\gamma \rightarrow A_\gamma$$

is an isomorphism.

THEOREM 2.18. ([2]) *Let A be an algebra. The following conditions are equivalent:*

- (1) *The algebra A is subdirectly irreducible.*
- (2) *The lattice of congruences of the algebra A has a least non-trivial element.*

COROLLARY 2.19. *An algebra $A = (\{A_i\}, \Omega)$ is subdirectly irreducible if and only if there exists a sort A_j and a pair (a, b) of its elements such that, for all $i \in I$, and for all elements $c \neq d$ of the sort A_i , $cg(a, b) \subseteq cg(c, d)$.*

THEOREM 2.20. ([2]) *Let A be an algebra. Let φ_γ , for $\gamma \in \Gamma$, be a family of congruences on A such that $\bigcap_{\gamma \in \Gamma} \varphi_\gamma = \hat{A}$, where $\hat{A}$ is the equality relation on A . Then the map*

$$h: A \rightarrow \prod_{\gamma \in \Gamma} A/\varphi_\gamma; h(a)(\gamma) = a/\varphi_\gamma$$

is a subdirect embedding.

3. Congruences of fibered automata

First recall, that a congruence of a fibered automaton (S, E) is a pair

$$\theta = \{\theta_S, \theta_E\},$$

of equivalence relations, where

$$((s, t) \in \theta_S \wedge (e, f) \in \theta_E) \Rightarrow (s \cdot e, t \cdot f) \in \theta_S$$

and

$$(s, t) \in \theta_S \Rightarrow ((s\delta, t\delta) \in \theta_S \wedge (s\varepsilon, t\varepsilon) \in \theta_E).$$

Instead of

$$(s, t) \in \theta_S, \quad (e, f) \in \theta_E,$$

we will often write

$$(s, t) \in \theta, \quad (e, f) \in \theta.$$

The pair of equality relations on a fibered automaton (S, E) is denoted by $\widehat{(S, E)}$, and is just called the equality relation.

We will describe some congruences of fibered automata with a non-empty state space.

Let (S, E) be a non-trivial, weakly connected permutational automaton. Let m be a natural number. Define a relation θ_S^m on the state space as follows:

$$s \theta_S^m t \Leftrightarrow \exists n \in \mathbb{Z}, s = t \delta^{nm}.$$

PROPOSITION 3.1. *Let (S, E) be a non-trivial weakly connected permutational automaton. Then*

$$\theta^m := \{\theta_S^m, \widehat{E}\}$$

is a congruence on (S, E) generated by $(s, s\delta^m)$, for any state s .

Proof. Let (S, E) be a weakly connected permutational fibered automaton. Let e be the unique elementary event. First, we show that θ_S^m is an equivalence relation. As $s = s\delta^0$, it follows that the relation θ_S^m is reflexive. If $(s, t) \in \theta_S^m$, then there exists an integer n such that $s = t\delta^{mn}$. If $n \geq 0$, then by Lemma 2.13,

$$t = t\delta^{mn}e^{mn} = se^{mn} = s\delta^{-mn}.$$

If $n < 0$, then by Lemma 2.13,

$$t = te^{-mn}\delta^{-mn} = t\delta^{mn}\delta^{-mn} = s\delta^{-mn}.$$

Hence $(t, s) \in \theta_S^m$ and the relation θ_S^m is symmetric. If $(s, t) \in \theta_S^m$ and $(t, v) \in \theta_S^m$, then there exist integers n and k such that $s = t\delta^{mn}$ and $t = v\delta^{mk}$. It follows that

$$s = t\delta^{mn} = v\delta^{mk}\delta^{mn} = v\delta^{mk+mn} = v\delta^{m(k+n)},$$

implying that $(s, v) \in \theta_S^m$, whence θ_S^m is transitive. Consequently, θ_S^m is an equivalence relation.

Now, we show that θ^m is a congruence of the fibered automaton (S, E) . Let $(s, t) \in \theta_S^m$. There exists an integer n such that $s = t\delta^{mn}$. By Lemma 2.13 and the equality (2.4),

$$s \cdot e = t\delta^{mn} \cdot e = t\delta^{mn-1} = (t \cdot e)\delta^{mn-1} = (t \cdot e)\delta^{mn}.$$

Hence $(s \cdot e, t \cdot e) \in \theta^m$. Moreover, $s\delta = t\delta^{mn+1}$, implying that $(s\delta, t\delta) \in \theta^m$ and $s\varepsilon = e = t\varepsilon$. Then $(s\varepsilon, t\varepsilon) \in \widehat{E}$. It follows that θ^m is a congruence.

Finally, we will show that

$$\theta^m = cg(s, s\delta^m),$$

for each $s \in S$. As $(s, s\delta^m) \in \theta^m$, it follows that $cg(s, s\delta^m) \subseteq \theta^m$. Assume that a pair (t, v) of states is in the congruence θ^m . There exists $n \in \mathbb{Z}$ such

that $v = t\delta^{mn}$. By Corollary 2.12, there exists $k \in \mathbb{Z}$ such that $t = s\delta^k$. Hence

$$t = s\delta^k \, cg(s, s\delta^m) \, s\delta^{k+m} \, cg(s, s\delta^m) \, s\delta^{k+2m} \, \dots \\ s\delta^{k+(n-1)m} \, cg(s, s\delta^m) \, s\delta^{k+nm} = t\delta^{mn} = v.$$

Then $(t, v) \in cg(s, s\delta^m)$, for any $(t, v) \in \theta^m$, and $\theta^m \subseteq cg(s, s\delta^m)$. It follows that the congruences θ^m and $cg(s, s\delta^m)$ are equal. ■

Now assume that (S, E) is a fibered automaton with at least 2 elementary events and a non-empty state space. Let $e \neq f$ be elementary events. Define a pair of relations on the sets S and E as follows

$$(c, d) \in \vartheta_E^{e,f} \Leftrightarrow c = d \vee \{c, d\} = \{e, f\}, \\ (s, t) \in \vartheta_S^{e,f} \Leftrightarrow \exists n \in \mathbb{N}, s\delta^n = t\delta^n \wedge \forall 0 \leq k < n, (s\delta^k e, t\delta^k e) \in \vartheta_E^{e,f}.$$

PROPOSITION 3.2. *Let (S, E) be a fibered automaton with at least 2 elementary events and a non-empty state space. Let $e \neq f \in E$. The relation*

$$\vartheta^{e,f} := \{\vartheta_S^{e,f}, \vartheta_E^{e,f}\}$$

is a congruence of (S, E) generated by (e, f) .

We omit the proof, which is similar to the proof of Proposition 3.1.

COROLLARY 3.3. *Let (S, E) be a fibered automaton with at least 3 elementary events. Let e, f, g be pairwise distinct elementary events. Then $\vartheta^{e,f} \cap \vartheta^{e,g}$ is the equality relation on (S, E) .*

Proof. By the definition of relations $\vartheta^{e,f}$ and $\vartheta^{e,g}$, a pair of distinct elementary events (c, d) is in the relation $\vartheta^{e,f}$ if $\{c, d\} = \{e, f\}$, and in the relation $\vartheta^{e,g}$ if $\{c, d\} = \{e, g\}$. Hence

$$(\vartheta^{e,f} \cap \vartheta^{e,g})_E = \widehat{E}.$$

If S is empty, then obviously

$$(\vartheta^{e,f} \cap \vartheta^{e,g})_S = \widehat{S}.$$

Assume that S is non-empty,

$$(s, t) \in (\vartheta^{e,f} \cap \vartheta^{e,g})_S,$$

and $s \neq t$. Then $(s, t) \in \vartheta^{e,f}$. There exists a positive number k such that $s\delta^k = t\delta^k$. Let k be the least such number. If

$$s\delta^{k-1}e = t\delta^{k-1}e,$$

then by Lemma 2.10 (1),

$$s\delta^{k-1} = s\delta^k s\delta^{k-1}e = t\delta^k t\delta^{k-1}e = t\delta^{k-1},$$

which contradicts the assumption that k is the least such number. Hence

$$s\delta^{k-1}\varepsilon \neq t\delta^{k-1}\varepsilon.$$

As

$$(s\delta^{k-1}\varepsilon, t\delta^{k-1}\varepsilon) \in \vartheta^{e,f},$$

it follows that

$$\{s\delta^{k-1}\varepsilon, t\delta^{k-1}\varepsilon\} = \{e, f\}.$$

Hence

$$(e, f) \in (\vartheta^{e,f} \cap \vartheta^{e,g})_E = \widehat{E},$$

a contradiction. Consequently the congruence $\vartheta^{e,f} \cap \vartheta^{e,g}$ is trivial. ■

PROPOSITION 3.4. *Let (S, E) be a fibered automaton. Let $T \subseteq S$. Then (T, E) is a subautomaton of the automaton (S, E) if and only if the induced labelled subgraph of the transition diagram of (S, E) with vertex set T is a disjoint union of weakly connected components of the transition diagram of (S, E) .*

DEFINITION 3.5. Let (S, E) be a fibered automaton. Let T be a subset of S such that the induced labelled subdigraph G of the transition diagram of (S, E) with vertex set T is a disjoint union of weakly connected components of the transition diagram of (S, E) . Then the fibered automaton (T, E) is said to be *determined by the digraph G* .

Let $G_i = (V_i, E_i)$, for $i \in I$, be a family of graphs. Then the graph $(\sum_{i \in I} G_i, \sum_{i \in I} E_i)$ is denoted by $\sum G_i$. For a cardinal number κ , the symbol κG denotes the disjoint union of κ copies of G .

EXAMPLE 3.6. Assume that a fibered automaton (S, E) has the transition diagram given in Figure 1.

Fig. 1. The fibered automaton with the graph $C_8 + C_2 + C_1$.

Then the fibered automaton with the transition diagram given in Figure 2 is the subautomaton of (S, E) determined by C_8 .

Fig. 2. The fibered automaton with the graph C_8 .

LEMMA 3.7. Let (T, E) be a subautomaton of a fibered automaton (S, E) . Let $\theta^{(T, E)}$ be a congruence of the fibered automaton (T, E) such that $\theta_E^{(T, E)} = \widehat{E}$. Then the relation $\overline{\theta^{(T, E)}}$ defined as follows

$$(s, t) \in \overline{\theta^{(T, E)}}_S \Leftrightarrow (s, t) \in \theta^{(T, E)}_T \vee s = t, \quad \overline{\theta^{(T, E)}}_E = \widehat{E}$$

is a congruence of the fibered automaton (S, E) .

Proof. The relation $\overline{\theta^{(T, E)}}$ is a congruence relation of the subautomaton (T, E) . On the remaining part of the automaton it is the pair of equality relations. Hence it is an equivalence relation. If two distinct states (s, t) are in the congruence $\overline{\theta^{(T, E)}}$, then they are in the subautomaton (T, E) , and $(s, t) \in \theta^{(T, E)}$. Hence $(s\delta, t\delta) \in \theta^{(T, E)}$, $(s\varepsilon, t\varepsilon) \in \theta^{(T, E)}$ and $(se, te) \in \theta^{(T, E)}$, for $e \in E$. Moreover, all of these pairs are in $\overline{\theta^{(T, E)}}$. Hence $\overline{\theta^{(T, E)}}$ is a congruence relation of the fibered automaton (S, E) . ■

LEMMA 3.8. Let (T, E) be a subautomaton of a fibered automaton (S, E) . Let $\alpha_\gamma^{(T, E)}$, for $\gamma \in \Gamma$, be a family of congruences, trivial on the event set, such that

$$\bigcap_{\gamma \in \Gamma} \alpha_\gamma^{(T, E)} = \widehat{(T, E)}.$$

Then

$$\bigcap_{\gamma \in \Gamma} \overline{\alpha_\gamma^{(T, E)}} = \widehat{(S, E)}.$$

Proof. A pair of states (s, t) is in $\bigcap_{\gamma \in \Gamma} \overline{\alpha_\gamma^{(T, E)}}$ if $(s, t) \in \overline{\alpha_\gamma^{(T, E)}}$. Hence either $s = t$ or $(s, t) \in \alpha_\gamma^{(T, E)}$. It follows that either $s = t$ or $(s, t) \in \bigcap_{\gamma \in \Gamma} \alpha_\gamma^{(T, E)} = \widehat{(T, E)}$. Finally, we obtain $s = t$. Consequently the congruence $\bigcap_{\gamma \in \Gamma} \overline{\alpha_\gamma^{(T, E)}}$ is the equality relation. ■

LEMMA 3.9. *Let (T_1, E) and (T_2, E) be subautomata of a fibered automaton (S, E) such that $|T_1 \cap T_2| \leq 1$. Let $\alpha^{(T_1, E)}$ and $\beta^{(T_2, E)}$ be congruences on the corresponding subalgebras, trivial on the event sets. Then $\overline{\alpha^{(T_1, E)}} \cap \overline{\beta^{(T_2, E)}} = \overline{(S, E)}$.*

Proof. If a pair of states (s, t) is in $\overline{\alpha^{(T_1, E)}} \cap \overline{\beta^{(T_2, E)}}$, then $(s, t) \in \overline{\alpha^{(T_1, E)}}$ and $(s, t) \in \overline{\beta^{(T_2, E)}}$. Hence either $s = t$ or both states are in both subalgebras. These subalgebras have at most one common state, hence $s = t$, and the congruence $\overline{\alpha^{(T_1, E)}} \cap \overline{\beta^{(T_2, E)}}$ is trivial. ■

4. Subdirectly irreducible fibered automata

In this section we will describe all subdirectly irreducible permutational automata.

LEMMA 4.1. *Fibered automata whose graphs are C_{p^k} , $2C_1$ or $C_{p^k} + C_1$, where p is a prime number, and k is a positive number, are subdirectly irreducible.*

Proof. Fibered automata with graphs $2C_1$ or C_2 have exactly two congruences. Hence they are subdirectly irreducible.

Let $(S, \{e\})$ be a fibered automaton with the graph C_{p^k} , where $p^k > 2$. We consider all congruences generated by pairs of states. Let s and t be states of the fibered automaton $(S, \{e\})$. As the fibered automaton is weakly connected, it follows by Corollary 2.12 that either $t = s\delta^m$ or $s = t\delta^m$, for some positive m . Without loss of generality, we can assume that $t = s\delta^m$. As the graph of (S, E) is C_{p^k} , it follows that $(S, \{e\})$ satisfies $x = x\delta^{p^k}$. If $m \geq p^k$, then there exist natural numbers d and r such that $0 \leq r < p^k$ and $m = dp^k + r$. Then

$$t = s\delta^m = s\delta^{dp^k+r} = s\delta^r.$$

Hence any pair of states has the form $(s, s\delta^m)$, for $0 < m < p^k$. By Proposition 3.1, the congruence generated by $(s, s\delta^m)$ is precisely the congruence θ^m .

We show that the congruence $\theta^{p^{k-1}}$ is the least non-trivial congruence of $(S, \{e\})$.

Let $s \neq t \in S$ and assume that $(s, t) \in \theta^{p^{k-1}}$. By Corollary 2.12, there exists $n \in \mathbb{Z}^+$ such that $s\delta^n = t$. By the definition of the congruence $\theta^{p^{k-1}}$, there exists $r \in \mathbb{Z}^+$ such that $s = t\delta^{p^{k-1}r}$. Let m be a natural number such that $p \nmid m$. By the Euclidean Algorithm, there exist integers b, c such that $bm + cp^k = n$. Then by the equality $x = x\delta^{p^k}$,

$$t = s\delta^n = s\delta^{bm+cp^k} = s\delta^{cp^k}\delta^{bm} = s\delta^{bm}\theta^m s.$$

Hence $(s, t) \in \theta^m$, and $\theta^{p^{k-1}} \subseteq \theta^m$. Let l be a natural number such that $0 \leq l < k - 1$. Consider the congruence θ^{p^l} . As

$$s = t\delta^{p^{k-1}r} = t\delta^{rp^{l+(k-l-1)}} = t\delta^{rp^{k-l-1}p^l},$$

it follows that $(s, t) \in \theta^{p^l}$. Hence $\theta^{p^{k-1}} \subseteq \theta^{p^l}$, for $0 \leq l < k - 1$. It follows that the congruence $\theta^{p^{k-1}}$ is contained in each congruence generated by two states. By Corollary 2.19, the fibered automaton $(S, \{e\})$ is subdirectly irreducible.

Let $(S, \{e\})$ be a fibered automaton with the graph $C_{p^k} + C_1$. We consider all congruences generated by pairs of states. Let $(T, \{e\})$ be a subautomaton of the fibered automaton $(S, \{e\})$ determined by C_{p^k} . As in the above consideration, the congruence $(\theta^{p^{k-1}})^{(T, \{e\})}$ is the least non-trivial congruence of $(T, \{e\})$. Then

$$\overline{(cg(s, t))^{(T, \{e\})}} \supseteq \overline{(\theta^{p^{k-1}})^{(T, \{e\})}}$$

also holds. Moreover, for all states $s, t \in T$,

$$cg(s, t) = \overline{(cg(s, t))^{(T, \{e\})}}.$$

Hence

$$cg(s, t) \supseteq \overline{(\theta^{p^{k-1}})^{(T, \{e\})}}.$$

Let s, v be states, where $s \in T$, and $v \in S \setminus T$. Then $v = v\delta^n$, for all natural n . Then for all $t \in T$, there exists natural n such that $t = s\delta^n$ and

$$t = s\delta^n cg(s, v) v\delta^n = v.$$

Consequently, the congruence $cg(s, v)$ is non-proper, and $\overline{(\theta^{p^{k-1}})^{(T, \{e\})}} \subseteq cg(s, v)$. By Corollary 2.19, the fibered automaton $(S, \{e\})$ is subdirectly irreducible. ■

PROPOSITION 4.2. *The only subdirectly irreducible fibered automata with a singleton event set are those with graphs:*

- (a) $2C_1$;
- (b) C_{p^k} for a prime power p^k ;
- (c) $C_{p^k} + C_1$ for a prime power p^k .

Proof. We show that any fibered automaton with a one-element event set, whose graph is not one of C_{p^k} , $2C_1$ or $C_{p^k} + C_1$, is not subdirectly irreducible. We consider all forms of the graphs of fibered automata, using Theorem 2.8 and Theorem 2.20.

First, we consider weakly connected fibered automata with graphs different from C_{p^k} . The graphs are C_∞ or C_{pqm} , where p and q are distinct prime numbers and $m \in \mathbb{Z}^+$.

Let $(S, \{e\})$ be a fibered automaton with the graph C_∞ . Let P be the set of all prime numbers. Consider $\bigcap_{p \in P} \theta^p$. Assume that s and t are in the congruence $\bigcap_{p \in P} \theta^p$. As s and t lie on the chain C_∞ , it follows by Corollary 2.12 that there exists $n \in \mathbb{N}$ such that $s = t\delta^n$ or $t = s\delta^n$. Without loss of generality, we can assume that $s = t\delta^n$. Moreover n is the unique such number, because otherwise $s = t\delta^n = t\delta^m$. Assume that $m > n$. Then $t\delta^n = t\delta^n\delta^{m-n}$ and by Lemma 2.14, the graph contains a cycle, a contradiction. There exists a prime number q greater than n . If $(s, t) \in \theta^q$, then $s = t\delta^{qm}$, for some integer m . If $m > 0$, then $s = t\delta^n = t\delta^{qm}$ and $qm > n$, what contradicts an assumption that n is the unique such number. If $m < 0$, then $s = t\delta^n = t\delta^{qm}$. By Corollary 2.13,

$$t = te^{-qm}\delta^{-qm} = t\delta^{qm}\delta^{-qm} = s\delta^{-qm}.$$

Hence

$$s = t\delta^n = s\delta^{-qm+n}$$

and $-qm + n > n$. Then by Lemma 2.14, the graph contains a cycle, a contradiction. Hence $m = 0$ and $s = t$. Consequently, the congruence $\bigcap_{p \in P} \theta^p$ is trivial.

Let $(S, \{e\})$ be a fibered automaton with the graph C_{pqm} , where p and q are distinct prime numbers, and $m \in \mathbb{Z}^+$. Let s, t be states. Assume that $(s, t) \in \theta^{pm} \cap \theta^{qm}$. By the definition of the congruences θ^{pm} and θ^{qm} , there exist natural numbers k and l such that $s = t\delta^{pmk}$ and $s = t\delta^{qml}$. The numbers k and l can be presented in the forms $k = aq + i$ and $l = bp + j$, for natural numbers a, b and $0 \leq i < q, 0 \leq j < p$. Hence

$$s = t\delta^{pm(aq+i)} = t\delta^{pqma}\delta^{pmi} = t\delta^{pmi}$$

and

$$s = t\delta^{qm(bp+j)} = t\delta^{pqmb}\delta^{qmj} = t\delta^{qmj}.$$

Assume that $qmj \neq pmi$. If $qmj < pmi$, then

$$t\delta^{qmj} = t\delta^{pmi} = t\delta^{qmj}\delta^{pmi-qmj}.$$

By Lemma 2.14, $t\delta^{qmj}$ is a vertex of a cycle with length dividing $pmi - qmj$. But $t\delta^{qmj}$ is a vertex of the cycle C_{pqm} , a contradiction. Similarly, if $qmj > pmi$, then

$$t\delta^{pmi} = t\delta^{qmj} = t\delta^{pmi}\delta^{qmj-pmi}.$$

By Lemma 2.14, $t\delta^{pmi}$ is a vertex of a cycle with length dividing $qmj - pmi$, a contradiction. Hence $pmi = qmj$ and $pi = qj$. As p does not divide q , it follows that p divide j , and $j < p$. Hence $j = i = 0$ and $s = t$. Consequently the congruence $\theta^{pm} \cap \theta^{qm}$ is trivial.

Now we consider fibered automata with no weakly connected graphs different from $C_{p^k} + C_1$. Such a graph has to satisfy at least one of the following conditions:

- 1) it contains a chain as a weakly connected component,
- 2) it contains at least two cycles of length greater than 1,
- 3) it contains at least two cycles C_1 and one longer cycle,
- 4) it has the form $C_{pqm} + C_1$,
- 5) it has the form κC_1 , for some cardinal number $\kappa > 2$.

Ad. 1) Let $(S, \{e\})$ be a fibered automaton with a graph containing C_∞ as a weakly connected component. Let $(T, \{e\})$ be a subautomaton of $(S, \{e\})$ determined by C_∞ . At the beginning of the proof we showed that

$$\bigcap_{p \in P} (\theta^p)^{(T, \{e\})} = \widehat{(T, \{e\})}.$$

By Lemma 3.8, the congruence

$$\bigcap_{p \in P} \overline{(\theta^p)^{(T, \{e\})}}$$

is trivial.

Ad. 2) Let $(S, \{e\})$ be a fibered automaton with a graph containing at least two cycles with length greater than 1. Let $(T, \{e\})$ be the subautomaton of $(S, \{e\})$ determined by one of these cycles, and let $(R, \{e\})$ be the subautomaton determined by the second. By Lemma 3.9, the congruence

$$\overline{(\theta^1)^{(T, \{e\})}} \cap \overline{(\theta^1)^{(R, \{e\})}}$$

is trivial.

Ad. 3) Let $(S, \{e\})$ be a fibered automaton whose graph contains two cycles of length 1, and at least one longer cycle. Let $(\{s, t\}, \{e\})$ be the subautomaton of $(S, \{e\})$ determined by the union of two cycles with length 1, and let $(T, \{e\})$ be the subautomaton determined by the longer cycle. By Lemma 3.9, the congruence

$$\overline{(\theta^1)^{(T, \{e\})}} \cap \overline{cg(s, t)^{(\{s, t\}, \{e\})}}$$

is trivial.

Ad. 4) Let $(S, \{e\})$ be a fibered automaton whose graph is $C_{pqm} + C_1$. Let $(T, \{e\})$ be the subautomaton of $(S, \{e\})$ determined by the graph C_{pqm} . At the beginning of the proof we showed that

$$(\theta^{pm})^{(T, \{e\})} \cap (\theta^{qm})^{(T, \{e\})} = \widehat{(T, \{e\})}.$$

By Lemma 3.8, the congruence

$$\overline{(\theta^{pm})^{(T, \{e\})}} \cap \overline{(\theta^{qm})^{(T, \{e\})}}$$

is trivial.

Ad. 5) Let $(S, \{e\})$ be a fibered automaton with the graph κC_1 , for $\kappa > 2$. Let s, t and v be vertices of three cycles. Let $(\{s, t\}, \{e\})$ and $(\{t, v\}, \{e\})$ be subautomata of $(S, \{e\})$ determined by disjoint unions of two cycles of

length 1. By Lemma 3.9, the congruence

$$\overline{cg(s, t)(\{s, t\}, \{e\})} \cap \overline{cg(t, v)(\{t, v\}, \{e\})}$$

is trivial.

By Theorem 2.20, no such automata can be subdirectly irreducible.

Hence and by Lemma 4.1, only fibered automata with graphs C_{p^k} , $2C_1$ and $C_{p^k} + C_1$ are subdirectly irreducible fibered automata with a one-element event set. ■

LEMMA 4.3. *A fibered automaton with an empty state space is subdirectly irreducible if and only if the event set has two elements.*

Proof. A fibered automaton with an empty state space and a two-element event set has exactly two congruences. Hence it is subdirectly irreducible. If a fibered automaton has at least three elementary events, then by Corollary 3.3, there are two non-trivial congruences such that the meet of them is trivial. By Theorem 2.18, this fibered automaton is not subdirectly irreducible. ■

THEOREM 4.4. *The only subdirectly irreducible permutational automata with a singleton event set are those with graphs:*

- (a) $2C_1$;
- (b) C_{p^k} for a prime power p^k ;
- (c) $C_{p^k} + C_1$ for a prime power p^k ;

and fibered automata with an empty state space and a two-element event set.

Proof. By the definition, permutational automata have a one-element event set or an empty state space. By Proposition 4.2, and Lemma 4.3, subdirectly irreducible permutational automata have an empty state space and a two-element event set, or graphs C_{p^k} , $2C_1$ or $C_{p^k} + C_1$, where p is a prime number and k is a positive number. ■

COROLLARY 4.5. *The only subdirectly irreducible permutational automata in the variety $\mathcal{P}_0$ are those with graphs:*

- (a) $2C_1$;
- (b) C_{p^k} for a prime power p^k ;
- (c) $C_{p^k} + C_1$ for a prime power p^k .

COROLLARY 4.6. *Let n be a positive number. The only subdirectly irreducible permutational automata in the variety $\mathcal{A}_n$ are those with graphs:*

- (a) $2C_1$;
- (b) C_{p^k} for a prime power p^k , where $p^k | n$;
- (c) $C_{p^k} + C_1$ for a prime power p^k , where $p^k | n$;

and fibered automata with an empty state space and a two-element event set.

Proof. By Theorem 2.15, a fibered automaton with a non-empty state space is in the variety $\mathcal{A}_n$ if and only if its graph is a disjoint union of cycles of length dividing n . Moreover $\mathcal{A}_n$ is a subvariety of the variety $\mathcal{P}_0$. Hence by Theorem 4.4, only C_{p^k} , $2C_1$ and $C_{p^k} + C_1$, where p^k divides n , are graphs of subdirectly irreducible fibered automata in the variety $\mathcal{A}_n$. All fibered automata with an empty state space are in the variety $\mathcal{A}_n$. By Lemma 4.3, subdirectly irreducible fibered automata with an empty state space have a two-element event set. ■

COROLLARY 4.7. *Let n be a positive number. The only subdirectly irreducible permutational automata in the variety $\mathcal{A}_n^0$ are those with graphs:*

- (a) $2C_1$;
- (b) C_{p^k} for a prime power p^k , where $p^k | n$;
- (c) $C_{p^k} + C_1$ for a prime power p^k , where $p^k | n$.

COROLLARY 4.8. *All subdirectly irreducible fibered automata in the variety $\mathcal{A}_0$ have an empty state space and a two-element event set.*

Proof. By the definition, the variety $\mathcal{A}_0$ consists of trivial fibered automata and fibered automata with an empty state space. By Lemma 4.3, subdirectly irreducible fibered automata with an empty state space have a two-element event set. ■

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