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COMMON FIXED POINTS FOR WEAKLY COMPATIBLE
MAPS SATISFYING IMPLICIT RELATIONS
WITHOUT CONTINUITY

Abstract. The purpose of this paper is to prove a common fixed point theorem for a set of four mappings on a complete metric space, using weak compatibility and a general implicit relation without appeal to continuity. Our results improve and generalize all the results obtained by A. Djoudi in a paper published in 2003.

1. Introduction

Throughout this paper, (X, d) will be a metric space. Let T and J be two self-mappings of X .

S. Sessa [21] defines T and J to be weakly commuting if $d(TJx, JT x) \leq d(Jx, Tx)$ for all $x \in X$.

G. Jungck [8] defines T and J to be compatible if $\lim_{n \rightarrow \infty} d(TJx_n, JT x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Jx_n = x$ for some $x \in X$. Clearly, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but no one of these implications is reversible (ex. 1 [20] and ex. 2.2 [8]).

In 1993, G. Jungck, P. P. Murthy, and Y. J. Cho [9] introduced the concept of compatible mappings of type (A) . Precisely, let T and J be two self-mappings of X . T and J are said to be compatible of type (A) (see [9]) if $\lim_{n \rightarrow \infty} d(TJx_n, J^2x_n) = 0$ and $\lim_{n \rightarrow \infty} d(JTx_n, T^2x_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Jx_n = x$ for some $x \in X$.

Examples are given to show that the two concepts of compatibility are independant (Ex. 2.1 and Ex. 2.2 [9]).

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In 1995, H. K. Pathak and M. S. Khan [15] defined T and J to be compatible of type (B) if

$$\lim_{n \rightarrow \infty} d(JTx_n, T^2x_n) \leq \frac{1}{2} \left[\lim_{n \rightarrow \infty} d(JTx_n, Jx) + \lim_{n \rightarrow \infty} d(Jx, J^2x_n) \right]$$

and

$$\lim_{n \rightarrow \infty} d(TJx_n, J^2x_n) \leq \frac{1}{2} \left[\lim_{n \rightarrow \infty} d(TJx_n, Tx) + \lim_{n \rightarrow \infty} d(Tx, T^2x_n) \right],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Jx_n = x$ for some $x \in X$. In [15], the authors also discussed the relations between the two concepts and proved a fixed point theorem of Greguš type for compatible mappings of type (B) in Banach spaces.

The concept of compatible mappings of type (B) was used in [15], [5] and [6] to prove results on common fixed points.

It is clear that compatible mappings of type (A) are compatible of type (B) . By example 2.4 [15], the implication is not reversible. One can find examples (see [5]) to show that the notions of compatible mappings and compatible mappings of type (A) (and consequently of type (B)) are different if the continuity is dropped. Indeed, there are examples (see [15]) showing that these two concepts are not equivalent.

In 1998, Jungck and Rhoades [10] introduced the notion of weak compatibility and showed that compatible maps are weakly compatible but converse need not be true. We recall that a pair $\{T, J\}$ (of maps T and J) is called weakly compatible if T and J commute at their coincidence points.

Let T and J be two selfmappings on a metric space (X, d) . We observe that if T and J are compatible, or compatible of type (B) then they are weakly compatible. So, the notion of weakly compatible mappings is weaker than all the other notions of compatibility considered before.

In fixed point theory, other types of compatibility are considered and used (see for example the papers [17], [16] and [3]).

In [6], A. Djoudi proved a unique common fixed point result for a set of four self-mappings on a complete metric space which are compatible of type (B) and satisfying an implicit relation. This result uses the continuity of at least one of these mappings.

The use of implicit relations in fixed point theory is now extensive. Many recent results on common fixed points for mappings are established via implicit relations. The reader is invited to consult the papers [18], [19], [7], [1], [2], [4] and [6]. The other papers listed in the references are also connected to the subject of this paper.

The implicit functions used in [6] are defined by the following.

Let $\mathbb{R}_+$ be the set of all non-negative real numbers and let $\mathcal{G}$ be the set of all continuous functions $G(t_1, \dots, t_6) : \mathbb{R}_+^6 \rightarrow \mathbb{R}$ satisfying the conditions:

(G_1): G is non decreasing in the coordinate variables t_5 and t_6 .

(G_2): There exists $\theta \in (1, \infty)$, such that for every $u, v \geq 0$ with

$$(G_a): G(u, v, u, v, u + v, 0) \geq 0 \text{ or}$$

$$(G_b): G(u, v, v, u, 0, u + v) \geq 0,$$

we have $u \geq \theta v$.

(G_3): $G(u, u, 0, 0, u, u) < 0$ for every $u > 0$.

Many examples of elements of $\mathcal{G}$ are given in [6].

Let (X, d) be a metric space. Let $G \in \mathcal{G}$, and let A, B, S, T be four self-mappings of X . For arbitrary elements $x, y \in X$, we consider the following inequality:

$$(1.1) \quad G(d(Ax, By), d(Sx, Ty), d(Ax, Sx), \\ d(By, Ty), d(Ax, Ty), d(Sx, By)) \geq 0.$$

The main result of [6] is the following theorem.

THEOREM 1.1. *Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the following conditions:*

- (i) A, B are surjective.
- (ii) The pairs $\{A, S\}$ and $\{B, T\}$ are compatible of type (B) .
- (iii) The inequality (1.1) above holds for all $x, y \in X$ for some $G \in \mathcal{G}$.
- (iv) One of A, B, S, T is continuous.

Then A, B, S and T have a unique common fixed point in X .

The purpose of this paper, is to prove a fixed point theorem for weakly compatible maps satisfying the condition (1.1) but without appeal to continuity. So, we give an improvement to Theorem 1.1 by weakning the assumptions (i) and (ii) and removing the condition (iv). Also, all the results obtained in [6] will be improved.

2. The results

The main result of this paper is the following theorem.

THEOREM 2.1. *Let $G \in \mathcal{G}$ and let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the following conditions:*

- (C1) A and B are surjective.
- (C2) The pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible.
- (C3) For all $x, y \in X$, we have

$$(2.1) \quad G(d(Ax, By), d(Sx, Ty), d(Ax, Sx), \\ d(By, Ty), d(Ax, Ty), d(Sx, By)) \geq 0.$$

Then A, B, S and T have a unique common fixed point $z \in X$.

Proof. Let x_0 be an arbitrary point in X . Choose, by condition (C1), an element $x_1 \in X$ such that $y_0 = Tx_0 = Ax_1$. For the element x_1 , we

choose, by condition (C1), a point $x_2 \in X$ such that $y_1 = Sx_1 = Bx_2$. By induction, we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying for each nonnegative integer n ,

$$(2.2) \quad y_{2n} = Tx_{2n} = Ax_{2n+1} \quad \text{and} \quad y_{2n+1} = Sx_{2n+1} = Bx_{2n}.$$

It is easy to prove (see [6]) that the sequence $\{y_n\}$ is a Cauchy sequence. Since (X, d) is complete, then there exists a point $z \in X$ such that

$$(2.3) \quad z = \lim_{n \rightarrow \infty} Ax_{2n+1} = \lim_{n \rightarrow \infty} Sx_{2n+1} = \lim_{n \rightarrow \infty} Tx_{2n} = \lim_{n \rightarrow \infty} Bx_{2n}.$$

Since A is surjective, then there exists $u \in X$ such that $z = Au$. By applying the inequality (2.1), we get

$$(2.4) \quad G(d(Au, Bx_{2n}), d(Su, Tx_{2n}), d(Au, Su), \\ d(B_{2n}, Tx_{2n}), d(Au, Tx_{2n}), d(Su, B_{2n})) \geq 0,$$

for all nonnegative integer n . By letting $n \rightarrow \infty$ in (2.4) and using the continuity of G , we obtain

$$(2.5) \quad G(0, d(Su, z), d(z, Su), 0, 0, d(Su, z)) \geq 0,$$

By (2.5) and the condition (G_b) we get $d(Su, z) = 0$. That is $Su = z$. Thus, we have $z = Au = Su$. Since A and S are weakly compatible, then we get

$$(2.6) \quad Az = Sz.$$

Since B is surjective, then there exists $v \in X$ such that $z = Bv$. By applying the inequality (2.1), we get

$$(2.7) \quad G(d(Ax_{2n+1}, Bv), d(Sx_{2n+1}, Tv), d(Ax_{2n+1}, Sx_{2n+1}), \\ d(Bv, Tv), d(Ax_{2n+1}, Tv), d(Sx_{2n+1}, Bv)) \geq 0,$$

for all nonnegative integer n . By letting $n \rightarrow \infty$ in (2.7) and using the continuity of G , we obtain

$$(2.8) \quad G(0, d(z, Tv), 0, d(z, Tv), d(z, Tv), 0) \geq 0.$$

By (2.8) and the condition (G_a) we get $d(z, Tv) = 0$. That is $Tv = z$. Thus, we have $z = Bv = Tv$. Since B and T are weakly compatible, then we get

$$(2.9) \quad Bz = Tz.$$

Now, we show that $z = Az$. To this end, we set $x = z$ and $y = v$ in the inequality (2.1). We get

$$G(d(Az, Bv), d(Sz, Tv), d(Az, Sz), d(Bv, Tv), d(Az, Tv), d(Sz, Tv)) \geq 0,$$

which gives

$$(2.10) \quad G(d(Az, z), d(Az, z), 0, 0, d(Az, z), d(Az, z)) \geq 0,$$

(2.10) with the condition (G_3) will imply that $Az = z$. Thus, we have

$z = Az = Sz$. To finish the proof, we show that $z = Bz$. To this end, we let $x = u$ and $y = z$ in the inequality (2.1). We obtain

$$G(d(Au, Bz), d(Su, Tz), d(Au, Su), d(Bz, Tz), d(Au, Tz), d(Su, Bz)) \geq 0,$$

which gives

$$(2.11) \quad G(d(z, Bz), d(z, Bz), 0, 0, d(z, Bz), d(z, Bz)) \geq 0,$$

(2.11) with the condition (G_3) satisfied by G will imply that $Bz = z$. Thus, we have $z = Bz = Tz$. We conclude that z is a common fixed point for A, B, S and T .

Suppose that y is another common fixed point for the mappings A, B, S and T , such that $y \neq z$. Then, by applying the condition (2.1), we obtain

$$\begin{aligned} G(d(Ay, Bz), d(Sy, Tz), d(Ay, Sy), d(Bz, Tz), d(Ay, Tz), d(Sy, Bz)) \\ = G(d(y, z), d(y, z), 0, 0, d(y, z), d(y, z)) \geq 0, \end{aligned}$$

which is a contradiction with the condition (G_3) . This completes the proof. ■

EXAMPLE 2.1. Let $q > 0$ and Let $h(x, y) : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$ satisfying the conditions:

(h_1) : h is continuous non decreasing in the coordinate variables x and y .

(h_2) : $h(x, 0) = h(0, y) = 0$ for all $x, y \in \mathbb{R}_+$.

(h_3) : There exists $\beta \geq 0$, such that for every $x \geq 0$ we have $h(x, x) \leq \beta x^q$.

Since these conditions are depending on q , we denote $\mathcal{H}_q$ the set of functions h satisfying the conditions (h_1) , (h_2) and (h_3) . We set $\mathcal{H} := \bigcup_{q>0} \mathcal{H}_q$.

An example of functions in the class $\mathcal{H}_q$ is being given by setting $h(x, y) = \beta x^{\alpha q} y^{(1-\alpha)q}$, $\forall x, y \geq 0$, where $\alpha \in [0, 1]$ and $\beta > 0$.

If $f \in \mathcal{H}_q$, we set $\beta_q(h) := \sup\{\frac{h(x, x)}{x^q} : x > 0\}$. Let $h \in \mathcal{H}_q$. We set

$$G(t_1, \dots, t_6) = t_1^q - [at_2^{pq} + bt_3^{pq} + ct_4^{pq}]^{\frac{1}{p}} + h(t_5, t_6),$$

where $a > (1 + \beta_q(h))^p$ and $0 \leq b, c < 1$.

It is easy to see that $G \in \mathcal{G}$.

COROLLARY 2.1. Let $q > 0$ and $p \geq 1$. Let A, B, S and T be mappings from a complete metric space (X, d) into itself satisfying the following conditions:

(C1) A and B are surjective.

(C2) The pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible.

(C3) For all $x, y \in X$, we have

$$\begin{aligned} (2.12) \quad d^{pq}(Ax, By) + ph(d(Ax, Ty), d(Sx, By))d^{(p-1)q}(Ax, By) \\ \geq ad^{pq}(Sx, Ty) + bd^{pq}(Ax, Sx) + cd^{pq}(Ty, By), \end{aligned}$$

where $h \in \mathcal{H}_q$ and a, b, c are real numbers such that $a > (1 + \beta_q(h))^p$ and $0 \leq b, c < 1$.

Then A, B, S and T have a unique common fixed point in X .

Proof. The proof is obtained by applying Theorem 2.1 to the mappings A, B, S and T under the implicit relation G given by

$$G(t_1, \dots, t_6) = t_1^q - [at_2^{pq} + bt_3^{pq} + ct_4^{pq}]^{\frac{1}{p}} + h(t_5, t_6),$$

and using the elementary inequality $(u + v)^p \geq u^p + pu^{p-1}v$ valid for all non-negative numbers u, v . ■

This corollary generalizes corollary 1 in [6].

THEOREM 2.2. *Let $G \in \mathcal{G}$. Let A, B and $\{S_n\}$ be mappings from a complete metric space (X, d) into itself satisfying the following conditions:*

(C1) A, B are surjective.

(C2) The pairs $\{A, S_n\}$ and $\{B, S_n\}$ are weakly compatible for each non-negative integer n .

(C3) For all nonnegative integer n and for all $x, y \in X$, we have

$$(2.13) \quad G[d(Ax, By), d(S_nx, S_{n+1}y), d(Ax, S_nx), d(S_{n+1}y, By), \\ d(Ax, S_{n+1}y), d(S_nx, By)] \geq 0.$$

Then A, B, S and $\{S_n\}$ have a unique common fixed point in X .

Proof. Letting $n = 0$ in the inequality (2.13), we get exactly the assumptions of Theorem 2.1 for the mappings $A, B, S = S_0$ and $T = S_1$. So these mappings have a unique common fixed point called z_0 .

We have $Fix\{A, B, S_0\} = \{z_0\}$. Indeed, suppose that there exists a point (say) $y \in X$ such that $y \in Fix\{A, B, S_0\}$ and $y \neq z_0$. Then, by using (2.13) for $n = 0$, we get

$$G(d(Ay, Bz_0), d(S_0y, S_1z_0), d(Ay, S_0y), d(Bz_0, S_1z_0), d(Ay, S_1z_0), d(S_0y, Bz_0)) \\ = G(d(y, z_0), d(y, z_0), 0, 0, d(y, z_0), d(y, z_0)) \geq 0.$$

By the condition (G_3) , the last inequality implies that $y = z_0$. By a similar method, one can prove that z_0 is the unique common fixed point for both A, B and S_1 .

Now, by letting $n = 1$ in the inequality (2.13), we get exactly the assumptions of Theorem 2.1 for the mappings $A, B, S = S_1$ and $T = S_2$. So these mappings have a unique common fixed point called z_1 . Analogously, we prove that

$$\{z_1\} = Fix\{A, B, S_1\} = Fix\{A, B, S_2\}.$$

Thus we have $z_0 = z_1$. By induction, we prove, for every nonnegative integer n , that $\{z_0\} = Fix\{A, B, S_n\}$. We conclude that z_0 is the unique common fixed point for the mappings A, B, S_n for $n = 0, 1, \dots$. This completes the proof. ■

THEOREM 2.3. *Let $G \in \mathcal{G}$. Let S, T and $\{A_n\}$ be mappings from a complete metric space (X, d) into itself satisfying the following conditions:*

(C1) A_n is surjective for each nonnegative integer n .

(C2) The pairs $\{A_n, S\}$ and $\{A_n, T\}$ are weakly compatible for each nonnegative integer n .

(C3) For all nonnegative integer n and for all $x, y \in X$, we have

$$(2.14) \quad G[d(A_n x, A_{n+1} y), d(Sx, Ty), d(A_n x, Sx), d(A_{n+1} y, Ty), \\ d(A_n x, Ty), d(A_{n+1} y, Sx)] \geq 0.$$

Then S, T and $\{A_n\}$ have a unique common fixed point in X .

This theorem improves Theorem 3 of [6]. Its proof is similar to the proof made for Theorem 2.1. So we omit the details.

3. A related result

In non complete metric spaces, by arguments similar to those used in the proof of Theorem 2.1, we obtain the following theorem where the surjectivity of A and B are not needed.

THEOREM 3.1. *Let $G \in \mathcal{G}$ and let A, B, S and T be mappings from a metric space (X, d) into itself satisfying the following conditions:*

(C1) $TX \subset AX$ and $SX \subset BX$.

(C2) The pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible.

(C3) For all $x, y \in X$, we have

$$(3.1) \quad G(d(Ax, By), d(Sx, Ty), d(Ax, Sx), d(By, Ty), \\ d(Ax, Ty), d(Sx, By)) \geq 0.$$

(C4) One of the subsets TX, AX, SX or BX is a complete subspace of X .

Then A, B, S and T have a unique common fixed point $z \in X$.

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