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GENERALIZED INTEGRATION OPERATORS FROM BLOCH TYPE SPACES TO WEIGHTED BMOA SPACES

Abstract. In this paper, we consider a generalized integration operator

$$I_{g,\varphi}^{(n)}f(z) = \int_0^z f^{(n)}(\varphi(\zeta))g(\zeta)d\zeta$$

induced by holomorphic maps g and φ of the open unit disk $\mathbb{D}$, where $\varphi(\mathbb{D}) \subset \mathbb{D}$ and n is a positive integer. We characterize boundedness and compactness of $I_{g,\varphi}^{(n)}$ from Bloch type spaces to weighted BMOA spaces by using logarithmic Carleson measure characterization of the weighted BMOA spaces.

1. Introduction

Let $\mathbb{D}$ be the open unit disk and $\partial\mathbb{D}$ be the unit circle in the complex plane. Denote by $H(\mathbb{D})$ the set of all holomorphic maps on $\mathbb{D}$. For $a \in \mathbb{D}$, let the Möbius transformation $\eta_a : \mathbb{D} \rightarrow \mathbb{D}$ be defined by

$$\eta_a(z) = \frac{a - z}{1 - \bar{a}z}.$$

It is well known that inverse of η_a under composition is η_a , that is, $(\eta_a \circ \eta_a)(z) = z$ for $z \in \mathbb{D}$ and $|\eta'_a(z)| = (1 - |a|^2)/|1 - \bar{a}z|^2$. Moreover,

$$1 - |\eta_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2} = (1 - |z|^2)|\eta'_a(z)|.$$

A function $f \in H(\mathbb{D})$ belongs to the weighted BMOA space $\text{BMOA}_{\log,2}$ if

$$M(f) = \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} |f'(z)|^2 (1 - |\eta_a(z)|^2) dA(z) < \infty.$$

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The expression $M(f)^{1/2}$ defines a semi-norm, while the norm is given by

$$\|f\|_{\text{BMOA}_{\log,2}} = |f(0)| + M(f)^{1/2},$$

which makes $\text{BMOA}_{\log,2}$ a Banach space. The space $\text{BMOA}_{\log,2}$ was introduced by R. Zhao in [25]. In fact, he characterized multipliers of BMOA as follows:

$$f \cdot \text{BMOA} \subset \text{BMOA} \text{ if and only if } f \in H^\infty \cap \text{BMOA}_{\log,2}.$$

Let $\text{VMOA}_{\log,2}$ denote the subspace of $\text{BMOA}_{\log,2}$ consisting of those $f \in \text{BMOA}_{\log,2}$ for which

$$\lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} |f'(z)|^2 (1 - |\eta_a(z)|^2) dA(z) = 0.$$

Let $dA(z) = \frac{1}{\pi} dx dy = \frac{1}{\pi} r dr d\theta$ be the normalized area measure on $\mathbb{D}$. For each $\alpha \in (-1, \infty)$, we set $d\nu_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z)$, $z \in \mathbb{D}$. Then $d\nu_\alpha$ is a probability measure on $\mathbb{D}$. For $0 < p < \infty$, let $\mathcal{A}^p$ be the Bergman space with the norm defined as

$$\|f\|_{\mathcal{A}^p} = \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{1/p} < \infty.$$

Let $\alpha > 0$. A function $f \in H(\mathbb{D})$ is in α -Bloch space $\mathcal{B}^\alpha$ if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty$$

and in the little α -Bloch Space $\mathcal{B}_0^\alpha$ if

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |f'(z)| = 0.$$

For $f \in \mathcal{B}^\alpha$ define

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|.$$

With this norm $\mathcal{B}^\alpha$ is a Banach space, and the little α -Bloch space $\mathcal{B}_0^\alpha$ is a closed subspace of the α -Bloch space. Note that $\mathcal{B}^1 = \mathcal{B}$ is the usual Bloch space.

For any $\alpha > 0$, the weighted Bergman spaces of controlled growth $\mathcal{A}_{\infty}^{-\alpha}$ consists of analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{\mathcal{A}_{\infty}^{-\alpha}} = \sup\{(1 - |z|^2)^\alpha |f(z)| : z \in \mathbb{D}\} < \infty.$$

Each $\mathcal{A}_{\infty}^{-\alpha}$ is a non-separable Banach space with the norm defined above and contains all bounded analytic functions on $\mathbb{D}$. The closure in $\mathcal{A}_{\infty}^{-\alpha}$ of the set of polynomials will be denoted by $\mathcal{A}_{\infty,0}^{-\alpha}$, which is a separable Banach space and consists of exactly those functions f in $\mathcal{A}_{\infty}^{-\alpha}$ with

$$\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^\alpha |f(z)| = 0.$$

We write $\mathcal{LA}_{\infty}^{-\alpha}$ for logarithmic α -Growth space of holomorphic functions f on $\mathbb{D}$ for which

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} \log \left(\frac{2}{1 - |z|^2} \right) |g(z)| < \infty.$$

Likewise we write $\mathcal{LA}_{\infty,0}^{-\alpha}$ for logarithmic little α -Growth space of holomorphic functions f on $\mathbb{D}$ for which

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^{\alpha} \log \left(\frac{2}{1 - |z|^2} \right) |g(z)| = 0.$$

In what follows, we make extensive use of p -logarithmic s -Carleson measures, so we give a short introduction to p -logarithmic s -Carleson measures first.

Suppose $I \subset \partial\mathbb{D}$ is an arc. Denote by $|I|$ the normalized arc length of I , so that $|\partial\mathbb{D}| = 1$. Let $S(I)$ be the Carleson box defined by

$$S(I) = \{z : 1 - |I| < |z| < 1, z/|z| \in I\}.$$

Let μ be a positive Borel measure on $\mathbb{D}$, $0 < p < \infty$ and $0 < s < \infty$. Then we say that μ is a p -logarithmic s -Carleson measure if there is a constant $K > 0$ such that for any arc $I \subset \partial\mathbb{D}$

$$\mu(S(I)) \leq K \frac{|I|^s}{\log(2/|I|)^p}.$$

If $s = 1$ and $p = 2$, we call μ a logarithmic Carleson measure.

The following characterizations of the weighted BMOA spaces $\text{BMOA}_{\log,2}$ and the weighted VMOA spaces $\text{VMOA}_{\log,2}$ are well known (see, for example [8] and [26]).

THEOREM 1.1. *Let $d\mu_f = |f'(z)|^2(1 - |z|^2)dA(z)$. Then the following are equivalent:*

- (1) $f \in \text{BMOA}_{\log,2}$.
- (2) $d\mu_f$ is a logarithmic Carleson measure.
- (3) $\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} |f'(z)|^2(1 - |z|^2) |\eta'_a(z)| dA(z) < \infty$.

THEOREM 1.2. *Let $d\mu_f = |f'(z)|^2(1 - |z|^2)dA(z)$. Then the following are equivalent:*

- (1) $f \in \text{VMOA}_{\log,2}$.
- (2) $d\mu_f$ is a vanishing logarithmic Carleson measure.
- (3) $\sup_{|a| \rightarrow 1} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} |f'(z)|^2(1 - |z|^2) |\eta'_a(z)| dA(z) = 0$.

Finally, we define hyperbolic $\text{BMOA}_{\log,2}^{h,\alpha}$ and $\text{VMOA}_{\log,2}^{h,\alpha}$ spaces as follows: Let $\alpha > 0$. Then a function $f \in H(\mathbb{D})$ such that $|f(z)| < 1$ belongs to

hyperbolic BMOA space $\text{BMOA}_{\log,2}^{h,\alpha}$ if

$$\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \frac{|f'(z)|^2}{(1-|f(z)|^2)^{2\alpha}} (1-|\eta_a(z)|^2) dA(z) < \infty$$

and a function $f \in \text{BMOA}_{\log,2}^{h,\alpha}$ belongs to $\text{VMOA}_{\log,2}^{h,\alpha}$ if

$$\lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \frac{|f'(z)|^2}{(1-|f(z)|^2)^{2\alpha}} (1-|\eta_a(z)|^2) dA(z) = 0.$$

It is well known (see [7]) that an operator T is a surjective isometry of the Dirichlet space $\mathcal{D}^p$, $1 \leq p < \infty, p \neq 2$ with respect to the norm

$$\|f\|_{\mathcal{D}^p}^p = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p dA(z)$$

if and only if there is an automorphism β of $\mathbb{D}$ such the

$$(1.1) \quad (Tf)(z) = \lambda_1 f(0) + \lambda_2 \int_0^z (\beta'(\zeta)^{2/p} f'(\beta'(\zeta))) d\zeta$$

for every $f \in \mathcal{D}^p$. Motivated by (1.1), for a non-negative integer n , φ and g in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$, we define a generalized integration operator as

$$I_{g,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta)) g(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

The operator $I_{g,\varphi}^{(n)}$ is a generalization of the Riemann–Stieltjes operator I_h induced by h , defined as

$$I_h f(z) = \int_0^z f(\zeta) h'(\zeta) d\zeta, \quad z \in \mathbb{D}.$$

Ch. Pommerenke [11] initiated the study of Riemann–Stieltjes operator I_h on H^2 , where he showed that I_h is bounded on H^2 if and only if h is in BMOA. This was extended to other Hardy spaces H^p , $1 \leq p < \infty$, in [1] and [2] where compactness of I_h on H^p and Schatten class membership of I_h on H^2 was also completely characterized in terms of the symbol h . In fact, the operator $I_{g,\varphi}^{(n)}$ induces many known operators. For example, if $n = 1$, then $I_{g,\varphi}^{(n)}$ reduces to an integration operator recently defined by Yoneda in [24]. If $n = 1$ and $g(z) = \varphi'(z)$, then we get the composition operator C_φ induced by φ , defined as

$$C_\varphi f = f \circ \varphi - f(\varphi(0)), \quad f \in H(\mathbb{D}).$$

For general back ground on composition operators, we refer [5], [14] and references therein. Let D be the differentiation operator. Then for a positive

integer n , we have

$$D^n f = f^{(n)}, \quad f \in H(\mathbb{D}).$$

If $n = m + 1$, m a non-negative integer and $g(z) = \varphi'(z)$, then we get the operator $C_\varphi D^m$ defined as $C_\varphi D^m f(z) = f^{(m)}(\varphi(z)) - f^{(m)}(\varphi(0))$. The operator $C_\varphi D^m$ is the product of the composition operator C_φ and the differentiation operator D^m . The operators of the type $C_\varphi D^m$ considered by a number of authors (see [6], [9] and [28]). In this paper, we provide a function theoretic characterization of when g and φ induces bounded and compact integration operators between Bloch and $\text{BMOA}_{\log,2}$ spaces. Problems of this kind for Integration operators were studied recently in [1]–[3], [13], [15], [16], [17], [21] and [22].

Throughout this paper constants are denoted by C , they are positive and not necessarily the same at each occurrence.

2. Boundedness and compactness of

$$I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \rightarrow \text{BMOA}_{\log,2}(\text{VMOA}_{\log,2})$$

In this section, we characterize boundedness and compactness of $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \rightarrow \text{BMOA}_{\log,2}$ and $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \rightarrow \text{VMOA}_{\log,2}$. In the formulation of our Theorems we denote

$$d\mu_{\varphi,g}^{n,\alpha}(z) = \frac{|g(z)|^2}{(1 - |\varphi(z)|^2)^{2(\alpha+n-1)}} (1 - |z|^2) dA(z).$$

THEOREM 2.1. *Let $\alpha > 0$ be a real number, and n be a positive integer. Suppose that g and φ are in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the following statements are equivalent:*

- (1) $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded.
- (2) $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded.
- (3) $\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} \frac{(1 - |\eta_a(z)|^2)}{(1 - |\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) < \infty$.
- (4) $d\mu_{\varphi,g}^{n,\alpha}(z)$ is a logarithmic Carleson measure.

To prove the Theorem 2.1, we need several lemmas. The first among these can be found in [29].

LEMMA 2.2. *Let $0 < p < \infty$ and suppose that n_k is an increasing sequence of positive integers with Hadamard gaps, that is,*

$$\frac{n_{k+1}}{n_k} \geq \lambda > 1$$

for all k . Then there exists constants C_1 and C_2 , depending upon p and λ , such that

$$C_1 \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} \leq \left(\frac{1}{2\pi} \left| \sum_{k=1}^{\infty} a_k e^{in_k \theta} \right|^p d\theta \right)^{1/p} \leq C_2 \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2},$$

for any scalars $a_1, a_2, \dots$ with $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.

The growth of functions and their derivatives in the α -Bloch spaces $\mathcal{B}^\alpha$ is essential to our study. To this end, the following estimates will be useful.

LEMMA 2.3. [27] For $f \in \mathcal{B}^\alpha$,

$$|f(z)| \leq C \begin{cases} \|f\|_{\mathcal{B}^\alpha} & \text{if } 0 < \alpha < 1; \\ \log \frac{2}{1-|z|^2} \|f\|_{\mathcal{B}^\alpha} & \text{if } \alpha = 1; \\ \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha-1}} & \text{if } \alpha > 1; \end{cases}$$

and

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+n-1}} \quad \text{if } n \geq 1.$$

LEMMA 2.4. [12] Let $\alpha > 0$ be a real number. Then there are two holomorphic maps $f_1, f_2 : \mathbb{D} \rightarrow \mathbb{C}$ in $\mathcal{B}^\alpha$ such that

$$|f_1'(z)| + |f_2'(z)| \geq \frac{C}{(1-|z|^2)^\alpha}, \quad z \in \mathbb{D}.$$

Proof of Theorem 2.1. (3) $\Leftrightarrow$ (1). Let $f \in \mathcal{B}^\alpha$ with $\|f\|_{\mathcal{B}^\alpha} \leq 1$. If $f \in \mathcal{B}^\alpha$, then $I_{g,\varphi}^{(n)} f(0) = 0$ for all $n \geq 1$. Thus by Lemma 2.3, we have

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} |(I_{g,\varphi}^{(n)} f)'(z)|^2 (1-|\eta_a(z)|^2) dA(z) \\ & \leq \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} |f^{(n)}(\varphi(z)|^2 |g(z)|^2 (1-|\eta_a(z)|^2) dA(z) \\ & \leq C \|f\|_{\mathcal{B}^\alpha} \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \frac{|g(z)|^2}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} (1-|\eta_a(z)|^2) dA(z). \end{aligned}$$

Thus (3) $\Rightarrow$ (1). Conversely, suppose (1) holds. Let $f \in \mathcal{B}^\alpha$. Then

$$\|I_{g,\varphi}^{(n)} f\|_{\text{BMOA}_{\log,2}} \leq \|I_{g,\varphi}^{(n)}\| \|f\|_{\mathcal{B}^\alpha}.$$

By Lemma 2.4, there exist two Bloch type functions f_1 and f_2 such that

$$\frac{C}{(1-|z|^2)^\alpha} \leq |f_1'(z)| + |f_2'(z)|, \quad z \in \mathbb{D}.$$

We choose $g_1(z) = f_1(z) - z f_1'(0)$ and $g_2(z) = f_2(z) - z f_2'(0)$.

By the well known result (see [27]), we have

$$(1 - |z|^2)^{\alpha+1} |f''(z)| + |f'(0)| \approx (1 - |z|^2)^\alpha |f'(z)|,$$

and so, there exist $g_1, g_2 \in \mathcal{B}^\alpha$ such that

$$\frac{C}{(1 - |z|^2)^{\alpha+1}} \leq |g_1''(z)| + |g_2''(z)|, \quad z \in \mathbb{D}.$$

Proceeding this way, there exist $h_1, h_2 \in \mathcal{B}^\alpha$ such that

$$\frac{C}{(1 - |z|^2)^{\alpha+n-1}} \leq |h_1^{(n)}(z)| + |h_2^{(n)}(z)|, \quad z \in \mathbb{D}.$$

Using the elementary inequality $(x + y)^\alpha \leq 2^\alpha (x^\alpha + y^\alpha)$, which holds for $x, y \geq 0$ and $\alpha > 0$, we have

$$\begin{aligned} \frac{C|g(z)|^2}{(1 - |\varphi(z)|^2)^{2(\alpha+n-1)}} &\leq |g(z)h_1^{(n)}(\varphi(z))|^2 + |g(z)h_2^{(n)}(\varphi(z))|^2 \\ &= |(I_{g,\varphi}^{(n)}h_1)'(z)|^2 + |(I_{g,\varphi}^{(n)}h_2)'(z)|^2, \end{aligned}$$

and so

$$\begin{aligned} C \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} \frac{|g(z)|^2}{(1 - |\varphi(z)|^2)^{2(\alpha+n-1)}} (1 - |\eta_a(z)|^2) dA(z) \\ \leq \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} |(I_{g,\varphi}^{(n)}h_1)'(z)|^2 (1 - |\eta_a(z)|^2) dA(z) \\ + \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} |(I_{g,\varphi}^{(n)}h_2)'(z)|^2 (1 - |\eta_a(z)|^2) dA(z) \\ \leq \|I_{g,\varphi}^{(n)}h_1\|_{\text{BMOA}_{\log,2}} + \|I_{g,\varphi}^{(n)}h_2\|_{\text{BMOA}_{\log,2}} < \infty. \end{aligned}$$

(3) $\Leftrightarrow$ (4). By Theorem 1.1, (3) is equivalent to (4). It is obvious that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3). Assume that (2) holds. We adopt the methods of Theorem 16 in [4], as well as its extension in [10].

Given $a \in \mathbb{D}$, let $\gamma_j \in (1/2, 1)$ be such that $\gamma_j \rightarrow 1$ as $j \rightarrow \infty$ and let

$$f_{j,t}(z) = \sum_{k=1}^{\infty} 2^{k(\alpha-1)} (\gamma_j e^{it} z)^{2k} = \sum_{j=1}^{\infty} a_k z^{2k}.$$

Since $2^{k(1-\alpha)}|a_k| \rightarrow 0$ as $k \rightarrow \infty$ and $\gamma_j \in (0, 1)$, by Theorem 1 in [23], we have that $f_{j,t} \in \mathcal{B}_0^\alpha$ and $\|f_{j,t}\|_{\mathcal{B}^\alpha} \leq C < \infty$, where $C > 0$ is a constant independent of n and t . Since $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded, it follows

that

$$(2.2) \quad C \|I_{g,\varphi}^{(n)}\|_{\text{BMOA}_{\log,2}} \geq \|I_{g,\varphi}^{(n)} f_{j,t}\|_{\mathcal{B}^\alpha} \\ \geq \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} |f_{j,t}^{(n)}(\varphi(z))|^2 |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z).$$

Integrating above inequality with respect to t , applying Fubini's Theorem, Lemma 2.2 and the inequality

$$\prod_{m=0}^{n-1} (2^k - m) \geq \frac{2^{nk}}{n!}, \quad n \in \mathbb{N}, k \in \mathbb{N}, 2^k \geq n,$$

we obtain

$$C \|I_{g,\varphi}^{(n)}\|_{\text{BMOA}_{\log,2}} \\ \geq \frac{1}{2\pi} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \\ \times \left(\int_0^{2\pi} \left(\sum_{k \geq [\log_2 n]}^\infty \prod_{m=0}^{n-1} (2^k - m) 2^{k(\alpha-1)} (\varphi(z))^{2^k-n} (\gamma_j e)^{it2^k} |^p dt \right) \right. \\ \times |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z) \\ \geq C \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \left(\sum_{k \geq [\log_2 n]}^\infty \left(\prod_{m=0}^{n-1} (2^k - m) \right)^2 2^{2k(\alpha-1)} (\gamma_j |\varphi(z)|)^{2(2^k-n)} \right) \\ \times |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z) \\ \geq \frac{C}{(n!)^2} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \left(\sum_{k \geq [\log_2 n]}^\infty 2^{2k(\alpha+n-1)} (\gamma_j |\varphi(z)|)^{2(2^k-n)} \right) \\ \times |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z),$$

where $[x]$ is the greatest integer less than or equal to x . By Lemma 3.1 of [10] there is a constant C depending only on α and such that

$$\sum_{k=1}^\infty 2^{2k\alpha} r^{2^{k+1}} > \frac{C(\alpha)}{(1-r^2)^{2\alpha}}$$

for $r \in (0, 1)$ and $\alpha > 0$. Now since $\alpha + n - 1 > 0$, there is an $r'_0 \in (0, 1)$

$$(2.3) \quad \sum_{k=1}^\infty 2^{2k(\alpha+n-1)} (r_j |\varphi(z)|)^{2^{k+1}} \geq \frac{C}{(1 - (r_j |\varphi(z)|)^2)^{2(\alpha+n-1)}}$$

for all $r'_0 \leq r_j |\varphi(z)| < 1$. Also, there is an $r''_0 \in (0, 1)$ such that

$$(2.4) \quad \sum_{1 \leq k < \lfloor \log_2 n \rfloor} 2^{2k(\alpha+n-1)} (r_j |\varphi(z)|)^{2^{k+1}} \leq \frac{C}{2(1 - (r_j |\varphi(z)|)^2)^{2(\alpha+n-1)}}$$

for all $r''_0 \leq r_j |\varphi(z)| < 1$. Combining (2.3) and (2.4) we obtain

$$\sum_{k \geq \lfloor \log_2 n \rfloor} 2^{2k(\alpha+n-1)} (r_j |\varphi(z)|)^{2^{k+1}} \geq \frac{C'}{2(1 - (r_j |\varphi(z)|)^2)^{2(\alpha+n-1)}}.$$

Substituting in (2.2) we get

$$C' \int_{\mathbb{D}} \frac{|g(z)|^2}{(1 - (r_j |\varphi(z)|)^2)^{2(\alpha+n-1)}} (1 - |\eta_a(z)|^2) dA(z) \leq C \|I_{g,\varphi}^{(n)}\|_{\text{BMOA}_{\log,2}}.$$

Applying Fatou's Lemma to above inequality, we get

$$\begin{aligned} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} \frac{|g(z)|^2}{(1 - |\varphi(z)|^2)^{2(\alpha+n-1)}} (1 - |\eta_a(z)|^2) dA(z) \\ \leq C \|I_{g,\varphi}^{(n)}\|_{\text{BMOA}_{\log,2}}. \end{aligned}$$

Since $a \in \mathbb{D}$ is arbitrary, (3) holds. ■

COROLLARY 2.5. *Let $\alpha > 0$ be a real number, and n be a non-negative integer. Suppose that $\varphi \in H(\mathbb{D})$ is such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the following statements are equivalent:*

- (1) $C_\varphi D^n : \mathcal{B}^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded.
- (2) $C_\varphi D^n : \mathcal{B}_0^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded.
- (3) $\varphi \in \text{BMOA}_{\log,2}^{h,\alpha+n}$
- (4) $d\mu_{\varphi,\varphi'}^{n+1,\alpha}(z)$ is a logarithmic Carleson measure.

COROLLARY 2.6. *Let $\alpha > 0$ be a real number, and n be a positive integer. Suppose that g and φ are in $H(\mathbb{D})$ such that $|\varphi(z)| \leq K < 1$ and $h \in \text{BMOA}_{\log,2}$, where $h' = g$, and K a constant. Then*

$$I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \rightarrow \text{BMOA}_{\log,2}$$

is bounded.

COROLLARY 2.7. *Let $\alpha > 0$ be a real number, and n be a positive integer. Suppose that*

$$g_a(z) = \left(\log \frac{2}{1 - |a|} \right)^{-1} h(z),$$

where $h \in \mathcal{A}^2$ and $\varphi \in H(\mathbb{D})$ is such that $|\varphi(z)| \leq K < 1$. Then $I_{g_a,\varphi}^{(n)} : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \rightarrow \text{BMOA}_{\log,2}$ is bounded.

COROLLARY 2.8. *Let $\alpha > 0$ be a real number, and n be a positive integer such that $\alpha + n < 2$. Suppose that g and φ are in $H(\mathbb{D})$ such that $\varphi(z) = z$ and $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded. Then $g \in \mathcal{LA}_\infty^{-(2-(\alpha+n))}$.*

Proof. Since $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded and $\varphi(z) = z$, by Theorem 2.1, we have

$$(2.5) \quad \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} (1-|z|^{3-2(\alpha+n)}) |\eta'_a(z)| |g(z)|^2 dA(z) < \infty.$$

For a fixed $a \in \mathbb{D}$, let

$$\Omega(a) = \{z \in \mathbb{D} : |z-a| \leq (1-|a|^2)/2\}.$$

Then $\Omega(a) \subset \mathbb{D}$. By the subharmonicity of $|\eta'_a(z)g(z)^2|$, we have

$$(1-|a|^2)^2 |\eta'_a(a)g(a)^2| \leq \int_{\Omega(a)} |\eta'_a(z)| |g(z)|^2 dA(z).$$

Thus, we have

$$(2.6) \quad (1-|a|^2)^{4-2(\alpha+n)} \left(\log \frac{2}{1-|a|} \right)^2 |g(a)|^2 \leq \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} (1-|z|^{3-2(\alpha+n)}) |\eta'_a(z)| |g(z)|^2 dA(z).$$

Therefore, by (2.5) and (2.6), we have

$$\sup_{z \in \mathbb{D}} (1-|z|^2)^{2-(\alpha+n)} \left(\log \frac{2}{1-|z|} \right) |g(z)| < \infty$$

ans so $g \in \mathcal{LA}_\infty^{-(2-(\alpha+n))}$. ■

COROLLARY 2.9. *Let $\alpha > 0$ be a real number and n be a positive integer. Suppose that $g_a(z) = \left(\log \frac{2}{1-|a|}\right)^{-1} h(z)$ and φ are in $H(\mathbb{D})$ such that $\varphi(z) = z$ and $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded. Then $h \in X$, where*

$$X = \begin{cases} \mathcal{A}_\infty^{-(2-\alpha+n)} & \text{if } \alpha + n < 2, \\ H^\infty & \text{if } \alpha + n = 2, \\ \{0\} & \text{if } \alpha + n > 2. \end{cases}$$

Proof. Suppose that $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded. Proceeding as in Corollary 2.8, we have

$$(2.7) \quad \sup_{z \in \mathbb{D}} (1-|z|^2)^{2-(\alpha+n)} |h(z)| < \infty.$$

Thus for $\alpha + n < 2$, $h \in \mathcal{A}_\infty^{-(2-\alpha-n)}$ and for $\alpha + n = 2$, $h \in H^\infty$. Again if $\alpha + n > 2$, then (2.7) implies that there is a positive constant C such that $|h(z)| \leq C(1 - |z|^2)^{\alpha+n-2}$ for all $z \in \mathbb{D}$. It follows that $|h(z)| \rightarrow 0$ as $|z| \rightarrow 1^-$, so by the maximum modulus theorem, we have $h \equiv 0$. ■

COROLLARY 2.10. *Let $\alpha > 0$ be a real number and n be a positive integer. Suppose that $\varphi = \eta'_a$, $g \in H(\mathbb{D})$ and $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded. Then $g \in \mathcal{LA}_\infty^{-1}$.*

Proof. Once again proceeding as in Corollary 2.8, we get the desired result. We omit the details. ■

THEOREM 2.11. *Let $\alpha > 0$ be a real number and n be a positive integer. Suppose that g and φ are in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the following statements are equivalent:*

- (1) $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{VMOA}_{\log,2}$ is bounded.
- (2) $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{VMOA}_{\log,2}$ is compact.
- (3) $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{VMOA}_{\log,2}$ is compact.
- (4) $\lim_{|a| \rightarrow 1} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} \frac{(1 - |\eta_a(z)|^2)}{(1 - |\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) = 0$.
- (5) $d\mu_{\varphi,g}^{h,\alpha}(z)$ is a vanishing logarithmic Carleson measure.

Once again to prove the Theorem 2.11, we need several lemmas.

Recall that the dual of Bergman space $\mathcal{A}^1$ is isomorphic to $\mathcal{B}^\alpha$, that is, $(\mathcal{A}^1)^* \cong \mathcal{B}^\alpha$ and $(\mathcal{B}_0^\alpha)^* \cong \mathcal{A}^1$ with respect to the pairing

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} d\nu_\alpha(z).$$

We also know that $\mathcal{A}^1 \cong l^1$ and that l^1 has the Schur property (see [19]). Moreover, $\text{BMOA}_{\log,2}$ and $\text{VMOA}_{\log,2}$ are Banach spaces, so we have the following lemma.

LEMMA 2.12. *Let $\alpha > 0$ be a real number and n be a positive integer. Suppose that g and φ are in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then*

$$I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{BMOA}_{\log,2}(\text{VMOA}_{\log,2})$$

is weakly compact if and only if $I_{g,\varphi}^{(n)}$ is compact.

Let X and Y be Banach spaces. Then a linear operator $T : X \rightarrow Y$ is compact if and only if $T^{**}(X^{**}) \subset Y$ from Gantmacher's theorem [25], where T^{**} and X^{**} denote the second adjoint of T and second dual of X respectively.

The next lemma can be proved in the same way as in [19, Lemma 2.3], we omit the details.

LEMMA 2.13. *If*

$$\left(\log \frac{2}{1-|a|}\right)^2 \int_{\mathbb{D}} \frac{|g(z)|^2}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} (1-|\eta_a(z)|^2) dA(z)$$

is finite at some point $a \in \mathbb{D}$, *then*

$$\left(\log \frac{2}{1-|a|}\right)^2 \int_{\mathbb{D}} \frac{|g(z)|^2}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} (1-|\eta_a(z)|^2) dA(z)$$

is continuous function of a *on* $\mathbb{D}$.

The following criterion for compactness is a useful tool to us and it follows from standard arguments, for example, to those outlined in Proposition 3.11 of [5].

LEMMA 2.14. *Let* $\alpha > 0$ *be a real number and* n *be a positive integer. Suppose that* g *and* φ *are in* $H(\mathbb{D})$ *such that* $\varphi(\mathbb{D}) \subset \mathbb{D}$. *Then* $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{BMOA}_{\log,2}$ *is compact if and only if for any sequence* $\{f_m\}$ *in* $\mathcal{B}^\alpha$ *with* $\sup_n \|f_n\|_{\mathcal{B}^\alpha} = M < \infty$ *and which converges to zero locally uniformly on* $\mathbb{D}$, *we have* $\lim_{m \rightarrow \infty} \|I_{g,\varphi}^{(n)} f_m\|_{\text{BMOA}_{\log,2}} = 0$.

Proof of Theorem 2.11. By Theorem 1.2, we have (4) is equivalent to (5). Again by Lemma 2.12 and Gantmacher's theorem, $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{VMOA}_{\log,2}$ is compact if and only if $I_{g,\varphi}^{(n)} f \in \text{VMOA}_{\log,2}$, since $(\mathcal{B}_0^\alpha)^{**} \cong \mathcal{B}^\alpha$. Hence (1) is equivalent (3). Also (2) obviously implies (1). Thus to complete the proof we will prove that (4) is equivalent to (1) and (4) implies (2).

(1) $\Leftrightarrow$ (4). First suppose that (1) holds. Let $f_1, f_2 \in \mathcal{B}^\alpha$ be two functions such that

$$\frac{C}{(1-|z|^2)^{\alpha+n-1}} \leq |f_1^{(n)}(z)| + |f_2^{(n)}(z)|, \quad z \in \mathbb{D}.$$

Thus by using the elementary inequality $(x+y)^\alpha \leq 2^\alpha(x^\alpha + y^\alpha)$, which holds for $x, y \geq 0$ and $\alpha > 0$, we have

$$\begin{aligned} \left(\log \frac{2}{1-|a|}\right)^2 \int_{\mathbb{D}} \frac{(1-|\eta_a(z)|^2)}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) \\ \leq C' (\|I_{g,\varphi}^{(n)} f_1\|_{\text{BMOA}_{\log,2}}^2 + \|I_{g,\varphi}^{(n)} f_2\|_{\text{BMOA}_{\log,2}}^2). \end{aligned}$$

Thus (4) follows from boundedness of $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{VMOA}_{\log,2}$.

Conversely, we suppose that (4) holds. Then by Theorem 2.1, we have

$$\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|}\right)^2 \int_{\mathbb{D}} \frac{(1-|\eta_a(z)|^2)}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) < \infty.$$

Hence $I_{g,\varphi}^{(n)}f \in \text{BMOA}_{\log,2}$ for $f \in \mathcal{B}^\alpha$. Further, by (4)

$$\lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \frac{(1-|\eta_a(z)|^2)}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) = 0$$

and so by Lemma 2.3, we have

$$\begin{aligned} & \lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} |(I_{g,\varphi}^{(n)}f)'(z)|^2 (1-|\eta_a(z)|^2) dA(z) \\ & \leq \|f\|_{\mathcal{B}^\alpha} \lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \frac{(1-|\eta_a(z)|^2)}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) = 0. \end{aligned}$$

Finally, we prove that (4) implies (2). Let $\|f_m\|_{\mathcal{B}^\alpha} \leq 1$ be such that $\|f_m\|_{\mathcal{B}^\alpha} \rightarrow 0$ as $m \rightarrow \infty$ on compact subsets of $\mathbb{D}$. To complete the proof, we only need to prove that

$$\|I_{g,\varphi}^{(n)}f_m\|_{\text{BMOA}_{\log,2}} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Since

$$\lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \frac{(1-|\eta_a(z)|^2)}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) = 0.$$

Thus there is $r_0 > 0$ such that

$$\left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \frac{(1-|\eta_a(z)|^2)}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) < \epsilon$$

for every $\epsilon > 0$ and $a \in \{z \in \mathbb{D} : |z| > r\}$. Hence for any compact subset $K \subset \mathbb{D}$, we have

$$(2.8) \quad \Phi(a) = \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D} \setminus K} \frac{(1-|\eta_a(z)|^2)}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) < \epsilon,$$

for $a \in \{z \in \mathbb{D} : |z| > r\}$. By Lemma 2.13, $\Phi(a)$ is continuous also. For any given a_i , there is a compact set $K_i \subset \mathbb{D}$ such that (2.8) holds. The continuity of $\Phi(a)$ guaranteed that there is a neighbourhood $U(a)$ such that (2.8) holds. By the compactness of $\{z \in \mathbb{D} : |z| \leq r\}$, we choose finite $U(a_i)$ ($i = 1, 2, \dots, n$) such that $\{z \in \mathbb{D} : |z| \leq r\} = \bigcup_{i=1}^n K_i$, then for all $a \in \{z \in \mathbb{D} : |z| \leq r\}$, (2.8) holds for all $a \in \mathbb{D}$. There exists a number N such that for $m > N$,

$$\sup_k [f_m^{(n)}(\varphi(z))]^2 (1-|\varphi(z)|^2)^{2(\alpha+n-1)} < \epsilon.$$

Then for $m > N$ and any $a \in \mathbb{D}$, we have

$$\begin{aligned}
& \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} |(I_{g,\varphi}^{(n)} f)'(z)|^2 (1-|\eta_a(z)|^2) dA(z) \\
& \leq \epsilon \left(\log \frac{2}{1-|a|} \right)^2 \int_K (1-|\eta_a(z)|^2) |g(z)|^2 dA(z) \\
& \quad + \epsilon \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D} \setminus K} \frac{(1-|\eta_a(z)|^2)}{(1-|\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) < C\epsilon,
\end{aligned}$$

where C is a constant. Hence (4) implies (2). ■

COROLLARY 2.15. *Let $\alpha > 0$ be a real number, and n be a positive integer. Suppose that $\varphi \in H(\mathbb{D})$ is such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the following statements are equivalent:*

- (1) $C_\varphi D^n : \mathcal{B}^\alpha \rightarrow \text{VMOA}_{\log,2}$ is bounded.
- (2) $C_\varphi D^n : \mathcal{B}^\alpha \rightarrow \text{VMOA}_{\log,2}$ is compact.
- (3) $C_\varphi D^n : \mathcal{B}_0^\alpha \rightarrow \text{VMOA}_{\log,2}$ is compact.
- (4) $\varphi \in \text{VMOA}_{\log,2}^{h,\alpha+n}$.
- (5) $d\mu_{\varphi,\varphi'}^{n+1,\alpha}(z)$ is a vanishing logarithmic Carleson measure.

COROLLARY 2.16. *Let $\alpha > 0$ be a real number, and n be a positive integer. Suppose that g and φ are in $H(\mathbb{D})$ such that $|\varphi(z)| \leq K < 1$, and $h \in \text{VMOA}_{\log,2}$, where $h' = g$ and K is a constant. Then $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \rightarrow \text{VMOA}_{\log,2}$ is bounded as well as compact.*

COROLLARY 2.17. *Let $\alpha > 0$ be a real number, and n be a positive integer such that $\alpha + n < 2$. Suppose that g and φ are in $H(\mathbb{D})$ such that $\varphi(z) = z$, and $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{VMOA}_{\log,2}$ is bounded or compact. Then $g \in \mathcal{LA}_{\infty,0}^{-(2-(\alpha+n))}$.*

Proof. Since $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{VMOA}_{\log,2}$ is bounded and $\varphi(z) = z$, by Theorem 2.11, we have

$$(2.9) \quad \lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} (1-|z|^{3-2(\alpha+n)}) |\eta'_a(z)| |g(z)|^2 dA(z) = 0.$$

Now proceeding as in Corollary 2.8, we have

$$\begin{aligned}
(2.10) \quad & (1-|a|^2)^{4-2(\alpha+n)} \left(\log \frac{2}{1-|a|} \right)^2 |g(a)|^2 \\
& \leq \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} (1-|z|^{3-2(\alpha+n)}) |\eta'_a(z)| |g(z)|^2 dA(z).
\end{aligned}$$

Therefore, by (2.9) and (2.10), we have

$$\lim_{|a| \rightarrow 1} (1 - |z|^2)^{2-(\alpha+n)} \left(\log \frac{2}{1 - |z|} \right) |g(z)| = 0$$

ans so $g \in \mathcal{LA}_{\infty,0}^{-(2-(\alpha+n))}$. ■

COROLLARY 2.18. *Let $\alpha > 0$ be a real number and n be a positive integer. Suppose that $\varphi = \eta'_a$, $g \in H(\mathbb{D})$ and $I_{g,\varphi}^{(n)} : \mathcal{B}^\alpha \rightarrow \text{VMOA}_{\log,2}$ is bounded or compact. Then $g \in \mathcal{LA}_{\infty,0}^{-1}$.*

Proof. Once again proceeding as in Corollary 2.8, we get the desired result. We omit the details. ■

THEOREM 2.19. *Let $\alpha > 0$ be a real number and n be a positive integer. Suppose that g and φ are in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then*

- (1) $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{VMOA}_{\log,2}$ is bounded.
- (2) $\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} \frac{(1 - |\eta_a(z)|^2)}{(1 - |\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z) < \infty$
and
 $\lim_{|a| \rightarrow 1} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z) = 0.$

Proof. Suppose that conditions in (2) hold and $f \in \mathcal{B}_0^\alpha$. Then by Theorem 2.1, $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{BMOA}_{\log,2}$ is bounded. We only show that $I_{g,\varphi}^{(n)} f \in \text{VMOA}_{\log,2}$. For any $\epsilon > 0$, we can choose $r \in (0, 1)$ such that

$$|f^{(n)}(z)|^2 (1 - |z|^2)^{2(\alpha+n-1)} < \epsilon \quad \text{for } |z| > r.$$

Thus

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{|\varphi(z)| > r} |f^{(n)}(\varphi(z))|^2 |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z) \\ & \leq \epsilon \sup_{a \in \mathbb{D}} \left(\log \frac{2}{1 - |a|} \right)^2 \int_{\mathbb{D}} \frac{(1 - |\eta_a(z)|^2)}{(1 - |\varphi(z)|^2)^{2(\alpha+n-1)}} |g(z)|^2 dA(z). \end{aligned}$$

Again for r as above, we have

$$\begin{aligned}
& \lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{|\varphi(z)| \leq r} |f^{(n)}(\varphi(z))|^2 |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z) \\
& \leq \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-r^2)^{2(\alpha+n-1)}} \lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{|\varphi(z)| \leq r} |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z) \\
& \leq \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-r^2)^{2(\alpha+n-1)}} \lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z).
\end{aligned}$$

Combining the above two inequalities, we get that $I_{g,\varphi}^{(n)} f \in \text{VMOA}_{\log,2}$.

Conversely suppose $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{VMOA}_{\log,2}$ is bounded. Let $f(z) = z$ be in $\mathcal{B}_0^\alpha$. Then boundedness of $I_{g,\varphi}^{(n)}$ implies that $I_{g,\varphi}^{(n)} f \in \text{VMOA}_{\log,2}$. Thus

$$\lim_{|a| \rightarrow 1} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} |g(z)|^2 (1 - |\eta_a(z)|^2) dA(z) = 0.$$

Again, if $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{VMOA}_{\log,2}$ is bounded, then $I_{g,\varphi}^{(n)} : \mathcal{B}_0^\alpha \rightarrow \text{BMOA}_{\log,2}$ is also bounded. Thus by Theorem 2.1, we get

$$\sup_{a \in \mathbb{D}} \left(\log \frac{2}{1-|a|} \right)^2 \int_{\mathbb{D}} \frac{|g(z)|^2 (1 - |\eta_a(z)|^2)}{(1 - |\varphi(z)|^2)^{2(\alpha+n-1)}} dA(z) < \infty. \blacksquare$$

COROLLARY 2.20. *Let $\alpha > 0$ be a real number and n be a positive integer. Suppose that g and φ are in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then the following statements are equivalent:*

- (1) $C_\varphi D^n : \mathcal{B}_0^\alpha \rightarrow \text{VMOA}_{\log,p}^s$ is bounded.
- (2) $\varphi \in \text{BMOA}_{\log,2}^{h,\alpha+n} \cap \text{VMOA}_{\log,2}$.

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