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SOME GENERAL RESULTS AND EXTREME POINTS OF p -VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

Abstract. In this paper, we further investigate the class of functions $\phi^*(n, p, \lambda, \alpha)$ which are analytic in the open unit disk $\Delta = \{z : |z| < 1\}$, and involve the combinations of the representations of p -valently starlike and convex functions. We obtain several generalized results on the modified-Hadamard product of the class $\phi^*(n, p, \lambda, \alpha)$, which extend the corresponding results obtained by Altıntaş et al. [Computers Math. Applic. 30(2), 9-16, (1995)]; Darwish, Aouf [Mathematical and Computer Modelling(2007), doi:10.1016/j.mcm.2007.08.016]. Further, by fixing the second coefficient of functions in $\phi^*(n, p, \lambda, \alpha)$, we introduce the special class $\phi_c^*(n, p, \lambda, \alpha)$ and discuss the extreme points.

1. Introduction

Let $S(n, p)$ denote the class of functions of the form

$$(1.1) \quad f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0, n, p \in N = \{1, 2, \dots\})$$

which are analytic and p -valent in the unit disk $\Delta = \{z : |z| < 1\}$.

We denote the class of p -valent starlike functions of order α and the p -valent convex functions of order α by $T_n(p, \alpha)$, $C_n(p, \alpha)$, $0 \leq \alpha < p$, respectively, where

$$T_n(p, \alpha) = \left\{ f(z) : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, f(z) \in S(n, p), z \in \Delta \right\}$$

and

$$C_n(p, \alpha) = \left\{ f(z) : \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha, f(z) \in S(n, p), z \in \Delta \right\}.$$

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Owa [1] have studied the classes $T_n(p, \alpha)$ and $C_n(p, \alpha)$, and H. E. Darwish, M. K. Aouf [2] further investigate them. Special cases of them have been considered by Srivastava et al. [3] and Domokos [4] with $T_n(1, \alpha)$, $C_n(1, \alpha)$.

A function $f(z) \in S(n, p)$ is said to be in the class $\phi^*(n, p, \lambda, \alpha)$ if and only if

$$(1.2) \quad \operatorname{Re} \left\{ \frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1 - \lambda)f(z)} \right\} > \alpha \quad (0 \leq \alpha < p, 0 \leq \lambda \leq 1, z \in \Delta).$$

The class $\phi^*(n, p, \lambda, \alpha)$ was studied by Altıntaş et al. [5] and Altıntaş et al. [6]. In particular, $\phi^*(n, 1, \lambda, \alpha) \equiv p(n, \lambda, \alpha)$ was considered earlier by Altıntaş [7]. We can note that $\phi^*(n, p, 0, \alpha) \equiv T_n(p, \alpha)$, $\phi^*(n, p, 1, \alpha) \equiv C_n(p, \alpha)$, therefore, the class $\phi^*(n, p, \lambda, \alpha)$ is the generalization of $C_n(p, \alpha)$ and $T_n(p, \alpha)$.

Now we can fix the second coefficient of functions in $\phi^*(n, p, \lambda, \alpha)$ and obtain the following class:

$$\phi_c^*(n, p, \lambda, \alpha) = \left\{ f(z) : f(z) = z^p - a_{n+p} z^{n+p} - \sum_{k=n+p+1}^{\infty} a_k z^k \in \phi^*(n, p, \lambda, \alpha), \right. \\ \left. a_{n+p} = \frac{c(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]}, 0 \leq c \leq 1 \right\}.$$

In the present paper, we will obtain several generalized results on $C_n(p, \alpha)$ and $T_n(p, \alpha)$ by further investigating $\phi^*(n, p, \lambda, \alpha)$, which mainly extend the corresponding results obtained by Altıntaş et al. [5], Owa [1], H. E. Darwish, M. K. Aouf [2], Domokos [4]. Also we discuss the extreme points of the $\phi_c^*(n, p, \lambda, \alpha)$.

2. The general results on two modified-Hadamard products

In order to investigate the class $\phi^*(n, p, \lambda, \alpha)$, we begin with the following Lemma given by Altıntaş et al. [5]:

LEMMA 1. *If the function $f(z) \in S(n, p)$, then $f(z)$ is in the class $\phi^*(n, p, \lambda, \alpha)$ if and only if*

$$(2.1) \quad \sum_{k=n+p}^{\infty} (k - \alpha)[\lambda(k - 1) + 1] a_k \leq (p - \alpha)[\lambda(p - 1) + 1] \quad (n, p \in N).$$

Let

$$f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \quad (a_{k,j} \geq 0; j = 1, 2, \dots, m; p, n \in N).$$

We define the generalized modified-Hadamard product $f_1 * f_2^*, \dots, * f_m(z)$ by

$$f_1 * f_2^*, \dots, * f_m(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,1} a_{k,2} \dots a_{k,m} z^k.$$

THEOREM 1. *If the functions $f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \in \phi^*(n, p, \lambda, \alpha_j)$, $j=1, 2, \dots, m$, then $f_1 * f_2^* \dots * f_m(z) \in \phi^*(n, p, \lambda, \gamma_m)$ when $n \geq p$, where γ_m is determined by*

$$\begin{cases} \gamma_m = \alpha_1, & m = 1, \\ \gamma_m = p - \frac{A}{B}, & m > 1, \end{cases}$$

where $A = n(\lambda p - \lambda + 1)(p - \gamma_{m-1})(p - \alpha_m)$, $B = [\lambda(n+p) - \lambda + 1](n+p - \gamma_{m-1})(n+p - \alpha_m) - (\lambda p - \lambda + 1)(p - \gamma_{m-1})(p - \alpha_m)$.

The result is sharp for the functions $f(z)$ given by

$$f(z) = z^p - \frac{(p - \alpha_j)[\lambda(p - 1) + 1]}{(n + p - \alpha_j)[\lambda(n + p - 1) + 1]} z^{n+p}, \quad j = 1, 2, \dots, m; \quad n, p \in \mathbb{N}.$$

Proof. First we need to prove the theorem when $m = 2$.

Suppose the functions

$$(2.2) \quad f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \in \phi^*(n, p, \lambda, \alpha_j), \quad j = 1, 2,$$

then from Lemma 1, we have

$$\sum_{k=n+p}^{\infty} \frac{(k - \alpha_j)[\lambda(k - 1) + 1]}{(p - \alpha_j)[\lambda(p - 1) + 1]} a_{k,j} \leq 1, \quad j = 1, 2,$$

moreover

$$\left\{ \sum_{k=n+p}^{\infty} \frac{(k - \alpha_j)[\lambda(k - 1) + 1]}{(p - \alpha_j)[\lambda(p - 1) + 1]} a_{k,j} \right\}^{\frac{1}{2}} \leq 1, \quad j = 1, 2.$$

By using the Holder inequality, we get

$$\sum_{k=n+p}^{\infty} \left[\frac{(k - \alpha_1)(\lambda k - \lambda + 1)}{(p - \alpha_1)(\lambda p - \lambda + 1)} \right]^{\frac{1}{2}} \left[\frac{(k - \alpha_2)(\lambda k - \lambda + 1)}{(p - \alpha_2)(\lambda p - \lambda + 1)} \right]^{\frac{1}{2}} a_{k,1}^{\frac{1}{2}} a_{k,2}^{\frac{1}{2}} \leq 1.$$

So

$$(2.3) \quad \sum_{k=n+p}^{\infty} \frac{\lambda k - \lambda + 1}{\lambda p - \lambda + 1} \left(\frac{k - \alpha_1}{p - \alpha_1} \right)^{\frac{1}{2}} \left(\frac{k - \alpha_2}{p - \alpha_2} \right)^{\frac{1}{2}} \sqrt{a_{k,1} a_{k,2}} \leq 1$$

and

$$(2.4) \quad \frac{1}{\sqrt{a_{k,1}a_{k,2}}} \geq \frac{\lambda k - \lambda + 1}{\lambda p - \lambda + 1} \left(\frac{k - \alpha_1}{p - \alpha_1} \right)^{\frac{1}{2}} \left(\frac{k - \alpha_2}{p - \alpha_2} \right)^{\frac{1}{2}} \quad (k \geq n + p).$$

We have to find the largest γ_2 such that

$$(2.5) \quad \sum_{k=n+p}^{\infty} \frac{(k - \gamma_2)[\lambda k - \lambda + 1]}{(p - \gamma_2)[\lambda p - \lambda + 1]} a_{k,1} a_{k,2} \leq 1.$$

From (2.3), (2.5) hold true if

$$\frac{k - \gamma_2}{p - \gamma_2} a_{k,1} a_{k,2} \leq \left(\frac{k - \alpha_1}{p - \alpha_1} \right)^{\frac{1}{2}} \left(\frac{k - \alpha_2}{p - \alpha_2} \right)^{\frac{1}{2}} \sqrt{a_{k,1} a_{k,2}} \quad (k \geq n + p)$$

or

$$(2.6) \quad \frac{k - \gamma_2}{p - \gamma_2} \leq \left(\frac{k - \alpha_1}{p - \alpha_1} \right)^{\frac{1}{2}} \left(\frac{k - \alpha_2}{p - \alpha_2} \right)^{\frac{1}{2}} \frac{1}{\sqrt{a_{k,1} a_{k,2}}} \quad (k \geq n + p).$$

From (2.4), (2.6) is equal to

$$\gamma_2 \leq p - \frac{(k - p)(\lambda p - \lambda + 1)(p - \alpha_1)(p - \alpha_2)}{(\lambda k - \lambda + 1)(k - \alpha_1)(k - \alpha_2) - (\lambda p - \lambda + 1)(p - \alpha_1)(p - \alpha_2)} \quad (k \geq n + p).$$

Now let

$$g(k) = p - \frac{(k - p)(\lambda p - \lambda + 1)(p - \alpha_1)(p - \alpha_2)}{(\lambda k - \lambda + 1)(k - \alpha_1)(k - \alpha_2) - (\lambda p - \lambda + 1)(p - \alpha_1)(p - \alpha_2)},$$

then $g'(k) = \frac{A(k)}{B(k)}$, where

$$\begin{aligned} B(k) &= [(\lambda k - \lambda + 1)(k - \alpha_1)(k - \alpha_2) - (\lambda p - \lambda + 1)(p - \alpha_1)(p - \alpha_2)]^2 > 0, \\ A(k) &= \lambda(k - p)(\lambda p - \lambda + 1)(p - \alpha_1)(p - \alpha_2)(k - \alpha_1)(k - \alpha_2) \\ &\quad + (\lambda p - \lambda + 1)^2 (p - \alpha_1)^2 (p - \alpha_2)^2 \\ &\quad + (\lambda p - \lambda + 1)(\lambda k - \lambda + 1)(p - \alpha_1)(p - \alpha_2)[k(k - 2p) + \alpha_1(p - \alpha_2) + p\alpha_2]. \end{aligned}$$

Since $n \geq p$, we have $k \geq n + p \geq 2p$, so $A(k) \geq 0$, it implies that $g(k)$ is increasing in k . This gives that

$$(2.7) \quad \gamma_2 \leq g(n + p) = p - \frac{n(\lambda p - \lambda + 1)(p - \alpha_1)(p - \alpha_2)}{[\lambda(n + p) - \lambda + 1](n + p - \alpha_1)(n + p - \alpha_2) - (\lambda p - \lambda + 1)(p - \alpha_1)(p - \alpha_2)}.$$

Therefore the result is true for $m = 2$.

When $m = 3$,

$$f_1 * f_2 * f_3(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,1}a_{k,2}a_{k,3}z^k = z^p - \sum_{k=n+p}^{\infty} \omega_k a_{k,3}z^k,$$

where $\omega_k = a_{k,1}a_{k,2}$.

From (2.7), we can know $f_1 * f_2(z) = z^p - \sum_{k=n+p}^{\infty} w_k z^k \in \phi^*(n, p, \lambda, \gamma_2)$,

use the ω_k and $a_{k,3}$ take place of the $a_{k,1}$ and $a_{k,2}$ of (2.2), respectively, therefore we have changed the case $m = 3$ to the case $m = 2$, it is easy to obtain

$$\gamma_3 \leq p - \frac{n(\lambda p - \lambda + 1)(p - \gamma_2)(p - \alpha_3)}{[\lambda(n+p) - \lambda + 1](n+p - \gamma_2)(n+p - \alpha_3) - (\lambda p - \lambda + 1)(p - \gamma_2)(p - \alpha_3)}.$$

Take the same process, we can prove the $\gamma_4, \gamma_5, \dots, \gamma_m$. So the

$$\gamma_m \leq p - \frac{n(\lambda p - \lambda + 1)(p - \gamma_{m-1})(p - \alpha_m)}{[\lambda(n+p) - \lambda + 1](n+p - \gamma_{m-1})(n+p - \alpha_m) - (\lambda p - \lambda + 1)(p - \gamma_{m-1})(p - \alpha_m)}$$

is true. ■

A special case of Theorem 1 when $\lambda = 0$ and $\lambda = 1$ yield

COROLLARY 1. If the functions $f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j}z^k \in T_n(p, \alpha_j)$, $j = 1, 2, \dots, m$, then $f_1 * f_2 * \dots * f_m(z) \in T_n(p, \gamma_m)$ when $n \geq p$, where γ_m is determined by

$$\begin{cases} \gamma_1 = \alpha_1, & m = 1, \\ \gamma_m = p - \frac{n(p - \gamma_{m-1})(p - \alpha_m)}{(n+p - \gamma_{m-1})(n+p - \alpha_m) - (p - \gamma_{m-1})(p - \alpha_m)}, & m > 1. \end{cases}$$

The result is sharp for the functions $f(z)$ given by

$$f(z) = z^p - \frac{p - \alpha_j}{n+p - \alpha_j} z^{n+p}, \quad j = 1, 2, \dots, m; \quad n, p \in \mathbb{N}.$$

COROLLARY 2. If the functions $f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j}z^k \in C_n(p, \alpha_j)$, $4j = 1, 2, \dots, m$, then $f_1 * f_2 * \dots * f_m(z) \in C_n(p, \gamma_m)$ when $n \geq p$, where γ_m is determined by

$$\begin{cases} \gamma_1 = \alpha_1, & m = 1, \\ \gamma_m = p - \frac{np(p - \gamma_{m-1})(p - \alpha_m)}{(n+p)(n+p - \gamma_{m-1})(n+p - \alpha_m) - p(p - \gamma_{m-1})(p - \alpha_m)}, & m > 1. \end{cases}$$

The result is sharp for the functions $f(z)$ given by

$$f(z) = z^p - \frac{p(p - \alpha_j)}{(n + p)(n + p - \alpha_j)} z^{n+p}, \quad j = 1, 2, \dots, m; n, p \in N.$$

REMARK 1. Taking $m = 2$, $\alpha_1 = \alpha_2 = \alpha$ in the Theorem 1, we can obtain the result obtained by O. Altıntaş et al. [5, Theorem 3.4].

THEOREM 2. If the functions $f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \in \phi^*(n, p, \lambda, \alpha_j)$, $j=1, 2, \dots, m$. Define $F_m(z) = z^p - \sum_{k=n+p}^{\infty} (\sum_{j=1}^m a_{k,j}^r) z^k$ ($z \in \Delta$; $n, p \in N$; $r > 1$), then when

$$(2.8) \quad p(n+p-\alpha)^r [\lambda(n+p-1)+1]^{r-1} \geq m(n+p)(p-\alpha)^r [\lambda(p-1)+1]^{r-1},$$

we have $F_m(z) \in \phi^*(n, p, \lambda, \gamma_m)$, where

$$\gamma_m = p - \frac{mn(p-\alpha)^r [\lambda(p-1)+1]^{r-1}}{(n+p-\alpha)^r [\lambda(n+p-1)+1]^{r-1} - m(p-\alpha)^r [\lambda(p-1)+1]^{r-1}},$$

$$\alpha = \min_{1 \leq j \leq m} \{\alpha_j\}.$$

The result is sharp for the functions $f(z)$ given by

$$f(z) = z^p - \frac{(p - \alpha_j)[\lambda(p-1)+1]}{(n+p-\alpha_j)[\lambda(n+p-1)+1]} z^{n+p}, \quad j = 1, 2, \dots, m; n, p \in N.$$

Proof. Suppose the functions $f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \in \phi^*(n, p, \lambda, \alpha_j)$, $j = 1, 2, \dots, m$, then from Lemma 1, we have

$$\sum_{k=n+p}^{\infty} \frac{(k - \alpha_j)[\lambda(k-1)+1]}{(p - \alpha_j)[\lambda(p-1)+1]} a_{k,j} \leq 1, \quad j = 1, 2, \dots, m.$$

By virtue of the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=n+p}^{\infty} \left\{ \frac{(k - \alpha_j)[\lambda(k-1)+1]}{(p - \alpha_j)[\lambda(p-1)+1]} \right\}^r a_{k,j}^r$$

$$\leq \left\{ \sum_{k=n+p}^{\infty} \frac{(k - \alpha_j)[\lambda(k-1)+1]}{(p - \alpha_j)[\lambda(p-1)+1]} a_{k,j} \right\}^r \leq 1, \quad j = 1, 2, \dots, m.$$

So

$$(2.9) \quad \sum_{k=n+p}^{\infty} \left\{ \frac{1}{m} \sum_{j=1}^m \left\{ \frac{(k - \alpha_j)[\lambda(k-1)+1]}{(p - \alpha_j)[\lambda(p-1)+1]} \right\}^r a_{k,j}^r \right\} \leq 1.$$

We must find the largest γ_m such that

$$(2.10) \quad \sum_{k=n+p}^{\infty} \frac{(k-\gamma_m)[\lambda(k-1)+1]}{(p-\gamma_m)[\lambda(p-1)+1]} \left(\sum_{j=1}^m a_{k,j}^r \right) \leq 1,$$

From (2.9), (2.10) hold true if

$$\begin{aligned} & \frac{(k-\gamma_m)[\lambda(k-1)+1]}{(p-\gamma_m)[\lambda(p-1)+1]} \left(\sum_{j=1}^m a_{k,j}^r \right) \\ & \leq \frac{1}{m} \sum_{j=1}^m \left\{ \frac{(k-\alpha_j)[\lambda(k-1)+1]}{(p-\alpha_j)[\lambda(p-1)+1]} \right\}^r a_{k,j}^r \quad (k \geq n+p). \end{aligned}$$

Let

$$\alpha = \min_{1 \leq j \leq m} \{\alpha_j\},$$

we have

$$\begin{aligned} \frac{1}{m} \sum_{j=1}^m \left\{ \frac{(k-\alpha_j)[\lambda(k-1)+1]}{(p-\alpha_j)[\lambda(p-1)+1]} \right\}^r a_{k,j}^r & \geq \frac{1}{m} \sum_{j=1}^m \left\{ \frac{(k-\alpha)[\lambda(k-1)+1]}{(p-\alpha)[\lambda(p-1)+1]} \right\}^r a_{k,j}^r \\ & = \frac{1}{m} \left\{ \frac{(k-\alpha)[\lambda(k-1)+1]}{(p-\alpha)[\lambda(p-1)+1]} \right\}^r \sum_{j=1}^m a_{k,j}^r. \end{aligned}$$

So (2.10) is equivalent to

$$(2.11) \quad \frac{(k-\gamma_m)[\lambda(k-1)+1]}{(p-\gamma_m)[\lambda(p-1)+1]} \leq \frac{1}{m} \left\{ \frac{(k-\alpha)[\lambda(k-1)+1]}{(p-\alpha)[\lambda(p-1)+1]} \right\}^r \quad (k \geq n+p).$$

Using (2.8), we can know that (2.11) is true if

$$\gamma_m \leq p - \frac{m(k-p)(p-\alpha)^r[\lambda(p-1)+1]^{r-1}}{(k-\alpha)^r[\lambda(k-1)+1]^{r-1} - m(p-\alpha)^r[\lambda(p-1)+1]^{r-1}} \quad (k \geq n+p).$$

Now let

$$g(k) = p - \frac{m(k-p)(p-\alpha)^r[\lambda(p-1)+1]^{r-1}}{(k-\alpha)^r[\lambda(k-1)+1]^{r-1} - m(p-\alpha)^r[\lambda(p-1)+1]^{r-1}},$$

then $g'(k) = \frac{C(k)}{D(k)}$, where

$$\begin{aligned} D(k) &= p(k-\alpha)[\lambda(k-1)+1] \\ & \quad \cdot \{[\lambda(k-1)+1]^{r-1}(k-\alpha)^r - m(p-\alpha)^r[\lambda(p-1)+1]^{r-1}\}^2 > 0, \\ C(k) &= p(k-\alpha)[\lambda(k-1)+1]m^2(p-\alpha)^{2r}[\lambda(p-1)+1]^{2(r-1)} \\ & \quad + pm(p-\alpha)^r[\lambda(p-1)+1]^{r-1}(k-\alpha)^r[\lambda(k-1)+1]^{r-1} \\ & \quad \cdot \{\lambda(k-p)(k-\alpha)(r-1) + [\lambda(k-1)+1][(k-p)(r-1) - (p-\alpha)]\}. \end{aligned}$$

It is clearly $C(k) \geq C(n+p)$, using (2.8), we obtain

$$C(k) \geq C(n+p) \geq m^2(p-\alpha)^{2r}[\lambda(p-1)+1]^{2(r-1)} \cdot \{[\lambda(n+p-1)+1][p(p-\alpha)+n(r-1)+p(n-1)]+\lambda n(n+p-\alpha)(r-1)\} \geq 0.$$

We see that $g'(k) = \frac{C(k)}{D(k)} \geq 0$, this implies $g(k)$ is increasing in k , which gives that

$$g(k) \geq g(n+p) = p - \frac{mn(p-\alpha)^r[\lambda(p-1)+1]^{r-1}}{(n+p-\alpha)^r[\lambda(n+p-1)+1]^{r-1} - m(p-\alpha)^r[\lambda(p-1)+1]^{r-1}}.$$

From $p(n+p-\alpha)^r[\lambda(n+p-1)+1]^{r-1} \geq m(n+p)(p-\alpha)^r[\lambda(p-1)+1]^{r-1}$, we can see that $0 \leq \gamma_m < p$, which complete the proof of Theorem 2. ■

A special case of Theorem 2 when $\lambda = 0$ and $\lambda = 1$ yield

COROLLARY 3. (See [2, Theorem 3].) *If the functions $f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \in T_n(p, \alpha_j)$, $j = 1, 2, \dots, m$, define $F_m(z) = z^p - \sum_{k=n+p}^{\infty} (\sum_{j=1}^m a_{k,j}^r) z^k$ ($z \in \Delta$; $n, p \in N$; $r > 1$), then when*

$$p(n+p-\alpha)^r \geq m(n+p)(p-\alpha)^r,$$

we have $F_m(z) \in T_n(p, \beta_m)$, where

$$\beta_m = p - \frac{mn(p-\alpha)^r}{(n+p-\alpha)^r - m(p-\alpha)^r}, \quad \alpha = \min_{1 \leq j \leq m} \{\alpha_j\}.$$

The result is sharp for the functions $f(z)$ given by

$$f(z) = z^p - \frac{p-\alpha_j}{n+p-\alpha_j} z^{n+p}, \quad j = 1, 2, \dots, m; \quad n, p \in N.$$

COROLLARY 4. (See [2, Theorem 4].) *If the functions $f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k \in C_n(p, \alpha_j)$, $j = 1, 2, \dots, m$, define $F_m(z) = z^p - \sum_{k=n+p}^{\infty} (\sum_{j=1}^m a_{k,j}^r) z^k$ ($z \in \Delta$; $n, p \in n$; $r > 1$), then when*

$$(n+p-\alpha)^r(n+p)^{r-2} \geq m(p-\alpha)^r p^{r-2},$$

we have $F_m(z) \in C_n(p, \xi_m)$, where

$$\xi_m = p - \frac{mn(p-\alpha)^r p^{r-1}}{(n+p-\alpha)^r(n+p)^{r-1} - m(p-\alpha)^r p^{r-1}}, \quad \alpha = \min_{1 \leq j \leq m} \{\alpha_j\}.$$

The result is sharp for the functions $f(z)$ given by

$$f(z) = z^p - \frac{p(p-\alpha_j)}{(n+p)(n+p-\alpha_j)} z^{n+p}, \quad j = 1, 2, \dots, m; \quad n, p \in N.$$

- REMARK 2.** (i). Putting $r = 2$ and $\alpha_j = \alpha$ ($j = 1, 2, \dots, m$) in Corollary 3 and Corollary 4, respectively, we obtain Owa [1, Theorem 5] and Owa [1, Theorem 6].
- (ii). Taking $r = 2$, $\alpha_j = \alpha$ ($j = 1, 2, \dots, m$) and $n = p = 1$ in Corollary 3 and Corollary 4, respectively, we obtain Owa [1, Corollary 9] and Owa [1, Corollary 10].
- (iii). Taking $r = 2$, $\alpha_j = \alpha$ ($j = 1, 2, \dots, m$) and $p = 1$ in Corollary 3 and Corollary 4, respectively, we obtain Domokos [4, Corollary 2] and Domokos [4, Corollary 3].

3. The extreme points of $\phi_c^*(n, p, \lambda, \alpha)$

Let K be a subset of a vector space X , the extreme points of K can be expressed as follows: $x_0 \in E(K)$ if and only if the condition $x, y, x_0 \in K$, $0 < t < 1$ and $tx + (1 - t)y = x_0$ can make sure $x = y = x_0$, where $E(K)$ denote the set of all extreme points of K .

LEMMA 2. *If the function*

$$f(z) = z^p - \frac{c(p - \alpha)[\lambda(p - 1) + 1]}{(n + p - \alpha)[\lambda(n + p - 1) + 1]} z^{n+p} - \sum_{k=n+p+1}^{\infty} a_k z^k \in S(n, p),$$

then $f(z)$ is in the class $\phi_c^(n, p, \lambda, \alpha)$ if and only if*

$$(3.1) \quad \sum_{k=n+p+1}^{\infty} (k - \alpha)[\lambda(k - 1) + 1]a_k \leq (1 - c)(p - \alpha)[\lambda(p - 1) + 1] \quad (0 \leq c \leq 1).$$

Proof. Using Lemma 1, we can obtain the result of Lemma 2 easily. ■

LEMMA 3. (See [8]). *Suppose X is a topological vector space. If K is a nonempty compact subset in X , then $K \subset \overline{\text{co}}(E(K))$. In particular, if K is a nonempty compact convex set in X , then K is the closed convex hull of the set of its extreme points.*

LEMMA 4. *Let*

$$f_{n+p}(z) = z^p - \frac{c(p - \alpha)[\lambda(p - 1) + 1]}{(n + p - \alpha)[\lambda(n + p - 1) + 1]} z^{n+p}$$

and

$$f_k(z) = z^p - \frac{c(p - \alpha)[\lambda(p - 1) + 1]}{(n + p - \alpha)[\lambda(n + p - 1) + 1]} z^{n+p} - \frac{(1 - c)(p - \alpha)[\lambda(p - 1) + 1]}{(k - \alpha)[\lambda(k - 1) + 1]} z^k$$

($k \geq n + p + 1$), then $f(z) \in \phi_c^(n, p, \lambda, \alpha)$ if and only if $f(z) = \sum_{k=n+p}^{\infty} \lambda_k f_k(z)$,*

where $\lambda_k \geq 0$ and $\sum_{k=n+p}^{\infty} \lambda_k = 1$.

Proof. Firstly, if $f(z) = \sum_{k=n+p}^{\infty} \lambda_k f_k(z)$, then we have

$$f(z) = z^p - \frac{c(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} z^{n+p} \\ - \sum_{k=n+p+1}^{\infty} \frac{(1-c)(p-\alpha)[\lambda(p-1)+1]}{(k-\alpha)[\lambda(k-1)+1]} \lambda_k z^k.$$

So

$$\sum_{k=n+p+1}^{\infty} (k-\alpha)[\lambda(k-1)+1] \frac{(1-c)(p-\alpha)[\lambda(p-1)+1]}{k(k-\alpha)[\lambda(k-1)+1]} \lambda_k \\ = (1-c)(p-\alpha)[\lambda(p-1)+1] \sum_{k=n+p+1}^{\infty} \lambda_k \leq (1-c)(p-\alpha)[\lambda(p-1)+1].$$

from Lemma 2, we can know $f(z) \in \phi_c^*(n, p, \lambda, \alpha)$.

Conversely, suppose $f(z) \in \phi_c^*(n, p, \lambda, \alpha)$, then from Lemma 1, it is easy to know that

$$(3.2) \quad a_k \leq \frac{(1-c)(p-\alpha)[\lambda(p-1)+1]}{(k-\alpha)[\lambda(k-1)+1]} \quad (k \geq n+p+1).$$

Setting

$$\lambda_k = \frac{(k-\alpha)[\lambda(k-1)+1]}{(1-c)(p-\alpha)[\lambda(p-1)+1]} a_k \quad (k \geq n+p+1), \\ \lambda_{n+p} = 1 - \sum_{k=n+p+1}^{\infty} \lambda_k,$$

then $f(z) = \sum_{k=n+p}^{\infty} \lambda_k f_k(z)$. ■

THEOREM 3. The extreme points of the class $\phi_c^*(n, p, \lambda, \alpha)$ are given by $f_k(z)$ that are defined in Lemma 4.

Proof. Letting

$$V = \left\{ f_k(z) : f_{n+p}(z) = z^p - \frac{c(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} z^{n+p}, \right. \\ f_k(z) = z^p - \frac{c(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]} z^{n+p} \\ \left. - \frac{(1-c)(p-\alpha)[\lambda(p-1)+1]}{(k-\alpha)[\lambda(k-1)+1]} z^k \quad (k \geq n+p+1) \right\}.$$

Suppose

$$z^p - \frac{c(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]}z^{n+p} - \frac{(1-c)(p-\alpha)[\lambda(p-1)+1]}{(k-\alpha)[\lambda(k-1)+1]}z^k \\ = tg_1(z) + (1-t)g_2(z),$$

where $0 < t < 1$, $g_i(z) = z^p - \frac{c(p-\alpha)[\lambda(p-1)+1]}{(n+p_\alpha)[\lambda(n+p-1)+1]}z^{n+p} - \sum_{k=n+p+1}^{\infty} a_{k,i}z^k$

and $g_i(z) \in \phi_c^*(n, p, \lambda, \alpha)$ ($i = 1, 2$), then we have

$$\frac{(1-c)(p-\alpha)[\lambda(p-1)+1]}{(k-\alpha)[\lambda(k-1)+1]} = ta_{k,1} + (1-t)a_{k,2}.$$

Since $g_i(z) \in \phi_c^*(n, p, \lambda, \alpha)$, (3.2) gives

$$a_{k,i} \leq \frac{(1-c)(p-\alpha)[\lambda(p-1)+1]}{(k-\alpha)[\lambda(k-1)+1]} \quad (i = 1, 2).$$

This implies that $a_{k,1} = a_{k,2} = \frac{(1-c)(p-\alpha)[\lambda(p-1)+1]}{(k-\alpha)[\lambda(k-1)+1]}$. Moreover, we can note that $a_{n,1} = a_{n,2} = 0$ when $n \neq k$ so $g_1(z) = g_2(z)$, this gives us

$$f_k(z) = z^p - \frac{c(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]}z^{n+p} \\ - \frac{(1-c)(p-\alpha)[\lambda(p-1)+1]}{(k-\alpha)[\lambda(k-1)+1]}z^k \in E\phi_c^*(n, p, \lambda, \alpha).$$

Taking the same process, we can obtain:

$$f_{n+p}(z) = z^p - \frac{c(p-\alpha)[\lambda(p-1)+1]}{(n+p-\alpha)[\lambda(n+p-1)+1]}z^{n+p} \in E\phi_c^*(n, p, \lambda, \alpha).$$

So $V \subset E\phi_c^*(n, p, \lambda, \alpha)$. From Lemma 4, we know $\phi_c^*(n, p, \lambda, \alpha) = HV$. Since V is a compact set, using Lemma 3, it gives $E\phi_c^*(n, p, \lambda, \alpha) = EHV \subset V$. So $E\phi_c^*(n, p, \lambda, \alpha) = V$. ■

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