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ON HOPF BIFURCATION OF LIU CHAOTIC SYSTEM

Abstract. In this paper, we analyze the dynamical behaviors of Liu system using the complementary-cluster energy-barrier criterion (CCEBC). Moreover, the Hopf bifurcation of this system is investigated using the first Lyapunov coefficient. Also, it is proved that this system has two Hopf bifurcation points, at which these Hopf bifurcations are non degenerate and subcritical.

1. Introduction

Since the pioneering work of Lorenz [1] and Rössler [2], it has been known that chaos can occur in systems of autonomous ordinary differential equations with as few as three variables and one or two quadratic nonlinearities. Many other chaotic systems have been discovered over the last years [3–7]. There have been extensive investigations on dynamical behaviors of these chaotic systems [8–14]. A bifurcation occurs where the solutions of a nonlinear system change their qualitative character as a parameter changes. In particular, bifurcation theory is about how the number of steady solutions of a system depends on parameters.

Recently Liu et al. [15] proposed a three dimensional autonomous system which relies on one multiplier and one quadratic term to introduce the nonlinearity necessary for folding trajectories. Over the last two years, there have been some detailed investigations and studies of the Liu system [15–18].

In this paper, we would like to investigate dynamical analysis of the Liu system using the complementary-cluster energy-barrier criterion (CCEBC) [19], Hopf bifurcation using the first Lyapunov coefficient [20, 21].

The Liu system can be described by the following differential equation:

$$(1) \quad \dot{x} = a(y - x), \quad \dot{y} = bx - kxz, \quad \dot{z} = -cz + hx^2,$$

where $a, c, k, h \in R^+$ and $b \in R$ this system exhibit chaotic attractor when

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$a = 10$, $b = 40$, $c = 2.5$, $k = 1$ and $h = 4$ (see Fig. (1)). The Liu system (1) has three equilibria

$$E_0 = (0, 0, 0) \text{ and } E_{1,2} = \left(\pm \sqrt{\frac{bc}{hk}}, \pm \sqrt{\frac{bc}{hk}}, \frac{b}{k} \right).$$

Fig. 1. Phase portraits of Liu system in the three-dimensional at $a = 10$, $b = 40$, $c = 2.5$, $k = 1$ and $h = 4$.

2. Dynamical analysis of Liu system

In the following, we extend the new practical (CCEBC) method [19] to distinguish the dynamical behaviors of Liu system.

Consider the first two differential equations in Liu system (1):

$$(2) \quad \dot{x} = a(y - x), \quad \dot{y} = bx - kxz,$$

where $z = z(t)$ is regarded as a known function of time t . When $t = t_0$, the system (2) is a two dimensional linear homogeneous system with four constants a , b , k and $z = z(t_0)$.

We discuss the dynamical behavior of the system (2) if $b \neq kz$. The system (2) has only one equilibrium point $O(0, 0)$ and its corresponding characteristic polynomial is

$$(3) \quad \lambda^2 + a\lambda + akz - ab = 0.$$

Depending up on the roots of Eq. (3), one can deduce the following results:

(1) If $z < \frac{b}{k}$ and $k > 0$, Eq. (3) has different real root such that $\lambda_1 < 0 < \lambda_2$ (i. e. $\lambda_1 \lambda_2 < 0$), then the equilibrium $O(0, 0)$ is saddle point in two dimensional xy -plane.

(2) If $\frac{b}{k} < z < \frac{b}{k} + \frac{a}{4k}$, $a > 0$ and $k > 0$, Eq. (3) has two different negative real root such that $\lambda_1 < 0$, $\lambda_2 < 0$, then the equilibrium $O(0, 0)$ is a stable node point in two dimensional xy -plane.

(3) If $\frac{b}{k} < z < \frac{b}{k} + \frac{a}{4k}$, $a < 0$ and $k < 0$, Eq. (3) has two different positive real roots such that $\lambda_1 > 0$, $\lambda_2 > 0$, then the equilibrium $O(0, 0)$ is unstable node point in two dimensional xy -plane.

(4) If $z > \frac{b}{k} + \frac{a}{4k}$, $a > 0$ and $k > 0$, Eq. (3) has two complex conjugate eigenvalues $\lambda_{1, 2} = \alpha \pm i\beta$ with negative real part $\alpha < 0$, then the equilibrium $O(0, 0)$ is stable focus in two dimensional xy -plane.

(5) If $z > \frac{b}{k} + \frac{a}{4k}$, $a < 0$ and $k > 0$, Eq. (3) has two complex conjugate eigenvalues $\lambda_{1, 2} = \alpha \pm i\beta$ with positive real part $\alpha > 0$, then the equilibrium $O(0, 0)$ is unstable focus in two dimensional xy -plane.

(6) In the case $z = \frac{b}{k} + \frac{a}{4k}$, Eq. (3) has two repeated real roots $\lambda_{1, 2} = -\frac{a}{2}$. Moreover, the Jacobian matrix of the system (2) admits two linearly independent eigenvectors. Therefore, the equilibrium $O(0, 0)$ is stable singular node if $a > 0$ or unstable singular node if $a < 0$ in the two dimensional xy -plane.

3. The first Lyapunov coefficient [20, 21]

We first introduce some preliminary definitions. Let C^n be a linear space defined on the complex number field C . The scalar product $\langle x, y \rangle$ for all $x, y \in C^n$ satisfies the following properties:

- (1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$, where $\langle x, y \rangle = x^T y = \sum_{i=1}^n \bar{x}_i y_i$;
- (2) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for any $\alpha, \beta \in C$ and $x, y, z \in C^n$;
- (3) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

If one introduces the norm $\|x\| = \sqrt{\langle x, x \rangle}$ in C^n , then the space C^n becomes the Hilbert space.

Consider the continuous-time nonlinear dynamical system

$$(4) \quad \dot{x} = Ax + N(x); \quad A = (a_{ij})_{n \times n} \text{ and } x \in R^n,$$

where $N(x) = o(\|x\|^2)$ is smooth function. Let the function $N(x)$ be written as:

$$(5) \quad N(x) = \frac{1}{2}B(x, x) + \frac{1}{6}C(x, x, x) + o(\|x\|^4),$$

where $B(x, y)$ and $C(x, y, z)$ are bilinear and trilinear functions, respectively.

Fig. 2. The possible phase portraits of (2): (a) saddle point, (b) stable node, (c) unstable node, (d) stable focus, (e) unstable focus, (f) stable singular node, (g) unstable singular node.

In coordinates, we have

$$\begin{aligned}
 B_i(x, y) &= \sum_{j, k=1}^n \left. \frac{\partial^2 N_i(\zeta)}{\partial \zeta_j \partial \zeta_k} \right|_{\zeta=0} x_j y_k \text{ and} \\
 (6) \quad C_i(x, y, z) &= \sum_{j, k, l=1}^n \left. \frac{\partial^3 N_i(\zeta)}{\partial \zeta_j \partial \zeta_k \partial \zeta_l} \right|_{\zeta=0} x_j y_k z_l.
 \end{aligned}$$

Suppose that A has a pair of complex eigenvalues on the imaginary axis: $\lambda_{1,2} = \pm i\omega$ ($\omega > 0$) and these eigenvalues are the only eigenvalues with $\text{Re } \lambda = 0$. Let $q \in C^n$ be a complex eigenvector corresponding to $\lambda_1 = i\omega$:

$$(7) \quad Aq = i\omega q \text{ and } A\bar{q} = -i\omega\bar{q}.$$

Introduce also, the adjoint eigenvector $p \in C^n$ admitting the properties:

$$(8) \quad A^T p = -i\omega p \text{ and } A^T \bar{p} = i\omega \bar{p}$$

and satisfying the normalization $\langle p, q \rangle = 1$.

The first Lyapunov coefficient at the origin is defined by:

$$(9) \quad l_1(0) = \frac{1}{2\omega} \text{Re} \left[\langle p, C(q, q, \bar{q}) \rangle - 2 \langle p, B(q, A^{-1}B(q, \bar{q})) \rangle \right. \\ \left. + \langle p, B(\bar{q}, (2i\omega E - A)^{-1}B(q, q)) \rangle \right]$$

where E is an identity matrix.

4. The Hopf-bifurcation analysis of Liu system

We will consider Hopf bifurcations of system (1) at the three equilibria by means of first Lyapunov coefficient:

The Jacobian matrix of system (1) at the point (x, y, z) is

$$J = \begin{pmatrix} -a & a & 0 \\ b - kz & 0 & -kx \\ 2hx & 0 & -c \end{pmatrix}.$$

In the following, we consider Hopf bifurcations of (1) at the three equilibria by means of the first Lyapunov coefficient:

Case 1: Equilibrium $E_0 = (0, 0, 0)$.

The Jacobian matrix of the system (1) at the equilibrium $E_0 = (0, 0, 0)$, is

$$(10) \quad J_{E_0} = \begin{pmatrix} -a & a & 0 \\ b & 0 & 0 \\ 0 & 0 & -c \end{pmatrix}$$

then, the characteristic equation is:

$$(\lambda + c)(\lambda^2 + a\lambda - ab) = 0.$$

If $a^2 + 4ab < 0$, $a > 0$ and $c > 0$, the eigenvalues of (10) are:

$$\lambda_1 = -c \text{ and } \lambda_{2,3} = \frac{-a \pm i\sqrt{a^2 + 4ab}}{2}.$$

Then, the equilibrium point $E_0 = (0, 0, 0)$ is stable.

When $a + 4b > 0$, $a > 0$, $b > 0$ and $c > 0$, the eigenvalues of Eq. (10) are:

$$\lambda_1 = -c \text{ and } \lambda_{2,3} = \frac{-a \pm \sqrt{a^2 + 4ab}}{2}.$$

Then, the equilibrium $E_0 = (0, 0, 0)$ is a saddle point. Therefore, the system (1) is unstable at the equilibrium point E_0 .

Case 2: Equilibrium $E_{1,2} = (\pm\sqrt{\frac{bc}{hk}}, \pm\sqrt{\frac{bc}{hk}}, \frac{b}{k})$.

The Jacobian matrix of the system (1) at the equilibrium $E_{1,2} = (\pm\sqrt{\frac{bc}{hk}}, \pm\sqrt{\frac{bc}{hk}}, \frac{b}{k})$, is:

$$(11) \quad J_{E_{1,2}} = \begin{pmatrix} -a & a & 0 \\ 0 & 0 & \mp k\sqrt{\frac{bc}{hk}} \\ \pm 2h\sqrt{\frac{bc}{hk}} & 0 & -c \end{pmatrix}.$$

By the straightforward calculations, we know that the characteristic equations of (1) at the equilibria $E_{1,2}$ admit the same form because of the symmetrically placed with respect to the z -axis, then it is sufficient to analyze the Hopf bifurcation of the system at the equilibrium point E_1

$$(12) \quad \lambda^3 + (a+c)\lambda^2 + ac\lambda + 2abc = 0.$$

Assume that (12) has a pair of pure imaginary root $\lambda_{1,2} = \pm i\omega$ ($\omega > 0$) which leads to

$$(13) \quad \mp i\omega^3 - (a+c)\omega^2 \pm iac\omega + 2abc = 0.$$

It follows from (13) that:

$$(14) \quad \mp\omega^3 \pm ac\omega = 0 \text{ and } -(a+c)\omega^2 + 2abc = 0,$$

which yields that:

$$\omega = \sqrt{ac}, \quad ac > 0,$$

and the bifurcation surface is:

$$(15) \quad ac(a+c) - 2abc = 0,$$

then:

$$b = b_E = \frac{a+c}{2}.$$

REMARK. By using Routh–Hurwitz criterion conditions, we get that, the equilibrium points $E_{1,2}$ are locally asymptotically stable (unstable) if and only if $b < b_E$ ($b > b_E$).

Since the equilibrium E_1 is not the origin, therefore, we firstly need to use the change of variable to translate the origin of the coordinates to this

equilibrium point E_1 in order to compute the first Lyapunov coefficient $l(0)$.

$$(16) \quad x(t) = X(t) + \sqrt{\frac{bc}{hk}}, \quad y(t) = Y(t) + \sqrt{\frac{bc}{hk}}, \quad z(t) = Z(t) + \frac{b}{k}.$$

These transformations transform the Liu system into the equivalent system

$$(17) \quad \begin{aligned} \frac{dX}{dt} &= a(Y - X), \\ \frac{dY}{dt} &= b \left(X + \sqrt{\frac{bc}{hk}} \right) - k \left(X + \sqrt{\frac{bc}{hk}} \right) \left(Z + \frac{b}{k} \right), \\ \frac{dZ}{dt} &= h \left(X + \sqrt{\frac{bc}{hk}} \right)^2 - c \left(Z + \frac{b}{k} \right). \end{aligned}$$

One can check that the characteristic equation of (17) at the origin is just (12). Therefore, a Hopf bifurcation takes place at the origin $O(0, 0, 0)$ of (17). It is very complicated to analyze the Hopf bifurcation of (1) on the bifurcation surface (15).

For convenience, we only consider the case $a = c$ and $h = k$, which leads to $\omega = b$. Thus the Jacobian matrix A of (17) at E_1 is:

$$A = \begin{pmatrix} -b & b & 0 \\ 0 & 0 & -b \\ 2b & 0 & -b \end{pmatrix}.$$

With the aid of Maple, by tedious calculations, we obtain the four vectors:

$$(18) \quad \begin{aligned} q &= \begin{pmatrix} -\frac{1}{2} + \frac{1}{2}i \\ -1 \\ i \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} -\frac{1}{2} - \frac{1}{2}i \\ -1 \\ -i \end{pmatrix}, \\ p &= \frac{1}{10} \begin{pmatrix} -2 + 4i \\ -4 - 2i \\ 1 + 3i \end{pmatrix} \quad \text{and} \quad \bar{p} = \frac{1}{10} \begin{pmatrix} -2 - 4i \\ -4 + 2i \\ 1 - 3i \end{pmatrix}, \end{aligned}$$

which satisfy the conditions (7), (8) and $\langle q, p \rangle = 1$.

There are only bilinear terms in the system (17). Therefore, the bilinear $B(x, y)$, defined for two vectors $x = (x_1, x_2, x_3)^T \in R^3$ and $y = (y_1, y_2, y_3)^T \in R^3$, can be expressed as:

$$B(x, y) = (0, -2kxz, hx^2)^T \quad \text{and} \quad C(x, y, z) = (0, 0, 0)^T.$$

When $h = k = 1$, then the straightforward and tedious calculations yield:
(19)

$$A^{-1} = \begin{pmatrix} 0 & \frac{-1}{2b} & \frac{1}{2b} \\ \frac{1}{b} & \frac{-1}{2b} & \frac{1}{2b} \\ 0 & \frac{-1}{b} & 0 \end{pmatrix}, \quad B(q, q) = \begin{pmatrix} 0 \\ 1+i \\ -\frac{1}{2}i \end{pmatrix}, \quad \text{and } B(q, \bar{q}) = \begin{pmatrix} 0 \\ -1-i \\ \frac{1}{2} \end{pmatrix}.$$

Then from (19), we have

$$(20) \quad s = A^{-1}B(q, \bar{q}) = \frac{1}{4b} \begin{pmatrix} 3+2i \\ 3+2i \\ 4+4i \end{pmatrix} \quad \text{and } B(q, s) = \frac{1}{4b} \begin{pmatrix} 0 \\ 8 \\ -\frac{5}{2} + \frac{1}{2}i \end{pmatrix}.$$

Therefore, we have:

$$(21) \quad \langle p, B(q, s) \rangle = \frac{-33+24i}{40b},$$

with the aid of Maple, we have the matrix $(2i\omega E - A)^{-1}$ when $\omega = b$,

$$(22) \quad (2i\omega E - A)^{-1} = \frac{1}{12b} \begin{pmatrix} 2-6i & -3-i & 1-i \\ 2-2i & -1-7i & 3+i \\ -4-4i & -2+2i & 2-6i \end{pmatrix}.$$

Then, we get:

$$(23) \quad r = (2i\omega E - A)^{-1}B(q, q) = \frac{1}{24b} \begin{pmatrix} -5-7i \\ 9-17i \\ -10+6i \end{pmatrix},$$

$$(24) \quad B(\bar{q}, r) = \frac{1}{24b} \begin{pmatrix} 0 \\ -16-4i \\ -1+6i \end{pmatrix},$$

and

$$(25) \quad \langle p, B(\bar{q}, r) \rangle = \frac{89-13i}{240b}.$$

Then, by substitution into the first Lyapunov coefficient (9) we get:

$$(26) \quad \begin{aligned} l_1(0) &= \frac{1}{2\omega} \operatorname{Re}[\langle p, C(q, q, \bar{q}) \rangle - 2\langle p, B(q, A^{-1}B(q, \bar{q})) \rangle \\ &\quad + \langle p, B(\bar{q}, (2i\omega E - A)^{-1}B(q, q)) \rangle] \\ &= \frac{1}{2b} \operatorname{Re}[-2\langle p, B(q, s) \rangle + \langle p, B(\bar{q}, r) \rangle] \\ &= \frac{1}{2b} \operatorname{Re}\left[\frac{66-48i}{40b} + \frac{89-13i}{240b}\right] \end{aligned}$$

$$= \frac{1}{480b^2} \operatorname{Re}[485 - 301i] = \frac{485}{480b^2} > 0.$$

Then, according to the method in Ref. [8], the Hopf bifurcation of Liu system at the equilibria points $E_{1,2} = (\pm\sqrt{\frac{bc}{hk}}, \pm\sqrt{\frac{bc}{hk}}, \frac{b}{k})$ is non degenerate and subcritical.

PROPOSITION 1. *The Hopf bifurcation of Liu system (1) at the equilibrium points $E_{1,2} = (\pm\sqrt{\frac{bc}{hk}}, \pm\sqrt{\frac{bc}{hk}}, \frac{b}{k})$ is non degenerate and subcritical.*

Fig. 3. Waveform diagram and asymptotically stable point of Liu system at $b = 5$ ($b < b_E$).

Fig. 4. Waveform diagram and Period-doubling bifurcations of Liu system at $b = 7$ ($b > b_E$).

5. Conclusions

In summary, we have analyzed dynamical behaviors of the Liu system using the complementary-cluster energy-barrier criterion. In addition, its Hopf bifurcation has been also investigated by the first Lyapunov coefficient. As a result, it has been proved that this system has two Hopf bifurcation points, at which the Hopf bifurcations are nondegenerate and subcritical.

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