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A. K. Bhuniya, K. Jana

SEMIGROUP OF k-bi-IDEALS OF A SEMIRING WITH SEMILATTICE ADDITIVE REDUCT

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Abstract. We associate a semigroup $\mathcal{B}(\mathcal{S})$ to every semiring S with semilattice additive reduct, namely the semigroup of all k-bi-ideals of S; and such semirings S have been characterized by this associated semigroup $\mathcal{B}(\mathcal{S})$. A semiring S is k-regular if and only if $\mathcal{B}(\mathcal{S})$ is a regular semigroup. For the left k-Clifford semirings S, $\mathcal{B}(\mathcal{S})$ is a left normal band; and consequently, $\mathcal{B}(\mathcal{S})$ is a semilattice if S is a k-Clifford semiring. Also we show that the set $\mathcal{B}_m(S)$ of all minimal k-bi-ideals of S forms a rectangular band and $\mathcal{B}_m(S)$ is a bi-ideal of the semigroup $\mathcal{B}(\mathcal{S})$.

1. Introduction

The notion of semirings was first introduced by Vandiver [16] as a universal algebra with two associative binary operations + and . which are connected by the ring like distributive laws. Like all other abstract structures, semirings also appeared in mathematics long before their axiomatic formulation, as the semirings of all ideals of a ring, the semirings of all endomorphisms on a commutative semigroup, the positive cone in ordered ring etc. Though the semirings had appeared long before, recently, they found their full place in mathematics e.g. in idempotent analysis [11, 12] which is now being used in theoretical physics, optimization etc. and in theoretical computer science and algorithm theory [7, 10].

In the initial phase of the semiring theory, it was treated as a generalization of rings; the perspectives that it is also a generalization of distributive lattices have been neglected. So the semirings, where addition is idempotent, have not been studied enough. On the other hand, the underlying semirings in both idempotent analysis (Maslov's dequatization semiring) and theoretical

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computer science (syntactic semiring) are such that their additive reduct is a semilattice. So it is worth to characterize this particular class of semirings.

Here we associate a semigroup $\mathcal{B}(\mathcal{S})$ of k-bi-ideals to each semiring S whose additive reduct is a semilattice with a view to use the semigroup theoretic properties of $\mathcal{B}(\mathcal{S})$ to characterize the semiring S. In 1952, R. A. Good and D. R. Hughes [6] first defined the notion of bi-ideals of a semigroup. We introduced k-bi-ideals of a semiring in [3] as an extension of the notion of bi-ideals in semigroups. Here we define a product of two k-bi-ideals of a semiring S, namely k-product of k-bi-ideals, such that the k-product of two k-bi-ideals is again a k-bi-ideal. Thus this k-product is a binary operation on $\mathcal{B}(\mathcal{S})$. The associativity of this k-product follows from that of the multiplicative reduct $(S, \cdot)$ of S. We show that the semigroup $\mathcal{B}(\mathcal{S})$ of k-bi-ideals characterizes different classes of semirings elegantly.

This introduction is followed by three sections. First of which deals with the basic definitions and preliminary results we need in this article. In Section 3, we characterize different subclasses of k-regular semirings by their semigroup of k-bi-ideals. The set $\mathcal{B}_m(S)$ of all minimal k-bi-ideals of a semiring S is a bi-ideal of the semigroup $\mathcal{B}(\mathcal{S})$. Moreover, $\mathcal{B}_m(S)$ is rectangular band. All these characterizations are made in Section 4.

2. Preliminaries

A semiring $(S, +, \cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the additive reduct $(S, +)$ and the multiplicative reduct $(S, \cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y + z) = xy + xz \quad \text{and} \quad (x + y)z = xz + yz.$$

A band is a semigroup in which every element is an idempotent. A commutative band is called a semilattice. Throughout this paper, unless otherwise stated, S is always a semiring whose additive reduct is a semilattice and the variety of all such semirings is denoted by SL^+.

A non-empty subset L of a semiring S is called a left ideal of S if $L+L \subseteq L$ and $SL \subseteq L$. The right ideals are defined dually.

For a non-empty subset A of S, the k-closure of A is defined by

$$\overline{A} = \{x \in S | x + a_1 = a_2;\ a_1, a_2 \in A\}.$$

Since $a + a = a$ for all $a \in S$, $A \subseteq \overline{A}$. Moreover, from $x + a_1 = a_2$ one gets $x + a_2 = x + x + a_1 = x + a_1 = a_2$, since (S,+) is assumed to be idempotent. So $\overline{A} = \{x \in S | x + a = a;$ for some $a \in A\}$ and $\overline{\overline{A}} = \overline{A}$. A is called a k-set if $\overline{A} = A$.

An left (resp. right) ideal A of S is a left (resp. right) k-ideal if it is a k-set.

Since the intersection of left (resp. right) k-ideals is again a left (resp. right) k-ideal (if it is nonempty), there is the smallest left (resp. right) k-ideal containing $a \in S$. This is called the left (resp. right) k-ideal generated by a and is denoted by $L_k(a)$ (resp. $R_k(a)$).

Sen and Bhuniya [13] introduced Green's relations $\overline{\mathcal{L}}, \overline{\mathcal{R}}$ and $\overline{\mathcal{H}}$ on a semiring S in the following way: for $a, b \in S$

$$a\overline{\mathcal{L}}b \quad \text{if } L_k(a) = L_k(b), \qquad a\overline{\mathcal{R}}b \quad \text{if } R_k(a) = R_k(b) \quad \text{and} \quad \overline{\mathcal{H}} = \overline{\mathcal{L}} \cap \overline{\mathcal{R}}.$$

These equivalences are additive congruences on S, whereas $\overline{\mathcal{L}}$ is multiplicative right and $\overline{\mathcal{R}}$ is multiplicative left congruence on S only.

A subsemiring A on S is called a k-bi-ideal of S if $ASA \subseteq A$ and $\bar{A} = A$.

Let S be a semiring and $a \in S$. Then the principal k-bi-ideal of S generated by a is given by [3],

$$B_k(a) = \{u \in S \mid u + a + a^2 + asa = a + a^2 + asa, \text{ for some } s \in S\}.$$

Now we show that the set $\mathcal{B}(\mathcal{S})$ of all k-bi-ideals, the set $\mathcal{L}(\mathcal{S})$ of all left k-ideals and the set $\mathcal{R}(\mathcal{S})$ of all right k-ideals are each a semigroup with respect to a product of subsets of S defined in usual way. Rather defining binary operations on these three sets separately, we consider them as subsemigroups of the semigroup of all subsets of S. Denote the set of all subsets of S by $P(S)$. Define a binary operation $* : P(S) \times P(S) \to P(S)$ by $A * B = \overline{AB}$, where $AB = \{ab | a \in A, b \in B\}$ and so $\overline{AB} = \{u \in S | u + a_1 b_1 = a_2 b_2, a_i \in A, b_i \in B, i = 1, 2\}$. Since $(S, +)$ is a semilattice, $\overline{AB} = \{u \in S | u + ab = ab,\ a \in A,\ b \in B\}$.

It remains to check that $(P(S), *)$ is a semigroup. Also since $\overline{L_1 L_2}$ is a left k-ideal for all left k-ideals L_1 and L_2, so $* : P(S) \times P(S) \to P(S)$ induces a binary operation on $\mathcal{L}(\mathcal{S})$, which we denote by the same symbol $*$, for the sake of simplicity. Thus $(\mathcal{L}(\mathcal{S}), *)$ is a semigroup. Similarly $(\mathcal{R}(\mathcal{S}), *)$ is a semigroup. Now we show that the same is true for $\mathcal{B}(\mathcal{S})$ also.

Consider two k-bi-ideals B and C; and $x, y \in B * C = \overline{BC}$. Then there exist $b \in B$ and $c \in C$ such that $x + bc = bc$ and $y + bc = bc$. This implies that $x + y + bc = bc$ and $xy + bcbc = bcbc$. Now $(bcb)c \in BC$ shows that $x + y, xy \in \overline{BC}$. Also for $s \in S$ we have $xsy + bcsbc = bcsbc$, which implies that $xsy \in \overline{BCSBC} \subseteq \overline{BC}$. Thus $\overline{BC}S\overline{BC} \subseteq \overline{BC}$. Again $\overline{BC}$ is a k-closure of BC and so a k-subset of S. Thus $\overline{BC}$ is a k-bi-ideal of S. Thus $(\mathcal{B}(\mathcal{S}), *)$ is a semigroup.

Also we refer to [7] for semiring theory and to [8] for the basics of semigroups.

3. Semigroup of k-bi-ideals in k-regular semirings

A semigroup S is called a regular semigroup if for each $a \in S$ there exists $x \in S$ such that $a = axa$. Bourne [5] defined a semiring S to be regular if for each $a \in S$ there exist $x, y \in S$ such that $a + axa = aya$. Adhikari, Sen and Weinert [1] renamed the Bourne regularity of a semiring as k-regularity. If the additive reduct $(S, +)$ of a k-regular semiring S is a semilattice, then $a + axa = aya$ implies (on addition of $axa + aya$ to both sides) that $a + a(x + y)a = a(x + y)a$. Thus a semiring $S \in SL^+$ is k-regular if and only if for each $a \in S$, there is $x \in S$ such that $a + axa = axa$. For k-regular semirings $S \in SL^+$, we have the following equivalent characterization.

Lemma 3.1. [3] *A semiring S is k-regular if and only if $B = \overline{BSB}$ for every k-bi-ideal B of S.*

Now $B = \overline{BSB} = B * S * B$ shows that the semigroup $(\mathcal{B}(S), *)$ is regular for every k-regular semiring S. In the following theorem we show that the converse is also true.

Proposition 3.2. *Let S be a semiring. Then the semigroup $(\mathcal{B}(S), *)$ is regular if and only if S is a k-regular semiring.*

Proof. First assume that $\mathcal{B}(S)$ is a regular semigroup. Let $a \in S$. Then $B_k(a) \in \mathcal{B}(S)$ and there is $C \in \mathcal{B}(S)$ such that $B_k(a) = \overline{B_k(a)CB_k(a)}$. Since $a \in B_k(a)$, there are $b \in B_k(a)$ and $c \in C$ such that $a + bcb = bcb$. Again $b+a+a^2+axa = a+a^2+axa$, for some $x \in S$. Then $a+bcb = bcb$ implies that $a+(b+a+a^2+asa)c(b+a+a^2+asa) = (b+a+a^2+asa)c(b+a+a^2+asa)$ which gives $a + (a + a^2 + asa)c(a + a^2 + asa) = (a + a^2 + asa)c(a + a^2 + asa)$. This can be regarded as $a + ata = ata$ for some $t \in S$. Hence S is a k-regular semiring.

The converse follows directly from Lemma 3.1. ∎

In the following we prove that the subsemigroups $\mathcal{R}(S)$ and $\mathcal{L}(S)$ are bands.

Lemma 3.3. *Let S be a k-regular semiring. Then $\mathcal{R}(S)$ ($\mathcal{L}(S)$) is a band.*

Proof. Let $R \in \mathcal{R}(S)$ and $a \in R$. Then there exists $x \in S$ such that $a + axa = axa$. Now $ax \in RS \subseteq R$ implies that $a \in R^2 = \overline{RR}$ and so $R \subseteq R^2$. Also $R^2 \subseteq R$. Thus $R^2 = R$. Hence $\mathcal{R}(S)$ is a band.

That $\mathcal{L}(S)$ is a band can be proved dually. ∎

Now we show that the semigroup $\mathcal{B}(S)$ of all k-bi-ideals is a product of the semigroups $\mathcal{R}(S)$ and $\mathcal{L}(S)$. Note that $\mathcal{R}(S)\mathcal{L}(S) = \{\overline{RL} \mid R \in \mathcal{R}(S), L \in \mathcal{L}(S)\}$.

Theorem 3.4. *Let S be a k-regular semiring. Then $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$.*

Proof. First consider $R \in \mathcal{R}(S)$ and $L \in \mathcal{L}(S)$ and denote $B = R * L = \overline{RL}$. Then B is a k-subsemiring of S.

Now $BSB = \overline{RL}S\overline{RL} \subseteq \overline{RLSRL} \subseteq \overline{RL} = B$ shows that $B \in \mathcal{B}(S)$. Hence $\mathcal{R}(S)\mathcal{L}(S) \subseteq \mathcal{B}(S)$.

Now consider $B \in \mathcal{B}(S)$. Denote $L = \overline{SB \cup B}$ and $R = \overline{BS \cup B}$. Then $SL = S\overline{SB \cup B} \subseteq \overline{SSB \cup SB} = \overline{SB} \subseteq \overline{SB \cup B} = L$ implies that L is a left k-ideal of S. Similarly, R is a right k-ideal of S. Now $\overline{RL} = \overline{\overline{BS \cup B}\ \overline{SB \cup B}} \subseteq \overline{(BS \cup B)(SB \cup B)} \subseteq \overline{BSB \cup B^2} \subseteq \overline{B} = B$. Also, $B \subseteq SB \cup B \subseteq \overline{SB \cup B} = L$ and $B \subseteq BS \cup B \subseteq \overline{BS \cup B} = R$. Since S is k-regular, by Lemma 3.1, $B = \overline{BSB}$ implies that $B \subseteq \overline{RSL} \subseteq \overline{RL}$. Hence $B = \overline{RL}$ and so $\mathcal{B}(S) \subseteq \mathcal{R}(S)\mathcal{L}(S)$. Thus $\mathcal{B}(S) = \mathcal{R}(S)\mathcal{L}(S)$. Hence the theorem. ∎

Now we consider those semirings $S \in SL^+$ for which $\overline{\mathcal{H}}$ is a congruence. Then $S/\overline{\mathcal{H}} = \{H_s | s \in S\}$ is a semiring in SL^+ where $+$ and $\cdot$ are given by: for $H_s, H_t \in S/\overline{\mathcal{H}}$, $H_s + H_t = H_{s+t}$ and $H_sH_t = H_{st}$. Moreover, if S is k-regular then for every $a \in S$, $a + axa = axa$ shows that $H_a + H_aH_xH_a = H_{a+axa} = H_{axa} = H_aH_xH_a$. Thus $S/\overline{\mathcal{H}}$ is also k-regular. Naturally $(\mathcal{B}(S/\overline{\mathcal{H}}), *)$ is a semigroup where $B' * C' = \{H_x \in S/\overline{\mathcal{H}} \mid H_x + H_bH_c = H_bH_c,\ H_b \in B',\ H_c \in C'\}$.

Below we attempt to characterize the k-bi-ideals of $S/\overline{\mathcal{H}}$ in terms of the k-bi-ideals of S. For a subset A of S we denote $H_A = \{H_a \mid a \in A\}$.

Lemma 3.5. *Let S be a semiring such that $\overline{\mathcal{H}}$ is a congruence on S. Then B' is a k-bi-ideal of $S/\overline{\mathcal{H}}$ if and only if there exists a k-bi-ideal B of S such that $B' = H_B$.*

Proof. Consider a k-bi-ideal B' of $S/\overline{\mathcal{H}}$. We denote $B = \{s \in S \mid H_s \in B'\}$. Suppose $b_1, b_2 \in B$. Then for any $s \in S$, $H_{b_1sb_2} = H_{b_1}H_sH_{b_2} \in B'\overline{S}B' \subseteq B'$ implies that $b_1sb_2 \in B$ where $\overline{S} = S/\overline{\mathcal{H}}$. Thus $BSB \subseteq B$. Similarly B is a subsemiring of S. Let $b \in B$, $s \in S$ be such that $s + b \in B$. Then $H_{s+b} \in B'$ implies that $H_s + H_b \in B'$. Then, as $H_b \in B'$, we have $H_s \in B'$ which implies that $s \in B$. Thus B is a k-bi-ideal of S. Also $B' = H_B$.

Conversely, we can similarly prove that if B is a k-bi-ideal of S, $B' = \{H_b : b \in B\}$ is a k-bi-ideal of $S/\overline{\mathcal{H}}$. ∎

Interestingly for every k-bi-ideals A and B of S, $H_{A*B} = H_A * H_B$. Consider $H_u \in H_{A*B}$. Then $u \in A * B$ implies that $u + ab = ab$ for some $a \in A$ and $b \in B$. Then $H_u + H_aH_b = H_aH_b$ implies that $H_u \in \overline{H_AH_B} = H_A * H_B$ and hence $H_{A*B} \subseteq H_A * H_B$. Again consider $H_c \in H_A * H_B$. Then there are $H_a \in H_A$ and $H_b \in H_B$ such that $H_c + H_aH_b = H_aH_b$ and hence $c + ab\overline{\mathcal{H}}ab$. Then there is $x \in S$ such that $c + ab + xab = xab$ and $c + ab + abx = abx$. Since S is k-regular, $c + ctc = ctc$ for some $t \in S$ and hence

$c + (ab + abx)(ab + xab) = (ab + abx)(ab + xab)$ implies $c + absab = absab$ where $s = t + xt + tx + xtx \in S$. Now $(absa)b \in ASAB \subseteq AB$ shows that $c \in \overline{AB} = A * B$ and hence $H_c \in H_{A*B}$. Thus $H_A * H_B \subseteq H_{A*B}$. Hence $H_{A*B} = H_A * H_B$.

This observation can be interpreted as $f : \mathcal{B}(S) \to \mathcal{B}(S/\overline{\mathcal{H}})$ given by $B \to H_B$ is a homomorphism of semigroups. Now we show that this is in fact an isomorphism.

Theorem 3.6. *Let S be a k-regular semiring such that $\overline{\mathcal{H}}$ is a congruence on S. Then $\mathcal{B}(S)$ and $\mathcal{B}(S/\overline{\mathcal{H}})$ are isomorphic.*

Proof. Define a mapping $f : \mathcal{B}(S) \mapsto \mathcal{B}(S/\overline{\mathcal{H}})$ by $f(B) = H_B = \{H_b : b \in B\}$, for all $B \in \mathcal{B}(S)$.

Let $x \in \bigcup_{b \in B} H_b$. There exists $b' \in B$ such that $x \in H_{b'}$. Then, $L_k(x) = L_k(b')$ and $R_k(x) = R_k(b')$. This implies that $x + s_1 b' = s_1 b'$ and $x + b' s_2 = b' s_2$ for suitable $s_1,\ s_2 \in S$. Since S is k-regular, there exists $s \in S$ such that $x + xsx = xsx$. This implies that $x + b'(s_3 s s_1)b' = b'(s_3 s s_1)b'$ and hence $x \in \overline{BSB} \subseteq \overline{B} = B$. Thus $\bigcup_{b \in B} H_b \subseteq B$ and finally $B = \bigcup_{b \in B} H_b$ which implies that f is one-to-one.

It follows from Lemma 3.5 that f is onto, thus f is an isomorphism.

Hence $\mathcal{B}(S)$ and $\mathcal{B}(S/\overline{\mathcal{H}})$ are isomorphic. ∎

3.1. Semigroup of k-bi-ideals in k-Clifford and left k-Clifford semiring. In this section, we characterize the semigroup of all k-bi-ideals of a k-Clifford semiring and a left k-Clifford semiring.

Sen and Bhuniya [14] introduced k-Clifford semirings as an analogous notion of the Clifford semigroups. A k-regular semiring S is called a k-Clifford semiring if $\overline{Sa} = \overline{aS}$, for all $a \in S$.

In this section, we need the following equivalent characterizations of k-Clifford semirings.

Lemma 3.7. [14] *For a k-regular semiring S the following conditions are equivalent:*

1. *S is k-Clifford semiring;*
2. *for all $a, b \in S$ there exists $x \in S$ such that $ab + bxa = bxa$;*
3. *for all $a, b \in S$ there exists $x \in S$ such that $ab + axa = axa$ and $ab + bxb = bxb$.*

Again in [9], Jana proved that for any two k-bi-ideals B_1 and B_2 of a k-Clifford semiring S, we always have $B_1 \cap B_2 = \overline{B_1 B_2} = B_1 * B_2$; and hence the semigroup $\mathcal{B}(S)$ of all k-bi-ideals of a k-Clifford semiring S is a semilattice. Now we show that $\mathcal{B}(S)$ is in fact a semilattice.

Theorem 3.8. *Let S be a semiring. Then the following statements are equivalent:*

1. *S is a k-Clifford semiring;*
2. *$(\mathcal{B}(S), *)$ is a semilattice;*
3. *$B_1 * B_2 = B_1 \cap B_2$ for all $B_1, B_2 \in \mathcal{B}(S)$.*

Proof. (1)⇒(3) and (3)⇒(2) follows from the above discussion.

(2)⇒(1) Assume that $\mathcal{B}(S)$ is a semilattice. Then S is a k-regular semiring. Consider $a, b \in S$. Then $ab \in \overline{B_k(a)B_k(b)} = \overline{B_k(b)B_k(a)}$ implies that $ab + vu = vu$ for some $u \in B_k(b)$ and $v \in B_k(a)$. Since S is k-regular, there are $s, t \in S$ such that $u + bsb = bsb$ and $v + ata = ata$. Then we have

$$\begin{aligned} ab + uv = uv &\Rightarrow ab + (u + bsb)(v + ata) = (u + bsb)(v + ata) \\ &\Rightarrow ab + b(sbat)a = b(sbat)a \\ &\Rightarrow ab + bxa = bxa \text{ where } x = sbat \in S. \end{aligned}$$

Hence S is k-Clifford semiring. ∎

In [2], Bhuniya introduced and characterized left k-Clifford semirings as a generalization of k-Clifford semirings. A k-regular semiring S is called a left k-Clifford semiring if $\overline{aS} \subseteq \overline{Sa}$, for all $a \in S$.

The following result is useful for the next part of this section.

Lemma 3.9. *A k-regular semiring is left k-Clifford if and only if for all $a, b \in S$, there is $x \in S$ such that*

$$ab + xa = xa.$$

We leave to the readers checking that $\mathcal{B}(S)$ is a band if S is left k-Clifford. Now we show that $\mathcal{B}(S)$ is a left normal band.

Theorem 3.10. *Let S be a semiring. Then $\mathcal{B}(S)$ is a left normal band if and only if S is a left k-Clifford semiring.*

Proof. Let S be a left k-Clifford semiring. Let $A, B, C \in \mathcal{B}(S)$ and $x \in A*B*C = \overline{ABC}$. Then for some $a \in A$, $b \in B$ and $c \in C$ we have $x + abc = abc$. Since S is k-regular, there is $s \in S$ such that $abc + abcsabc = abcsabc$ which implies that $x + abcsabc = abcsabc$. From which we get $x + atabctcsas_1b = atabctcsas_1b$ as $a + ata = ata$, $c + ctc = ctc$ and $bc + s_1b = s_1b$ for some $t, s_1 \in S$. Again since $ab + s_2a = s_2a$ and $csas_1 + s_3c = s_3c$ for some $s_2, s_3 \in S$, after rearranging the above expression we get $x + (ats_2a)(cts_3c)b = (ats_2a)(cts_3c)b$. Thus $x \in \overline{ACB} = A * C * B$. Therefore $A * B * C \subseteq A * C * B$. Similarly it can be proved that $A * C * B \subseteq A * B * C$. Hence $A * B * C = A * C * B$ and so $\mathcal{B}(S)$ is a left normal band.

Conversely, assume that $\mathcal{B}(S)$ is a left normal band. Then S is k-regular, by Proposition 3.2. Let $a, b \in S$. Then there is $x \in S$ such that $ab + abxab =$

$abxab$ which implies that $ab \in \overline{B_k(abx)B_k(a)B_k(b)} = \overline{B_k(abx)B_k(b)B_k(a)}$, since $\mathcal{B}(S)$ is a left normal band. Then $ab + uvw = uvw$ where $u \in B_k(abx)$, $v \in B_k(b)$ and $w \in B_k(a)$. Since S is k-regular, there is $s \in S$ such that $w + asa = asa$. Now $ab + uvw = uvw$ implies that $ab + (uvas)a = (uvas)a$. Thus S is left k-Clifford. ∎

4. Semigroup of minimal k-bi-ideals

In [4], we introduced and characterized minimal k-bi-ideals of a semiring. A k-bi-ideal B of S is called a minimal k-bi-ideal if there is no non-trivial k-bi-ideal A of S such that $A \subsetneq B$. Thus a k-bi-ideal B is minimal if and only if for all non-trivial k-bi-ideals A of S, $A \subseteq B$ implies that $A = B$.

Now we consider those semirings $S \in SL^+$ of which minimal left k-ideal, minimal right k-ideal and minimal k-bi-ideal exist. Denote the class of all minimal k-bi-ideals by $\mathcal{B}_m(S)$. Similarly, we define minimal left k-ideals and minimal right k-ideals and the class of all minimal left (right) k-ideals will be denoted by $\mathcal{L}_m(S)$ ($\mathcal{R}_m(S)$). Throughout this section, we assume that the semiring S is such that both $\mathcal{L}_m(S)$ and $\mathcal{R}_m(S)$ and hence $\mathcal{B}_m(S)$ are nonempty.

Consider two minimal k-bi-ideals B_1 and B_2. Let $C \in \mathcal{B}(S)$ be such that $C \subseteq \overline{B_1B_2}$. Then $\overline{CB_1}, \overline{B_2C} \in \mathcal{B}(S)$. Now $\overline{CB_1} \subseteq B_1B_2B_1 \subseteq B_1SB_1 \subseteq B_1$. Then by minimality of B_1 we have $\overline{CB_1} = B_1$. Similarly, $B_2C = B_2$. Then, $\overline{B_1B_2} = \overline{\overline{CB_1}\ \overline{B_2C}} \subseteq \overline{CB_1B_2C} \subseteq \overline{CSC} \subseteq \overline{C} = C$. Therefore $C = \overline{B_1B_2}$.

Thus $\mathcal{B}_m(S)$ is a subsemigroup of $\mathcal{B}(S)$. Similarly it can be proved that both $\mathcal{L}_m(S)$ and $\mathcal{R}_m(S)$ are subsemigroups of $\mathcal{B}(S)$. Now we characterize special properties of $\mathcal{L}_m(S)$, $\mathcal{R}_m(S)$ and $\mathcal{B}_m(S)$.

Theorem 4.1. (a) *If S is a semiring such that the set $\mathcal{L}_m(S)$ of all minimal left k-ideals is non-empty then $L * L_m = L_m$ for all $L \in \mathcal{L}(S)$ and $L_m \in \mathcal{L}_m(S)$. Hence $\mathcal{L}_m(S)$ is a left ideal of the semigroup $(\mathcal{L}(S), *)$, in particular a subsemigroup. Moreover, $(\mathcal{L}_m(S), *)$ is a right zero band.*

(b) *If S is a semiring such that the set $\mathcal{R}_m(S)$ of all minimal right k-ideals is non-empty then $R_m * R = R_m$ for all $R \in \mathcal{R}(S)$ and $R_m \in \mathcal{R}_m(S)$. Hence $\mathcal{R}_m(S)$ is a right ideal of the semigroup $(\mathcal{R}(S), *)$, in particular a subsemigroup. Moreover, $(\mathcal{R}_m(S), *)$ is a left zero band.*

Proof. (a) Consider the left k-ideal $L * L_m$ for some $L \in \mathcal{L}(S)$ and $L_m \in \mathcal{L}_m(S)$. Then $L * L_m = \overline{LL_m} \subseteq \overline{L_m} = L_m$ and minimality of L_m shows $L * L_m = L_m$. The rest is obvious.

(b) This follows dually. ∎

We already have that $\mathcal{B}_m(S)$ is a subsemigroup of $\mathcal{B}(S)$. Now we show that, in fact, $\mathcal{B}_m(S)$ is a bi-ideal of $\mathcal{B}(S)$.

Theorem 4.2. *If S is a semiring such that the set $\mathcal{B}_m(S)$ of all minimal k-bi-ideals is non-empty then $B_m * B * C_m \in \mathcal{B}_m(S)$, $B_m * B_m = B_m$ and $B_m * B * B_m = B_m$ for all $B_m, C_m \in \mathcal{B}_m(S)$ and $B \in \mathcal{B}(S)$. Hence $\mathcal{B}_m(S)$ is a bi-ideal of the semigroup $(\mathcal{B}(S), *)$, in particular a subsemigroup. Moreover, $(\mathcal{B}_m(S), *)$ is a rectangular band.*

Proof. Let $B_m, C_m \in \mathcal{B}_m(S)$ and $B \in \mathcal{B}(S)$ and denote the bi-ideal $B_m * B * C_m$ by C. Let $D \in \mathcal{B}(S)$ such that $D \subseteq C$. Then $D * B_m \subseteq \overline{B_m B C_m B_m} \subseteq B_m$ and minimality of B_m shows that $B_m = D * B_m$. Similarly, $C_m = C_m * D$. Therefore $C = B_m * B * C_m = D * B_m * B * C_m * D = \overline{D B_m B C_m D} \subseteq D$ and so $C = D$. Thus the k-bi-ideal $B_m * B * C_m \in \mathcal{B}_m(S)$. Also $\mathcal{B}_m(S)$ is a subsemigroup of $\mathcal{B}(S)$. Hence $\mathcal{B}_m(S)$ is a bi-ideal of $\mathcal{B}(S)$.

Consider the k-bi-ideal $B_m * B_m$ for some $B_m \in \mathcal{B}_m(S)$. Then $B_m * B_m = \overline{B_m B_m} \subseteq \overline{B_m} = B_m$ and minimality of B_m shows that $B_m * B_m = B_m$. Since also $B_m * B * B_m$ is a k-bi-ideal, from $B_m * B * B_m = \overline{B_m B B_m} \subseteq \overline{B_m} = B_m$ and minimality of B_m it follows that $B_m * B * B_m = B_m$. The rest follows obviously. ∎

In [4], we proved that a k-bi-ideal B is minimal if and only if $B = \overline{RL}$ for some minimal right k-ideal R and minimal left k-ideal L of S. Thus we have the following result:

Theorem 4.3. *Let S be a semiring. Then $\mathcal{B}_m(S) = \mathcal{R}_m(S)\mathcal{L}_m(S)$.*

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A. K. Bhuniya
DEPARTMENT OF MATHEMATICS
VISVA-BHARATI, SANTINIKETAN-731235, INDIA
E-mail: anjankbhuniya@gmail.com

K. Jana
DEPARTMENT OF MATHEMATICS
KATWA COLLEGE
KATWA-713130, INDIA
E-mail: kjana76@gmail.com

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