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CORRIGENDUM TO "GENERALIZATIONS OF OPIAL-TYPE INEQUALITIES IN SEVERAL INDEPENDENT VARIABLES" PUBLISHED IN DEMONSTRATIO MATH. 4(47) (2014), 324–335

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Abstract. The purpose of this corrigendum is to correct an error in the earlier paper by the authors: Generalizations of Opial-type inequalities in several independent variables, Demonstratio Math.

In the paper "Generalizations of Opial-type inequalities in several independent variables" published in Demonstratio Mathematica ([1]), we have considered certain multidimensional Opial-type inequalities, and for two of them, inequalities obtained in Theorem 2.1 (page 841) and Theorem 2.3 (page 844), we give corrigendum. Namely, the error was made in a final step of the proof of Theorem 2.1, in the equality (2.7). Here we made necessary corrections, which resulted from the need to observe the inequality on $\Omega = \prod_{j=1}^{m} [a_j, b_j]$ with boundary conditions only in $a = (a_1, \dots, a_m)$. Since applied Theorem 2.1 was used in Theorem 2.3, we made appropriate changes in Theorem 2.3, also.

Following notation is used:

Let $\Omega = \prod_{j=1}^{m} [a_j, b_j]$ and $vol(\Omega) = \prod_{j=1}^{m} (b_j - a_j)$. Let $t = (t_1, \dots, t_m)$ be a general point in Ω, $\Omega_t = \prod_{j=1}^{m} [a_j, t_j]$ and $dt = dt_1 \dots dt_m$. Further, let

$$Du(x) = \frac{d}{dx} u(x), \quad D_k u(t_1, \dots, t_m) = \frac{\partial}{\partial t_k} u(t_1, \dots, t_m)$$

and

$$D^k u(t_1, \dots, t_m) = D_1 \cdots D_k u(t_1, \dots, t_m),$$

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$1 \le k \le m$. Let $\Omega' = \prod_{j=2}^{m}[a_j, b_j]$ and $dt' = dt_2 \dots, dt_m$. Let

$$D^{jl}u(t_1, \dots, t_m) = \frac{\partial^{jl}}{\partial t_j^l \dots \partial t_1^l} u(t_1, \dots, t_m),$$

$1 \le j \le m$, $1 \le l \le n$.

Also, by $C^{mn}(\Omega)$ we denote the space of all functions u on Ω which have continuous derivatives $D^{jl}u$ for $j = 1, \dots, m$ and $l = 1, \dots, n$.

Proofs of corrected theorems follow the same step as in [1], but finish with the inequality using boundary conditions only in a. First, we give corrigendum to [1, Theorem 2.1]. Notice that the equation (2.2) from [1] is explained in more detail here.

Theorem 1. *Let $m, n, p \in \mathbb{N}$. Let f be a nonnegative and differentiable function on $[0, \infty)^p$, with $f(0, \dots, 0) = 0$. Further, for $i = 1, \dots, p$ let $x_i \in C^{mn}(\Omega)$ be such that $D^{jl}x_i(t)|_{t_j = a_j} = 0$, where $j = 1, \dots, m$ and $l = 0, \dots, n-1$. Also, let $D_i f$, $i = 1, \dots, p$, be nonnegative, continuous and nondecreasing on $[0, \infty)^p$. Then the following inequality holds*

$$\begin{aligned}(1)\qquad &\int_{\Omega} \left(\sum_{i=1}^{p} D_i f\left(|x_1(t)|, \dots, |x_p(t)|\right) |D^{mn}x_i(t)| \right) dt \\ &\le \frac{(n-1)!^m}{(vol(\Omega))^{n-1}} f\left(\frac{(vol(\Omega))^{n-1}}{(n-1)!^m} \int_{\Omega} |D^{mn}x_1(t)|\, dt, \dots, \right. \\ &\qquad \left. \frac{(vol(\Omega))^{n-1}}{(n-1)!^m} \int_{\Omega} |D^{mn}x_p(t)|\, dt \right).\end{aligned}$$

Proof. We extend technique used in [2, Theorem 2.1] on a multidimensional case. For continuous function $g : \Omega \to \mathbb{R}$, we should define $y : \Omega \to \mathbb{R}$ such that

$$(2)\qquad D^{mn}y(x_1, \dots, x_m) = \frac{\partial^{mn} y}{\partial x_m^n \cdots \partial x_1^n} = g(x_1, \dots, x_m)$$

and

$$(3)\qquad y(x_1, \dots, x_m) = \frac{1}{(n-1)!^m} \int_{\Omega_x} \prod_{j=1}^{m} (x_j - t_j)^{n-1} g(t_1, \dots, t_m)\, dt_1 \cdots dt_m,$$

where $\Omega_x = \prod_{j=1}^{m}[a_j, x_j]$.

Define

$$(4)\qquad y(x) = \int_a^x dt^1 \int_a^{t^1} dt^2 \cdots \int_a^{t^{n-2}} dt^{n-1} \int_a^{t^{n-1}} g(t^n)\, dt^n$$

or, in different notations

$$y(x) = \int\limits_{\Omega_x} dt^1 \int\limits_{\Omega_{t^1}} dt^2 \cdots \int\limits_{\Omega_{t^{n-2}}} dt^{n-1} \int\limits_{\Omega_{t^{n-1}}} g(t^n)\, dt^n, \tag{5}$$

where

$$a = (a_1, \ldots, a_m), \quad x = (x_1, \ldots, x_m), \quad t^i = (t^i_1, \ldots, t^i_m), \quad dt^i = dt^i_1 \cdots dt^i_m,$$

$i = 1, \ldots, n$ and

$$\Omega_{t^i} = \prod_{j=1}^{m} [a_j, t^i_j], \quad \Omega_{t^i} \subseteq \Omega_{t^{i-1}}, \quad i = 1, \ldots, n-1.$$

Since g is a continuous function, (2) obviously follows.

Obviously, integrals on the right-hand side of (4) or (5), can be written as iterations of the integrals of the form

$$\int_{a_j}^{x_j} dt^1_j \int_{a_j}^{t^1_j} dt^2_j \cdots \int_{a_j}^{t^{n-2}_j} dt^{n-1}_j \int_{a_j}^{t^{n-1}_j} \widetilde{g}(t^n_j)\, dt^n_j,$$

which are known (and easy to deduce by interchanging the order of integration) to be equal to

$$\frac{1}{(n-1)!} \int_{a_j}^{x_j} (x_j - t^n_j)^{n-1} \widetilde{g}(t^n_j)\, dt^n_j,$$

$j = 1, \ldots, m$, from which (3) easily follows.

Let

$$y_i(t) = \frac{1}{(n-1)!^m} \int\limits_{\Omega_t} \prod_{j=1}^{m} (t_j - s_j)^{n-1} |D^{mn} x_i(s)|\, ds, \tag{6}$$

for $t \in \Omega$, $i = 1, \ldots, p$. Hence

$$D^{mn} y_i(t) = |D^{mn} x_i(t)| \quad \text{and} \quad y_i(t) \geq |x_i(t)|.$$

It is easy to conclude that for each $l = 0, \ldots, n-1$ we have $D^{jl} y_i(t) \geq 0$ and nondecreasing on Ω ($i = 1, \ldots, p$ and $j = 1, \ldots, m$). From $D^{jl} y_i(t)|_{t_j = a_j} = 0$ follows

$$y_i(t) \leq \frac{(vol(\Omega))^{n-1}}{(n-1)!^m} D^{m(n-1)} y_i(t), \quad t \in \Omega.$$

Define

$$u_i(t) = \frac{(vol(\Omega))^{n-1}}{(n-1)!^m} D^{m(n-1)} y_i(t)$$

for $t \in \Omega$ and $i = 1, \ldots, p$. Since $D_i f$ are nonnegative, continuous and nondecreasing on $[0,\infty)^p$, it follows

$$\int_\Omega \Big[\sum_{i=1}^{p} D_i f\left(|x_1(t)|,\dots,|x_p(t)|\right)|D^{mn}x_i(t)|\Big]dt \le \int_\Omega \Big[\sum_{i=1}^{p} D_i f\left(y_1(t),\dots,y_p(t)\right) D^{mn}y_i(t)\Big]dt, \tag{7}$$

and

$$\begin{aligned}
&\int_\Omega \left[\sum_{i=1}^{p} D_i f\left(y_1(t),\dots,y_p(t)\right) D^{mn}y_i(t)\right] dt \\
&\le \int_\Omega \Bigg[\sum_{i=1}^{p} D_i f\Bigg(\frac{(vol(\Omega))^{n-1}}{(n-1)!^m} D^{m(n-1)}y_1(t),\dots, \\
&\qquad \frac{(vol(\Omega))^{n-1}}{(n-1)!^m} D^{m(n-1)}y_p(t)\Bigg) D^{mn}y_i(t)\Bigg] dt \\
&\le \int_{a_1}^{b_1} \left[\sum_{i=1}^{p} D_i f\left(u_1(t_1,b_2,\dots,b_m),\dots,u_p(t_1,b_2,\dots,b_m)\right) \times \int_{\Omega'} D^{mn}y_i(t)\,dt'\right] dt_1 \\
&\le \int_{a_1}^{b_1} \Bigg[\sum_{i=1}^{p} D_i f\left(u_1(t_1,b_2,\dots,b_m),\dots,u_p(t_1,b_2,\dots,b_m)\right) \\
&\qquad \times \frac{(n-1)!^m}{(vol(\Omega))^{n-1}} D_1 u_i(t_1,b_2\dots,b_m)\Bigg] dt_1 \\
&= \frac{(n-1)!^m}{(vol(\Omega))^{n-1}} \int_{a_1}^{b_1} \frac{d}{dt_1}\left[f\left(u_1(t_1,b_2,\dots,b_m),\dots,u_p(t_1,b_2,\dots,b_m)\right)\right] dt_1 \\
&= \frac{(n-1)!^m}{(vol(\Omega))^{n-1}} f\left(u_1(b_1,b_2,\dots,b_m),\dots,u_p(b_1,b_2,\dots,b_m)\right) \\
&= \frac{(n-1)!^m}{(vol(\Omega))^{n-1}} f \\
&\qquad \times \left(\frac{(vol(\Omega))^{n-1}}{(n-1)!^m}\int_\Omega |D^{mn}x_1(t)|\,dt,\dots,\frac{(vol(\Omega))^{n-1}}{(n-1)!^m}\int_\Omega |D^{mn}x_p(t)|\,dt\right). \blacksquare
\end{aligned}$$

Next comes a result for a convex function f. The proof follows the same steps as in [1, Theorem 2.3], again with the difference of observing the inequality on Ω with boundary conditions only in a. We will use the following lemma about convex function of several variables ([3, page 11]).

Lemma 1. *Suppose that f is defined on the open convex set $U \subset \mathbb{R}^n$. If f is (strictly) convex on U and the gradient vector $f'(x)$ exists throughout U, then f' is (strictly) increasing on U.*

Theorem 2. *Let $m, n, p \in \mathbb{N}$. Let f be a convex and differentiable function on $[0,\infty)^p$ with $f(0,\dots,0) = 0$. Further, for $i = 1,\dots,p$ let $x_i \in C^{mn}(\Omega)$ be such that $D^{jl}x_i(t)|_{t_j=a_j} = 0$, where $j = 1,\dots,m$ and $l = 0,\dots,n-1$. Then the following inequality holds*

$$\int\limits_{\Omega} \Big(\sum_{i=1}^{p} D_i f\left(|x_1(t)|,\dots,|x_p(t)|\right) |D^{mn}x_i(t)| \Big) dt \leq \frac{(n-1)!^m}{(vol(\Omega))^n} \int\limits_{\Omega} f\bigg(\frac{(vol(\Omega))^n}{(n-1)!^m} |D^{mn}x_1(t)|,\dots, \frac{(vol(\Omega))^n}{(n-1)!^m} |D^{mn}x_p(t)| \bigg) dt. \tag{8}$$

Proof. As in the proof of the previous theorem, we obtain (1) with the difference of applying Lemma 1 in (7) since f is a convex function. Then, from Jensen's inequality for integrals (see for example [3, page 51]), we have

$$\int\limits_{\Omega} \left[\sum_{i=1}^{p} D_i f\Big(|x_1(t)|,\dots,|x_p(t)| \Big) |D^{mn}x_i(t)| \right] dt$$

$$\leq \frac{(n-1)!^m}{(vol(\Omega))^{n-1}} f\bigg(\frac{(vol(\Omega))^{n-1}}{(n-1)!^m} \int\limits_{\Omega} |D^{mn}x_1(t)|\, dt,\dots, \frac{(vol(\Omega))^{n-1}}{(n-1)!^m} \int\limits_{\Omega} |D^{mn}x_p(t)|\, dt \bigg)$$

$$= \frac{(n-1)!^m}{(vol(\Omega))^{n-1}} f\bigg(\frac{1}{(vol(\Omega))} \int\limits_{\Omega} \frac{(vol(\Omega))^n}{(n-1)!^m} |D^{mn}x_1(t)|\, dt,\dots, \frac{1}{(vol(\Omega))} \int\limits_{\Omega} \frac{(vol(\Omega))^n}{(n-1)!^m} |D^{mn}x_p(t)|\, dt \bigg)$$

$$\leq \frac{(n-1)!^m}{(vol(\Omega))^n} \int\limits_{\Omega} f\left(\frac{(vol(\Omega))^n}{(n-1)!^m} |D^{mn}x_1(t)|,\dots, \frac{(vol(\Omega))^n}{(n-1)!^m} |D^{mn}x_p(t)| \right) dt. \blacksquare$$

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