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SEMI-SLANT SUBMERSIONS FROM ALMOST PRODUCT RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we introduce semi-slant submersions from almost product Riemannian manifolds onto Riemannian manifolds. We give some examples, investigate the geometry of foliations which are arisen from the definition of a Riemannian submersion. We also find necessary and sufficient conditions for a semi-slant submersion to be totally geodesic.

1. Introduction

Given a $C^\infty$-submersion $\pi$ from a Riemannian manifold $(M, g)$ onto a Riemannian manifold $(B, g')$, there are several kinds of submersions according to the conditions on it: e.g. Riemannian submersion ([5], [11]), slant submersion ([6], [12], [13]), almost Hermitian submersion [16], quaternionic submersion [7], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang–Mills theory ([3], [17]), Kaluza–Klein theory ([2], [8]), semi-invariant submersion([14]), supergravity and superstring theories ([9], [10]), etc. In [15], the author studied the slant and semi-slant submanifolds of an almost product Riemannian manifold. Let $(M, g, F)$ be an almost product Riemannian manifold. A Riemannian submersion $\pi : (M, g, F) \to (N, g')$ is called a slant submersion if the angle $\theta(X)$ between $FX$ and the space $\ker(\pi_*)_p$ is constant for any nonzero $X \in T_pM$ and $p \in M$ [6]. We call $\theta(X)$ a slant angle. The paper is organized as follows. In Section 2 we recall some notions needed for this paper. In Section 3 we give definition of semi-slant submersions and provide examples. We also investigate the geometry of leaves of the distributions. Finally, we give necessary and sufficient conditions for such submersions to be totally geodesic.

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2. Preliminaries

In this section, we define almost product Riemannian manifolds, recall the notion of Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions.

Let $M$ be a $m$-dimensional manifold with a tensor $F$ of type $(1,1)$ such that

$$F^2 = I, (F \neq I).$$

Then, we say that $M$ is an almost product Riemannian manifold with almost product structure $F$. We put

$$P_1 = \frac{1}{2}(I + F), \quad P_2 = \frac{1}{2}(I - F).$$

Then we get

$$P_1 + P_2 = I, \quad P_1^2 = P_1, \quad P_2^2 = P_2, \quad P_1 P_2 = P_2 P_1 = 0, \quad F = P_1 - P_2.$$ 

Thus $P_1$ and $P_2$ define two complementary distributions $P_1$ and $P_2$. We easily see that the eigenvalues of $F$ are $+1$ or $-1$.

If an almost product manifold $M$ admits a Riemannian metric $g$ such that

$$(1) \quad g(FX, FY) = g(X, Y)$$

for any vector fields $X$ and $Y$ on $M$, then $M$ is called an almost product Riemannian manifold, denoted by $(M, g, F)$.

Denote the Levi–Civita connection on $M$ with respect to $g$ by $\nabla$. Then, $M$ is called a locally product Riemannian manifold if $F$ is parallel with respect to $\nabla$, i.e.,

$$(2) \quad \nabla_X F = 0, X \in \Gamma(TM) \ [18].$$

Let $(M, g)$ and $(N, g')$ be two Riemannian manifolds. A surjective $C^\infty$-map $\pi : M \to N$ is a $C^\infty$-submersion if it has maximal rank at any point of $M$. Putting $\mathcal{V}_x = ker\pi_\ast x$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$, which is called vertical distribution and corresponds to the foliation of $M$ determined by the fibres of $\pi$. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric $g$, is called horizontal distribution. A $C^\infty$-submersion $\pi : M \to N$ between two Riemannian manifolds $(M, g)$ and $(N, g')$ is called a Riemannian submersion if, at each point $x$ of $M$, $\pi_\ast x$ preserves the length of the horizontal vectors. A horizontal vector field $X$ on $M$ is said to be basic if $X$ is $\pi$-related to a vector field $X'$ on $N$. It is clear that every vector field $X'$ on $N$ has a unique horizontal lift $X$ to $M$ and $X$ is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion
\( \pi : M \to N \) determines two \((1, 2)\) tensor fields \( T \) and \( A \) on \( M \), by the formulas:

\[
T(E, F) = T_E F = h \nabla_{vE} vF + v \nabla_{vE} hF
\]

and

\[
A(E, F) = A_E F = v \nabla_{hE} hF + h \nabla_{hE} vF
\]

for any \( E, F \in \Gamma(TM) \), where \( v \) and \( h \) are the vertical and horizontal projections (see [4]). From (3) and (4), one can obtain

\[
\nabla_U W = T_U W + \hat{\nabla}_U W;
\]

\[
\nabla_U X = T_U X + h(\nabla_U X);
\]

\[
\nabla_X U = v(\nabla_X U) + A_X U;
\]

\[
\nabla_X Y = A_X Y + h(\nabla_X Y),
\]

for any \( X, Y \in \Gamma((\ker \pi^\ast)^\perp) \), \( U, W \in \Gamma(\ker \pi^\ast) \). Moreover, if \( X \) is basic then

\[
h(\nabla_U X) = h(\nabla_X U) = A_X U.
\]

We note that for \( U, V \in \Gamma(\ker \pi^\ast) \), \( T_U V \) coincides with the second fundamental form of the immersion of the fibre submanifolds and for \( X, Y \in \Gamma((\ker \pi^\ast)^\perp) \), \( A_X Y = \frac{1}{2} v[X, Y] \) reflecting the complete integrability of the horizontal distribution \( \mathcal{H} \). It is known that \( A \) is alternating on the horizontal distribution: \( A_X Y = -A_Y X \), for \( X, Y \in \Gamma((\ker \pi^\ast)^\perp) \) and \( T \) is symmetric on the vertical distribution: \( T_U V = T_V U \), for \( U, V \in \Gamma(\ker \pi^\ast) \).

We now recall the following result which will be useful for later.

**Lemma 2.1.** (see [4], [11]) If \( \pi : M \to N \) is a Riemannian submersion and \( X, Y \) basic vector fields on \( M \), \( \pi \)-related to \( X' \) and \( Y' \) on \( N \), then we have the following properties

1. \( h[X, Y] \) is a basic vector field and \( \pi^\ast h[X, Y] = [X', Y'] \circ \pi \);
2. \( h(\nabla_X Y) \) is a basic vector field \( \pi \)-related to \( (\nabla'_X, Y') \), where \( \nabla \) and \( \nabla' \) are the Levi–Civita connection on \( M \) and \( N \);
3. \( [E, U] \in \Gamma(\ker \pi^\ast) \), for any \( U \in \Gamma(\ker \pi^\ast) \) and for any basic vector field \( E \).

Let \((M, g_M)\) and \((N, g_N)\) be Riemannian manifolds and \( \pi : M \to N \) is a smooth map. Then the second fundamental form of \( \pi \) is given by

\[
(\nabla_{\pi^\ast})(X, Y) = \nabla_{\pi^\ast X} \pi^\ast Y - \pi^\ast(\nabla_X Y)
\]

for \( X, Y \in \Gamma(TM) \), where we denote conveniently by \( \nabla \) the Levi–Civita connections of the metrics \( g_M \) and \( g_N \). Recall that \( \pi \) is said to be harmonic if \( \text{trace}(\nabla_{\pi^\ast}) = 0 \) and \( \pi \) is called a totally geodesic map if \( (\nabla_{\pi^\ast})(X, Y) = 0 \) for \( X, Y \in \Gamma(TM) \) [1]. It is known that the second fundamental form is symmetric.
3. Semi-slant submersions

In this section, we define semi-slant submersions from an almost product Riemannian manifold onto a Riemannian manifold, investigate the integrability of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map.

**Definition 3.1.** Let \((M, g, F)\) be an almost product Riemannian manifold and \((N, g')\) be a Riemannian manifold. A Riemannian submersion \(\pi : (M, g, F) \to (N, g')\) is called a semi-slant submersion if there is a distribution \(D_1 \subset \ker \pi^*\) such that

\[
\ker \pi^* = D_1 \oplus D_2, \quad F(D_1) = D_1,
\]

and the angle \(\theta = \theta(X)\) between \(FX\) and the space \((D_2)_q\) is constant for nonzero \(X \in (D_2)_q\) and \(q \in M\), where \(D_2\) is the orthogonal complement of \(D_1\) in \(\ker \pi^*_q\). We call the angle \(\theta\) a semi-slant angle.

First, we give some examples of semi-slant submersions.

**Example 1.** Let \(\pi\) be a slant submersion from an almost product Riemannian manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\) [6]. Then the map \(\pi\) is a semi-slant submersion with \(D_2 = \ker \pi^*_q\).

**Example 2.** Let \(\pi\) be a semi-invariant submersion from an almost product Riemannian manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\) [14]. Then the map \(\pi\) is a semi-slant submersion with the semi-slant angle \(\cos^{-1}(0)\).

Note that given an Euclidean space \(\mathbb{R}^{2n}\) with coordinates \((x_1, \ldots, x_{2n})\) on \(\mathbb{R}^{2n}\), we can naturally choose an almost product structure \(F\) on \(\mathbb{R}^{2n}\) as follows:

\[
F\left(\frac{\partial}{\partial x_{2i}}\right) = \frac{\partial}{\partial x_{2i-1}}, \quad F\left(\frac{\partial}{\partial x_{2i-1}}\right) = \frac{\partial}{\partial x_{2i}},
\]

where \(i = 1, \ldots, n\).

Throughout this section, we will use this notation.

**Example 3.** Define a map \(\pi : \mathbb{R}^6 \to \mathbb{R}^2\) by

\[
\pi(x_1, \ldots, x_6) = (x_1 \sin \alpha - x_3 \cos \alpha, x_4),
\]

where \(0 < \alpha < 90\). Then the map \(\pi\) is a semi-slant submersion such that

\[
D_1 = \left\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6} \right\rangle \quad \text{and} \quad D_2 = \left\langle \frac{\partial}{\partial x_2}, \cos \alpha \frac{\partial}{\partial x_1} + \sin \alpha \frac{\partial}{\partial x_3} \right\rangle
\]

with the semi-slant angle \(\cos^{-1}(\alpha)\).

**Example 4.** Define a map \(\pi : \mathbb{R}^8 \to \mathbb{R}^2\) by

\[
\pi(x_1, \ldots, x_8) = \left(\frac{x_1 - x_3}{\sqrt{2}}, x_4\right),
\]
Then the map $\pi$ is a semi-slant submersion such that

$$D_1 = \left\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle \quad \text{and} \quad D_2 = \left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \right\rangle$$

with the semi-slant angle $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

**Example 5.** Define a map $\pi : R^8 \to R^2$ by

$$\pi(x_1, \ldots, x_8) = (x_1 \cos \alpha - x_3 \sin \alpha, x_2 \sin \beta - x_4 \cos \beta),$$

where $\alpha$ and $\beta$ are constant. Then the map $\pi$ is a semi-slant submersion such that

$$D_1 = \left\langle \frac{\partial}{\partial x_5}, \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_8} \right\rangle \quad \text{and} \quad D_2 = \left\langle \sin \alpha \frac{\partial}{\partial x_1} + \cos \alpha \frac{\partial}{\partial x_3}, \cos \beta \frac{\partial}{\partial x_2} + \sin \beta \frac{\partial}{\partial x_4} \right\rangle$$

with the semi-slant angle $\theta$ with $\cos \theta = |\sin(\alpha + \beta)|$.

**Example 6.** Define a map $\pi : R^{10} \to R^4$ by

$$\pi(x_1, \ldots, x_{10}) = \left(\frac{x_4 - x_6}{\sqrt{2}}, x_9, \frac{x_5 - x_7}{\sqrt{2}}, x_{10}\right).$$

Then the map $\pi$ is a semi-slant submersion such that

$$D_1 = \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle \quad \text{and} \quad D_2 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_6}, \frac{\partial}{\partial x_8}, \frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_7} \right\rangle$$

with the semi-slant angle $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$.

**Example 7.** Define a map $\pi : R^8 \to R^2$ by

$$\pi(x_1, \ldots, x_8) = \left(x_1, x_2, \frac{x_5 + x_7}{\sqrt{2}}, \frac{x_6 - x_8}{\sqrt{2}}\right).$$

Then the map $\pi$ is a semi-slant submersion such that

$$D_1 = \left\langle \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\rangle \quad \text{and} \quad D_2 = \left\langle -\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7}, \frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_8} \right\rangle$$

with the semi-slant angle $\cos^{-1}(0)$.

Let $\pi : (M, g, F) \to (N, g')$ be a semi-slant submersion. Then there is a distribution $D_1 \subset \ker \pi_*$ such that

$$\ker \pi_* = D_1 \oplus D_2, \quad F(D_1) = D_1,$$

and the angle $\theta = \theta(X)$ between $FX$ and the space $(D_2)_q$ is constant for nonzero $X \in (D_2)_q$ and $q \in M$, where $D_2$ is the orthogonal complement of $D_1$ in $\ker \pi_*$. Then for $X \in \Gamma(\ker \pi_*)$, we have

$$X = PX + QX,$$
where $PX \in \Gamma(D_1)$ and $QX \in \Gamma(D_2)$. For $X \in \Gamma(ker\pi_*)$, we get

$$FX = \phi X + \omega X,$$

where $\phi X \in \Gamma(ker\pi_*)$ and $\omega X \in \Gamma((ker\pi_*)^\perp)$. For $Z \in \Gamma((ker\pi_*)^\perp)$, we obtain

$$FZ = BZ + CZ,$$

where $BZ \in \Gamma(ker\pi_*)$ and $CZ \in \Gamma((ker\pi_*)^\perp)$. For $U \in \Gamma(TM)$, we have

$$U = vU + hU,$$

where $vU \in \Gamma(ker\pi_*)$ and $hU \in \Gamma((ker\pi_*)^\perp)$. Then

$$(ker\pi_*)^\perp = \omega D_2 \oplus \mu,$$

where $\mu$ is the orthogonal complement of $\omega D_2$ in $(ker\pi_*)^\perp$ and is invariant under $F$. Furthermore,

$$\phi D_1 = D_1, \quad \omega D_1 = 0, \quad \phi D_2 \subset D_2, \quad B((ker\pi_*)^\perp) = D_2,$$

$$\phi^2 + B \omega = I, \quad C^2 + \omega B = I, \quad \omega \phi + C \omega = 0, \quad BC + \phi B = 0.$$

We define the covariant derivatives of $\phi$ and $\omega$ as follows

$$\nabla_X \phi Y = \hat{\nabla}_X \phi Y - \phi \hat{\nabla}_X Y$$

and

$$\nabla_X \omega Y = h\nabla_X \omega Y - \omega \hat{\nabla}_X Y$$

for $X, Y \in \Gamma(ker\pi_*)$, where $\hat{\nabla}_X Y = v\nabla_X Y$. Then we easily have

**Lemma 3.1.** Let $(M, g, F)$ be a locally product manifold and $(N, g')$ be a Riemannian manifold. Let $\pi : (M, g, F) \to (N, g')$ be a semi-slant submersion. Then we get

(a) \[ \hat{\nabla}_X \phi Y + T_X \omega Y = \phi \hat{\nabla}_X Y + BT_X Y \]

\[ T_X \phi Y + h\nabla_X \omega Y = \omega \hat{\nabla}_X Y + CT_X Y \]

for any $X, Y \in \Gamma(ker\pi_*)$.

(b) \[ v\nabla_Z BW + A_Z CW = \phi A_Z W + Bh\nabla_Z W \]

\[ A_Z BW + h\nabla_Z CW = \omega A_Z W + Ch\nabla_Z W \]

for $Z, W \in \Gamma((ker\pi_*)^\perp)$.

(c) \[ \hat{\nabla}_X BZ + T_X CZ = \phi T_X Z + Bh\nabla_X Z \]

\[ T_X BZ + h\nabla_X CZ = \omega T_X Z + Ch\nabla_X Z \]

for $X \in \Gamma(ker\pi_*)$ and $Z \in \Gamma((ker\pi_*)^\perp)$. 
**Theorem 3.1.** Let \( \pi \) be a semi-slant submersion from an almost product Riemannian manifold \((M,g,F)\) onto a Riemannian manifold \((N,g')\). Then the distribution \(D_1\) is integrable if and only if we have
\[
\omega(\hat{\nabla}_X Y - \hat{\nabla}_Y X) = C(T_Y X - T_X Y)
\]
for \(X, Y \in \Gamma(D_1)\).

**Proof.** For \(X, Y \in \Gamma(D_1)\) and \(Z \in \Gamma((\ker \pi_*)^\perp)\), since \([X, Y] \in \Gamma(\ker \pi_*), \) from (5), (12) and (13) we get
\[
g(F[X, Y], Z) = g(F\nabla_X Y - F\nabla_Y X, Z)
\]
\[
= g(FT_X Y + F\hat{\nabla}_X Y - FT_Y X - F\hat{\nabla}_Y X, Z)
\]
\[
= g(BT_X Y + CT_X Y + \phi \hat{\nabla}_X Y + \omega \hat{\nabla}_X Y
\]
\[
- BT_Y X - \phi \hat{\nabla}_Y X - CT_Y X - \omega \hat{\nabla}_Y X, Z)
\]
\[
= g(CT_X Y + \omega \hat{\nabla}_X Y - CT_Y X - \omega \hat{\nabla}_Y X, Z).
\]
Therefore, we have the result. □

**Theorem 3.2.** Let \( \pi \) be a semi-slant submersion from an almost product Riemannian manifold \((M,g,F)\) onto a Riemannian manifold \((N,g')\). Then the slant distribution \(D_2\) is integrable if and only if we get
\[
P(\phi(\hat{\nabla}_X Y - \hat{\nabla}_Y X) + B(T_X Y - T_Y X)) = 0
\]
for \(X, Y \in \Gamma(D_2)\).

**Proof.** For \(X, Y \in \Gamma(D_2)\) and \(Z \in \Gamma(D_1)\), since \([X, Y] \in \Gamma(\ker \pi_*), \) from (5), (12) and (13) we obtain
\[
g(F[X, Y], Z) = g(F\nabla_X Y - F\nabla_Y X, Z)
\]
\[
= g(BT_X Y + \phi \hat{\nabla}_X Y - BT_Y X - \phi \hat{\nabla}_Y X, Z)
\]
which proves assertion. □

**Lemma 3.2.** Let \( \pi \) be a semi-slant submersion from a locally product manifold \((M,g,F)\) onto a Riemannian manifold \((N,g')\). Then the distribution \(D_1\) is integrable if and only if we obtain \(Q(\hat{\nabla}_X \phi Y - \hat{\nabla}_Y \phi X) = 0\) and \(T_X \phi Y = T_Y \phi X\) for \(X, Y \in \Gamma(D_1)\).

**Proof.** For \(X, Y \in \Gamma(D_1)\), from (2), (5), (12) and (14) we obtain
\[
F[X, Y] = \nabla_X FY - \nabla_Y FX
\]
\[
= T_X \phi Y + \hat{\nabla}_X \phi Y - T_Y \phi X - \hat{\nabla}_Y \phi X
\]
which proves assertion. □
**Lemma 3.3.** Let \( \pi \) be a semi-slant submersion from a locally product manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). Then the slant distribution \(D_2\) is integrable if and only if we have
\[
P(\nabla_X \phi Y - \nabla_Y \phi X + T_X \omega Y - T_Y \omega X) = 0
\]
for \(X, Y \in \Gamma(D_2)\).

**Proof.** For \(X, Y \in \Gamma(D_2)\) and \(Z \in \Gamma(D_1)\), since \([X, Y] \in \Gamma(\ker \pi^*),\) from (2), (5), (6) and (12) we obtain
\[
g(F[X, Y], Z) = g(\nabla_X FY - \nabla_Y FX, Z)
= g(T_X \omega Y + \nabla_X \phi Y - T_Y \omega X - \nabla_Y \phi X, Z).
\]
Therefore, the result follows. \(\blacksquare\)

**Theorem 3.3.** Let \( \pi \) be a semi-slant submersion from an almost product Riemannian manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). Then we get
\[
\phi^2 X = \cos^2 \theta X
\]
for \(X \in \Gamma(D_2)\), where \(\theta\) denotes the semi-slant angle of \(D_2\).

**Proof.** For \(X \in \Gamma(D_2)\), we can write
\[
\cos \theta(X) = \frac{\|\phi X\|}{\|FX\|}.
\]
By using (12), (19) and (1) we get
\[
g(\phi^2 X, X) = g(\phi X, \phi X)
= \cos^2 \theta(X) g(FX, FX)
= \cos^2 \theta(X) g(X, X)
\]
for \(X \in \Gamma(D_2)\). Since \(g\) is Riemannian metric, from (20) we have
\[
\phi^2 X = \cos^2 \theta(X) X, \quad X \in \Gamma(D_2). \quad \blacksquare
\]

**Corollary 3.1.** Let \( \pi \) be a semi-slant submersion from an almost product Riemannian manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). Then we have
\[
g(\phi X, \phi Y) = \cos^2 \theta g(X, Y), \quad g(\omega X, \omega Y) = \sin^2 \theta g(X, Y)
\]
for \(X, Y \in \Gamma(D_2)\).

From (15) and (18) we have

**Corollary 3.2.** Let \( \pi \) be a semi-slant submersion from an almost product Riemannian manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). Then
\( \pi \) is a semi-slant submersion if and only if there exists a constant \( k \in [0, 1] \) such that
\[
B\omega = kI.
\]
If \( \pi \) is a semi-slant submersion, then \( k = \sin^2 \theta \), where \( \theta \) denotes the semi-slant angle of \( D_2 \).

**Theorem 3.4.** Let \( \pi \) be a semi-slant submersion from a locally product manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). Then the distribution \( \ker\pi_* \) defines a totally geodesic foliation if and only if
\[
\omega(\hat{\nabla}_X \phi Y + T_X \omega Y) + C(T_X \phi Y + h\nabla_X \omega Y) = 0
\]
for \( X, Y \in \Gamma(\ker\pi_*) \).

**Proof.** For \( X, Y \in \Gamma(\ker\pi_*) \), from (2), (5), (6) and (12) we obtain
\[
\nabla_X Y = F\nabla_X FY
\]
\[
= F(T_X \phi Y + \hat{\nabla}_X \phi Y + T_X \omega Y + h\nabla_X \omega Y).
\]
Using (12) and (13), we have
\[
\nabla_X Y = BT_X \phi Y + CT_X \phi Y + \phi \hat{\nabla}_X \phi Y + \omega \hat{\nabla}_X \phi Y
\]
\[
+ \phi T_X \omega Y + \omega T_X \omega Y + Bh\nabla_X \omega Y + Ch\nabla_X \omega Y.
\]
Thus, we get
\[
\nabla_X Y \in \Gamma(\ker\pi_*) \Leftrightarrow C(T_X \phi Y + h\nabla_X \omega Y) + \omega(\hat{\nabla}_X \phi Y + T_X \omega Y) = 0.
\]

From (2), (7), (8), (12) and (13) we have

**Theorem 3.5.** Let \( \pi \) be a semi-slant submersion from a locally product manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). Then the distribution \( (\ker\pi_*)^\perp \) defines a totally geodesic foliation if and only if
\[
\phi(v\nabla_X BY + A_X CY) + B(A_X BY + h\nabla_X CY) = 0
\]
for \( X, Y \in \Gamma((\ker\pi_*)^\perp) \).

In a similar way we have the following.

**Theorem 3.6.** Let \( \pi \) be a semi-slant submersion from a locally product manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). Then the distribution \( D_1 \) defines a totally geodesic foliation if and only if
\[
Q(\phi \hat{\nabla}_X \phi Y + BT_X \phi Y) = 0
\]
and \( CT_X \phi Y + \omega \hat{\nabla}_X \omega Y = 0 \), for \( X, Y \in \Gamma(D_1) \).

From Theorem 3.4 we have the following result.

**Theorem 3.7.** Let \( \pi \) be a semi-slant submersion from a locally product manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). Then the distribution
defines a totally geodesic foliation if and only if
\[ 0 = P(\phi(\hat{\nabla}_X \phi Y + T_X \omega Y) + B(T_X \phi Y + h \nabla_X \omega Y)), \]
\[ 0 = \omega(\hat{\nabla}_X \phi Y + T_X \omega Y) + C(T_X \phi Y + h \nabla_X \omega Y) \]
for \( X, Y \in \Gamma(D_2) \).

**Theorem 3.8.** Let \( \pi \) be a semi-slant submersion from a locally product manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). If the tensor \( \omega \) is parallel, then we have
\[ T_{\phi X} \phi X = \cos^2 \theta T_X X \]
for \( X \in \Gamma(\ker \pi_*) \).

**Proof.** For \( X, Y \in \Gamma(\ker \pi_*) \), from Lemma 3.1(a) we get
\[ T_X \phi Y = CT_X Y. \]
So that interchanging the role of \( X \) and \( Y \),
\[ T_Y \phi X = CT_Y X. \]
Hence
\[ T_X \phi Y = T_Y \phi X. \]
Substituting \( Y \) by \( \phi X \) and using (18),
\[ T_{\phi X} \phi X = \cos^2 \theta T_X X. \]

Finally we give necessary and sufficient conditions for a semi-slant submersion to be totally geodesic. Recall that a differentiable map \( \pi \) between Riemannian manifolds \((M, g)\) and \((B, g')\) is called a totally geodesic map if \((\nabla \pi_*)(X, Y) = 0 \) for all \( X, Y \in \Gamma(TM) \).

**Theorem 3.9.** Let \( \pi \) be a semi-slant submersion from a locally product manifold \((M, g, F)\) onto a Riemannian manifold \((N, g')\). Then \( \pi \) is a totally geodesic map if and only if
\[ 0 = \omega(\hat{\nabla}_X \phi Y + T_X \omega Y) + C(T_X \phi Y + h \nabla_X \omega Y), \]
\[ 0 = \omega(\hat{\nabla}_X BZ + T_X CZ) + C(T_X BZ + h \nabla_X CZ) \]
for \( X, Y \in \Gamma(\ker \pi_*) \) and \( Z \in \Gamma((\ker \pi_*)^\perp) \).

**Proof.** For \( Z_1, Z_2 \in \Gamma((\ker \pi_*)^\perp) \), since \( \pi \) is a Riemannian submersion, from (10) we obtain
\[ (\nabla \pi_*)(Z_1, Z_2) = 0. \]
For \( X, Y \in \Gamma(\ker \pi_*) \), using (2), (10) and (12) we have
\[ (\nabla \pi_*)(X, Y) = -\pi_*(\nabla_X Y) = -\pi_*(F \nabla_X (\phi Y + \omega Y)). \]
From (5), (6), (12) and (13) we get
\[
(\nabla_{\pi_*})(X, Y) = -\pi_*(\phi\nabla_X Y + \omega\nabla_X Y + BT_X Y + CT_X Y + \phi T_X Y + \omega T_X Y + Bh\nabla_X Y + Ch\nabla_X Y).
\]

Thus, we have
\[
(\nabla_{\pi_*})(X, Y) = 0 \iff \omega(\nabla_X Y + T_X Y) + C(T_X Y + h\nabla_X Y) = 0.
\]

For \(X \in \Gamma(ker\pi_*)\) and \(Z \in \Gamma((ker\pi_*)^\perp)\), using again (2), (10) and (12) we obtain
\[
(\nabla_{\pi_*})(X, Z) = (\nabla_{\pi_*})(Z, X) = -\pi_*(\nabla_X Z) = -\pi_*(F\nabla_X (BZ + CZ)).
\]

From (5), (6), (12) and (13) we get
\[
(\nabla_{\pi_*})(X, Z) = -\pi_*(\phi\nabla_X BZ + \omega\nabla_X BZ + BT_X BZ + CT_X BZ + \phi T_X CZ + \omega T_X CZ + Bh\nabla_X CZ + Ch\nabla_X CZ).
\]

Thus, we obtain
\[
(\nabla_{\pi_*})(X, Z) = 0 \iff \omega(\nabla_X BZ + T_X CZ) + C(T_X BZ + h\nabla_X CZ) = 0.
\]

References


