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Research Article

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On the geometric nature of the twin paradox in curved spacetimes

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Abstract: The famous „twin paradox" of special relativity is of purely geometric nature and formulated in curved spacetimes of general relativity motivates investigations of the timelike geodesic structure of these manifolds. Except for the maximally symmetric spacetimes the search for the longest timelike curves is hard, complicated and requires both advanced methods of global Lorentzian geometry and solving the intricate geodesic deviation equation. This article is a theoretical introduction to the problem. First we describe the procedure of determining the locally longest curves; it is algorithmic in the sense of consisting of a small number of definite steps and is effective if the geodesic deviation equation may be solved. Then we discuss the problem of globally maximal timelike curves; due to its nonlocal nature there is no prescription of how to solve it in finite number of steps. In the case of sufficiently high symmetry of the manifold also the globally longest curves may be found. Finally we briefly present some results recently found.

Keywords: Local and global Lorentzian geometry, Jacobi fields, Maximal timelike curves, Conjugate points, Cut points

MSC: 53B50, 53C80, 83-02, 83C99

This paper is dedicated to Rev. Professor Michal Heller in honor of his eightieth birthday.

1 Introduction

The twin paradox in special relativity (SR) theory was formulated by Paul Langevin in 1911 and was based on the famous Einstein's paper of 1905. In a popular formulation it is as follows: two twins (by definition two beings at the same age) separate at some instant of time and the twin A stays on Earth (which approximately is an inertial reference frame) while twin B makes a cosmic travel to a nearby star at a relativistic velocity and returns after ten or more years. At the reunion it turns out that the astronaut is physically younger than twin A. The paradox merely lies in the psychological amazement that the motion at relativistic velocities makes the twins' ages unequal. Logically there is no paradox at all and SR is consistent. The „paradox" (we shall use this traditional misnomer) may be discussed and explained on three levels of description. On the first, most elementary level of reasoning, one merely accounts for the difference of the ages; this explanation is formally correct and employs only school algebra, therefore it appears in most textbooks, though it works only in this simplest version of the paradox and provides no deeper understanding of the effect. On the second level one asks of why, contrary to intuition, the astronaut, who was subject to accelerations, gets younger than the twin. It is here that the geometry of flat Minkowski space plays the key role, for one invokes the reverse triangle inequality. The physical time of each of the twins is its proper time, or the

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length of its worldline. The difference of the twins' ages arises from the geometrical fact that two different timelike curves having common initial and endpoint have in general different length. This geometric interpretation enables one to correctly formulate the problem which replaces the alleged paradox: assuming that there is infinite number of twins („siblings") all travelling on timelike worldlines from a spacetime point P to some point Q, one should not seek for the shortest curve in this set because the infimum of their lengths is zero and this limit is physically inaccessible since it corresponds to a null line. One can seek for the longest worldline in this collection. In Minkowski space the solution is simple: the longest timelike curve from P to Q is the straight line connecting them.

On the third, highest level of understanding, one gives a generic form of the problem: in a curved spacetime (a Lorentzian manifold) of general relativity (GR) theory to determine the longest timelike curve between given points P and Q. This is the problem we shall discuss in this paper and we shall only marginally invoke the twin paradox which has initially motivated it. In a curved spacetime the problem is far more difficult than in the framework of SR and after a number of works one sees that it is still in an initial stage of being solved. A generic, partial and imprecise answer is given in an intermediate level textbook on GR [1]: „for a really reliable answer one has to know how to deal with accelerated systems; here GR is to be asked, and answer is: yes, travelling (deviating from geodesic motion) keeps you younger". Finding out a more precise answer is hard. The geometric problem is meaningful and directly leads to searching the geodesic structure of the spacetime and this is why it is worth studying.

The problem actually consists of two distinct problems, the local and the global one. In the local problem of maximum length one connects P to Q by a narrow bundle of nearby timelike curves and seeks for the longest of them. The longest one is a geodesic and since in general the two points may be connected by more than one geodesic, the locally longest geodesic is the one which contains no point conjugate to P on the segment PQ.

In the global problem one searches for the longest timelike curve in the whole space of timelike curves from P to Q. A locally longest geodesic γ need not be globally maximal because there may exist timelike curves not belonging to the bundle surrounding γ (beyond the endpoints they are distant from it) which are longer than it. Here the crucial notion is that of a future cut point of P on a geodesic.

This paper is an introduction, written in an expository style, to the problem of determining the locally and globally longest timelike curves in a Lorentzian manifold. We present the methods of dealing with both the problems. The approach is general, nonetheless we apply it to physically relevant solutions of Einstein field equations and if the manifold under consideration has high symmetries (isometries) the methods are more effective. Though the article is addressed to the mathematically oriented reader we do not attempt to preserve the mathematical rigour, we keep the rigour accepted in GR papers. Since the Lorentzian geometry is familiar to a minority of relativists and to few experts beyond this group, we provide a detailed and pedagogical exposition of those aspects of the geometry which are relevant to the problem. Our exposition is based on two monographs and an advanced textbook and we present theorems without proofs which can be found in these books.

The paper is organized as follows. Section 2 presents the necessary notions and four theorems concerning the local problem. In Section 3 the analytic tools for solving the local problem are discussed: the geodesic deviation equation and its first integrals generated by spacetime isometries are recast in a form most suitable for finding a solution. Then assuming that the equation may be solved we outline the procedure of determining the locally longest curve. In Section 4 we provide in a „nutshell" the current knowledge of global Lorentzian geometry relevant to the problem of globally maximal curves; this knowledge takes the form of six „existence theorems". Employing the methods presented here we have described in a number of recent works the geodesic structure of some simple and physically relevant spacetimes and these results are briefly sketched in Section 5.

2 The local problem

Almost all geometric foundations of the local problem of maximal timelike curves are contained in [2] and [3]. (In these texts it is not explicitly stated that the following theorems, except the first, apply only to the local problem.) Let U be a convex normal neighbourhood (according to the standard definition) of a point p in any Lorentzian spacetime. Then the following proposition holds:

Theorem 2.1 (Theorem 4.5.3 in [2]). *In any convex normal neighbourhood U, if p and q can be joined by a timelike curve then the unique timelike geodesic connecting them has length strictly greater than that of any other piecewise smooth timelike curve between the points.*

This proposition provides the solution for both the global and local problem. However in most of physically interesting situations the two points do not lie in a convex normal neighbourhood. A counterexample (which has misled some authors dealing with the twin paradox) is provided by the Schwarzschild spacetime generated by a static star or a static black hole of mass M. Take the standard coordinates (t, r, θ, ϕ) exhibiting that the spacetime is static and spherically symmetric. Let the twin A stay at rest at a point with the radial coordinate $r = r_0 > 3M$ (in natural units $G = 1 = c$) and the twin B moves on a timelike geodesic line with a circular orbit in the space at $r = r_0$. Initially the twins are at a spacetime point P_0 ($t = t_0, r = r_0$), then B flies away, makes a full circle around the centre and meets A again at P_1 ($t = t_1$, $r = r_0$). A simple calculation shows that the proper time s_A (i.e. the worldline length) of A between P_0 and P_1 is longer than the geodesic proper time s_B of B,

$$\frac{s_A}{s_B} = \left(\frac{r_0 - 2M}{r_0 - 3M}\right)^{1/2} > 1. \quad (1)$$

P_1 does not belong to a convex normal neighbourhood of P_0 and one finds that there is at least one timelike geodesic line joining P_0 and P_1 whose length is larger than that of B and longer than A.

Whenever the relevant neighbourhood is not convex normal and two points are connected by two or more timelike geodesics, the key notion is that of conjugate points and we recall the essential notions. Let γ be a timelike geodesic parametrized by its length s with tangent unit u^α (Greek letters denote spacetime indices). Any vector field $Z^\mu(s)$ which is a solution to the geodesic deviation equation (described below) on γ is called a Jacobi field on γ. A pair of points p and q on γ are said to be *conjugate* if there exists a Jacobi field Z^μ on γ which is not identically zero and $Z^\mu(p) = Z^\mu(q) = 0$ ([2] chapter 4). Points p and q are conjugate if an infinitesimally nearby geodesic intersects γ at both p and q. Existence of conjugate points on a geodesic shows that it is neither the unique geodesic nor the longest curve connecting its endpoints. More precisely, if a timelike geodesic γ joining points p_1 and p_2 has a point q conjugate to p_1 belonging to the segment $p_1 p_2$, then there exists a nearby timelike curve τ (not necessarily a geodesic one) with endpoints p_1 and p_2, which is longer than γ, $s(\tau) > s(\gamma)$. Conversely, if there is no conjugate points on a timelike geodesic, then it is the longest curve (in a bundle of close lines) between its endpoints, independently of whether the endpoints lie in some convex normal neighbourhood, or not. It is stated in

Theorem 2.2 (Theorem 4.5.8 in [2]). *A timelike geodesic γ has the maximal length from p_1 to p_2 if and only if there is no point conjugate to p_1 on the segment $p_1 p_2$.*

In anti–de Sitter space selected pairs of points may be connected by infinite number of timelike geodesics of the same length. Further, in this spacetime two given points may be joined by various timelike curves, but none of these is a geodesic and none of these attains maximal length. The longest curve always exists in globally hyperbolic spacetimes.

Theorem 2.3 (Theorem 9.4.4 in [3]). *Let $(M, g_{\alpha\beta})$ be a globally hyperbolic spacetime. Let p_1 and p_2 be connected by timelike curves. Then there exists a timelike geodesic γ from p_1 to p_2 having the maximal length.*

The existence of conjugate points is determined by (here $R_{\alpha\beta}$ is the Ricci tensor)

Theorem 2.4 (Theorem 4.4.2 in [2]). *If $R_{\alpha\beta}\, u^\alpha\, u^\beta \geq 0$ on a timelike geodesic γ and if the tidal force $R_{\mu\alpha\nu\beta}\, u^\alpha\, u^\beta \neq 0$ at some point p_0 on γ, there will be conjugate points p and q on γ, providing that the geodesic can be extended sufficiently far.*

Returning to the problem of the rate of ageing of the twins one sees that in a curved spacetime there is a great variety of motions and for a given pair of worldlines there is no generic criterion stating which of these is longer; one must explicitly compute their lengths.

For a given pair of spacetime points one applies physical arguments to find out a timelike geodesic connecting them and seeks for conjugate points on it to determine whether it may be locally the longest worldline. In physically interesting spacetimes the strong energy condition [2] holds implying $R_{\alpha\beta}\, u^\alpha\, u^\beta \geq 0$ and in most cases the tidal forces do not vanish, hence Theorem 2.4 indicates that the geodesic under consideration contains somewhere conjugate points. It is then crucial to establish whether the conjugate points belong to the relevant segment of the geodesic. To determine all conjugate points it is necessary to find out the generic Jacobi field on the segment.

3 Jacobi fields and conjugate points

We recall that a Jacobi field on a given timelike geodesic γ with a unit tangent vector field $u^\alpha(s)$ is any vector field $Z^\mu(s)$ being a solution of the geodesic deviation equation (GDE) on γ,

$$\frac{D^2}{ds^2}\, Z^\mu = R^\mu{}_{\alpha\beta\gamma}\, u^\alpha\, u^\beta\, Z^\gamma, \tag{2}$$

which is orthogonal to the geodesic, $Z^\mu\, u_\mu = 0$. Here D/ds is the absolute derivative with respect to the length s of a tensor field defined on γ. Geometrically Z^μ is a connecting vector joining γ to an infinitesimally close geodesic γ_ε given by $\bar{x}^\mu(s,\varepsilon) = x^\mu(s) + \varepsilon Z^\mu(s)$, where $x^\mu(s)$ are coordinates of points of γ and $|\varepsilon| \ll 1$. (In other terms $Z^\mu(s)$ joins the points of the two geodesics for the same value of s.) The GDE is derived in the linear approximation in ε. If $Z^\mu(0) = 0 = Z^\mu(s_0)$ for $s_0 \neq 0$ and Z^μ does not vanish identically, then it is said that γ_ε intersects γ at points $\gamma(0)$ and $\gamma(s_0)$. Actually γ_ε needs not to intersect γ at $\gamma(s_0)$ and $Z^\mu(s_0) = 0$ means that for $s = s_0$ the two geodesics are close of order higher than ε. A geodesic nearby to the given γ for not very small ε may be determined by expanding the difference between its coordinates and the coordinates of γ in a series of deviations,

$$\bar{x}^\mu(s,\varepsilon) = x^\mu(s) + \varepsilon Z^\mu(s) + \frac{1}{2!}\delta^2 x^\mu(s) + \ldots + \frac{1}{n!}\delta^n x^\mu(s) + \ldots,$$

where the $n-$th deviation $\delta^n x^\mu$ is of order ε^n and for $n > 1$ it is not a vector. These deviations give rise to higher order geodesic deviation equations. Yet in the search for locally maximal curves the equation (2) derived in the lowest approximation and the conjugate points determined by its solutions are fully sufficient. (It is also worth noticing that this equation also describes in a similar way the motion of nearby free test particles in spacetimes of any dimension $D > 4$ [4].) Due to the presence of the second absolute derivative D^2/ds^2 the GDE is very complicated and one can simplify it by removing this derivative and replacing it by the ordinary ones. To this end one expands Z^μ in a basis consisting of three spacelike orthonormal vector fields $e_a{}^\mu(s)$, $a = 1, 2, 3$ on γ, which are orthogonal to γ and are parallelly transported along the geodesic, i.e.

$$e_a{}^\mu\, e_{b\mu} = -\delta_{ab}, \qquad e_a{}^\mu\, u_\mu = 0, \qquad \frac{D}{ds} e_a{}^\mu = 0, \tag{3}$$

here δ_{ab} is the Kronecker symbol. (Since we are dealing with timelike curves it is convenient to apply the metric signature $+---$.) Then $Z^\mu = \sum_a Z_a e_a{}^\mu$ and the covariant vector equation (2) is reduced to three scalar second order ODEs for the scalar functions[1] $Z_a(s)$ (Jacobi scalars),

$$\frac{d^2}{ds^2}\, Z_a = -e_a{}^\mu\, R_{\mu\alpha\beta\gamma}\, u^\alpha\, u^\beta \sum_{b=1}^{3} Z_b\, e_b{}^\gamma. \tag{4}$$

A general Jacobi field depends on 6 integration constants appearing as a result of solving eqs. (4). Though simpler than eqs. (2), they are still intractable in a general spacetime and a hope to deal with them arises only if the metric admits some isometries. Any Killing vector field K^μ of the spacetime generates a first integral of eq. (1) of the form [5]

$$K_\mu \frac{D}{ds} Z^\mu - Z^\mu \frac{D}{ds} K_\mu = \text{const.} \tag{5}$$

1 The vector index of a Jacobi vector field will always be written as a superscript and the number of the Jacobi scalar — as a subscript.

One verifies by a direct calculation that the function on the LHS of (5) is constant along the given geodesic. The integral of motion may be recast in terms of the scalars Z_a. To this end one extends the vector triad $e_a{}^\mu(s)$ to a spacetime tetrad $e_A{}^\mu$, $A = 0, 1, 2, 3$, along γ by supplementing the triad by $e_0{}^\mu \equiv u^\mu$. The tetrad is orthonormal,

$$e_A{}^\mu \, e_{B\mu} = \eta_{AB} = \mathrm{diag}(1, -1, -1, -1) \tag{6}$$

and parallelly transported along γ. Expanding $K^\mu = \sum_{A=0}^{3} K_A \, e_A{}^\mu$ in the tetrad and Z^μ in the triad and inserting them into (5) one gets

$$\sum_{a=1}^{3} \left(Z_a \frac{dK_a}{ds} - \frac{dZ_a}{ds} K_a \right) = \text{const}, \tag{7}$$

where $K_a = -K^\mu \, e_{a\mu}$. If the spacetime admits n linearly independent Killing vector fields one gets n integrals of motion (7). Besides simplest spacetimes, such as the maximally symmetric ones, the first integrals (7) are essential in solving equations (4).

There are two approaches to finding the Jacobi vector fields. Bażański [6] gave a generic algorithm for solving the geodesic deviation equation in cases where one knows a complete integral of the Hamilton-Jacobi equation for timelike geodesics. In a subsequent work [7] the formalism was applied in Schwarzschild spacetime. This case shows that this elegant formalism is of restricted practical use: it does not apply to circular geodesics. If one wishes to apply the algorithm to a particular type of geodesic lines, e.g. radial ones, it is necessary to first find the general solution of the geodesic deviation equation and then carefully take appropriate limits in it to this type, what makes the procedure rather cumbersome. Furthermore, at least in the Schwarzschild metric, the algorithm works in the case of radial geodesics only for worldlines escaping to the spatial infinity, what excludes finite geodesics (reaching maximal height and then falling down). This is why our approach is closer to that of Fuchs, who directly solved the geodesic deviation equation in static spherically symmetric spacetimes [8]. A general formula for the Jacobi field is given in his work in terms of four integrals of expressions made up of Killing vectors and constants of motion they generate. It is our experience that employing this formula is not considerably simpler than solving the equation for radial geodesics from the very beginning. Also the Fuchs' formula does not apply to the circular geodesics and this case must be dealt with separately [9]. It is therefore more practical not to employ the Fuchs' integral solutions and solve the GDE independently in each case under study.

To summarize, the procedure is as follows.

i) Choose an interesting spacetime with some isometries (Killing vectors).
ii) Choose a geometrically interesting (and possibly simple, e.g. radial or circular) timelike geodesic γ explicitly given, $x^\alpha = x^\alpha(\tau)$, where τ is a scalar parameter. In some cases, e. g. in static spherically symmetric spacetimes τ is different from s on radial geodesics.
iii) Choose the spacelike triad $e_a{}^\mu(s)$ on γ with the properties (3). It is clear that the triad is not uniquely determined by eqs. (3) and should be properly chosen as to render the equations (4) as simple as possible.
iv) Solve the GDE (4) applying the first integrals and find a generic solution $Z_a(\tau)$. If $\tau \neq s$ one must appropriately transform the LHS of (4).
v) Consider all possible special solutions with $Z_a(0) = 0$ and seek for their zeros, $Z_a(\tau_0) = 0$ for $\tau_0 > 0$.

Then the geodesic γ with $x^\alpha = x^\alpha(\tau)$ is *uniquely locally maximal* on the segment $0 \le \tau < \tau_0$ and is non–uniquely locally maximal on the segment $0 \le \tau \le \tau_0$. For some $\tau_1 > \tau_0$, there is a timelike curve (not necessarily geodesic) joining the points $\gamma(0)$ and $\gamma(\tau_1)$ which is longer than γ.

This is an algorithmic and effective procedure for checking whether the given geodesic is the unique locally longest curve between its fixed endpoints. We emphasize that the procedure is algorithmic in the sense that one has to do a finite number of definite steps culminating with solving GDE and it is effective providing that one is capable to solve the concrete GDE. Clearly solving this equation is not an algorithmic process and limitations in finding out the solution are the main obstacle in determining locally maximal curves.

4 The global problem of maximal curves

The difference between the global and local maximum of length of a timelike curve is essential both conceptually and in practice, i.e. in our ability to computationally establish a maximal curve.

In the case of globally maximal length one takes into account *all* timelike curves connecting p and q in the spacetime (actually the rigorous definitions and theorems require to take all the future directed nonspacelike curves from p to q; for our purposes it is usually sufficient to include only future directed timelike curves). Our approach consists in presenting excerpts from the unique monograph on global Lorentzian geometry [10]. Let $\Omega_{p,q}$ denote the path space of all future directed timelike piecewise smooth curves between p and q; each curve λ has then the well defined length $s(\lambda) > 0$. Here the key notion is that of *Lorentzian distance function* $d(p,q)$ of any two points. It is defined as follows ([10], Chap. 4). If q does not lie in the causal future $J^+(p)$ of p, then $d(p,q) = 0$ and if q is in $J^+(p)$, then $d(p,q) \equiv \sup\{s(\lambda) : \lambda \in \Omega_{p,q}\}$. The distance is nonzero, $d(p,q) > 0$, if and only if q is in the chronological future $I^+(p)$ of p. The distance function is nonsymmetric, $d(p,q) \neq d(q,p)$ and if $0 < d(p,q) < \infty$, then $d(q,p) = 0$; in some spacetimes, e.g. Reissner-Nordström one, there are points such that $d(p,q) = \infty$ and in totally vicious spacetimes there is $d(p,p) = \infty$ for all p. The curve $\lambda \in \Omega_{p,q}$ is said to be *globally maximal* (or shortly *maximal*) if it is the longest one in the set $\Omega_{p,q}$, i.e. if $s(\lambda) = d(p,q)$. The maximal curve (usually non unique) is always a timelike geodesic (Theorem 4.13 of [10]). The definition does not imply that in an arbitrary spacetime the maximal geodesic does exist between any chronologically related points, as the counterexample of anti–de Sitter spacetime shows. Yet in globally hyperbolic spacetimes for any pair of chronologically related points p and q ($p \prec\prec q$) there is a maximal future directed geodesic segment $\gamma \in \Omega_{p,q}$ with $s(\gamma) = d(p,q)$ (Theorem 6.1 in [10]); usually it is not unique.

If a timelike geodesic is complete (it is defined for all real values of the canonical length parameter, $-\infty < s < +\infty$), it usually is not maximal beyond some segment from p to q. A Riemannian example: a great circle arc on a sphere emanating from the north pole is maximal (in this case „maximal" means „globally the shortest") on the half-circle up to the south pole since points on the arc lying beyond this segment may be connected to the north pole by a shorter geodesic. This gives rise to the notion of the cut point on a geodesic. Let $\gamma : [0,a) \to M$ be a future directed, future inextendible, timelike geodesic parameterized by its length s in a spacetime (M,g). Set

$$s_0 \equiv \sup\{s \in [0,a) : d(\gamma(0), \gamma(s)) = s\}.$$

If $0 < s_0 < a$, then $\gamma(s_0)$ is said to be the *future timelike cut point* of $\gamma(0)$ along γ. For all $0 < s < s_0$ the geodesic γ is the unique globally maximal timelike curve from $\gamma(0)$ to $\gamma(s)$ and is globally maximal (not necessarily unique) on the segment from $\gamma(0)$ to $\gamma(s_0)$, while for $s_1 > s_0$ there exists a future directed timelike curve σ from $\gamma(0)$ to $\gamma(s_1)$ with $s(\sigma) > s(\gamma)$. In other terms s_0 is the length of the longest maximal segment of the given geodesic (for a fixed initial point).

Theorem 4.1 (Theorem 9.10 in [10])**.** *A timelike geodesic is not maximal beyond the first conjugate point, or equivalently: the future cut point of $p = \gamma(0)$ along γ comes no later than the first future conjugate point to p.*

A closer connection between conjugate and cut points is revealed in

Theorem 4.2 (Theorem 9.12 in [10])**.** *Let (M,g) be globally hyperbolic. If $q = \gamma(s_0)$ is the future cut point of $p = \gamma(0)$ along the timelike geodesic γ from p to q, then either one or possibly both of the following hold:*
i) *the point q is the first future conjugate point to p;*
ii) *there exist at least two future directed maximal timelike geodesic segments from p to q.*

Now consider the set of *all* future directed timelike geodesics emanating from any point p. In general each of them has the cut point. The *future timelike cut locus* $C_t^+(p)$ of p in (M,g) is defined to be the set of cut points along all future directed timelike geodesic segments issuing from p.

One may ask whether the cut locus contains a point q which is the closest one to p, i.e. $d(p,q) \leq d(p,r)$ for all $r \in C_t^+(p)$. It turns out that

Theorem 4.3 (Theorem 9.24 in [10]). *If a point p in a globally hyperbolic spacetime has a closest cut point q, then q must be a point conjugate to p on a geodesic.*

In a noncompact complete Riemannian manifold at each point there is a direction (a tangent vector) such that the geodesic emanating from this point in this direction has no cut points. Something analogous occurs in specific spacetimes.

Theorem 4.4 (Theorem 9.23 in [10]).
- i) *In a strongly causal (M, g) at each point there is a future directed nonspacelike direction such that the geodesic issuing in this direction has no cut point.*
- ii) *In a globally hyperbolic spacetime given any point p, there is no farthest nonspacelike cut point of p.*

Some Riemannian manifolds are distinguished by satisfying the topological condition of being simply connected. For Lorentzian manifolds one introduces an analogous notion of a spacetime being *future one-connected* if for all pairs of chronologically related points, $p \prec\prec q$, any two future directed timelike curves from p to q are homotopic through smooth future directed timelike curves with fixed endpoints p and q. An example (R. Geroch, quoted in [10][2]) shows that the topological simple connectedness does not imply that the spacetime is one-connected.

Finally one deals with properties of Jacobi vector fields on a geodesic. Let $J_t(\gamma)$ denote the vector space of smooth Jacobi vector fields $Z^\mu(s)$ along the timelike geodesic $\gamma : [a, b] \to M$ with $Z^\mu(a) = Z^\mu(t) = 0$ for some $a < t \leq b$. Then the *order* of the conjugate point $\gamma(t)$ to p on the timelike geodesic γ with $\gamma(a) = p$ is defined as $\dim J_t(\gamma)$. Applying these two notions two following theorems were proved.

Theorem 4.5 (Theorem 10.30 in [10]). *Let (M, g) be future one-connected and globally hyperbolic. Suppose that for some p in M the first future conjugate point on every timelike geodesic emanating from p is of order two or greater. Then the future timelike cut locus of p and the locus of first future timelike conjugate points to p coincide. Equivalently: all future timelike geodesics from p are maximal up to the first future conjugate point.*

Theorem 4.6 (Theorem 11.16 in [10]). *Let (M, g) be a future one-connected globally hyperbolic spacetime with no future nonspacelike conjugate points. Then given any $p, q \in M$ with $p \prec\prec q$, there is exactly one future directed timelike geodesic from p to q (and is clearly maximal).*

These six theorems express our basic current knowledge about maximal timelike geodesics in various spacetimes. These are mathematical „existence theorems" stating the presence of some global properties if some global conditions are satisfied. They are not „constructive" in the sense that they do not indicate a computationally effective procedure for obtaining the interesting object, as is seen in the two most important cases. Firstly, given two chronologically related points p and q, one may indicate a geodesic connecting them and its segment is maximal if and only if the cut point of p is at q or farther. However, the location of the cut point cannot be found by investigating solely this geodesic. One may instead find by geometrical and/or physical arguments a number (or a continuous narrow class) of geodesics joining p and q and by the direct computation get the longest curve (one or more), that free of conjugate points on the segment. In this way one determines the locally longest curve and nothing more, even if the set under consideration contains curves distant from this one. In fact, the absence of conjugate points on the locally longest segment does not imply that it does not contain cut points and the distant curves belonging to the set may not include the maximal geodesic. There is no general way out of the problem and in the search for the maximal curve from p to q one must deal with the whole space $\Omega_{p,q}$ and the space cannot be examined in a finite number of steps.

Secondly, given point p, one may ask of which timelike geodesic emanating from p contains the longest maximal segment, i.e. which cut point $q \in C_t^+(p)$ is farthest from p. In an arbitrary spacetime in this problem again there is no shortcut and one must study all geodesics emanating from the point.

2 It is hard to find the original reference.

In conclusion, the difficulty lies in that there is no analytic tool, such as a differential equation, allowing one to find the cut point on the given geodesic in a finite number of steps and this is due to the very nonlocal nature of the notion. Quite the opposite, one should first study all geodesics in the space $\Omega_{p,q}$, compute their lengths, find points where they intersect and in this way determine their cut points. Then Theorems 4.2 and 4.6 in the first problem and Theorems 4.3, 4.4 and 4.5 in the second problem (given the initial point) will turn out to be a compact and geometrically elegant description of the results of all the computations. Without this huge work being done, the theorems are practically useless for any quantitative problem, e.g. the twin paradox.

All this sounds rather hopeless. Fortunately, if one restricts the research to spacetimes which not only are both globally hyperbolic and future one-connected, but also have some high isometries, the problem of globally maximal worldlines becomes more tractable. Our experience up to now shows that spherical symmetry is useful. Another approach is based on the use of the Gaussian normal geodesic (GNG) coordinates wherein timelike geodesics may be expressed in a very simple form. By joining the spherical symmetry to the use of the GNG coordinates one gets an effective tool in the research.

5 A few words on results

There is no space in this article to present methods of dealing with the geodesic deviation equation in the local problem and how the spherical symmetry works in dealing with the global problem. The reader interested in these technicalities is referred to our papers [11–14] containing also many references. These methods turn out relatively effective, at least in the case of high symmetry. Here we give a brief description of the most interesting results.

de Sitter space

This is maximally symmetric vacuum solution of the Einstein equations for a positive cosmological constant, $\Lambda = +3H^2$, where H has dimension (length)$^{-1}$ and $1/H$ determines the length scale of this spacetime. The first assumption of Theorem 2.4 does not hold since $R_{\alpha\beta}\, u^\alpha u^\beta = R/4 = -3H^2 < 0$, hence timelike geodesics have no conjugate points. Also null geodesics are free of conjugate points. Then all the assumptions of Theorem 4.6 hold and timelike geodesics are free of cut points, are unique geodesics between their endpoints and are globally maximal. This result may also be easily derived analytically. In fact, eq. (4) for the Jacobi scalars is reduced to

$$\frac{d^2 Z_a}{ds^2} - H^2 Z_a = 0 \tag{8}$$

and the general deviation vector field vanishing at $s = 0$ is

$$Z^\mu(s) = \sum_{a=1}^{3} C_a\, e_a^\mu(s) \sinh Hs. \tag{9}$$

Clearly neither conjugate nor cut points exist since the neighbouring geodesics exponentially diverge. Physically this means that the gravitation in this spacetime is repulsive.

anti–de Sitter (AdS) space

This is maximally symmetric vacuum solution for a negative cosmological constant, $\Lambda = -3/a^2$, where a has dimension of length and determines its length scale. The assumptions of Theorem 2.4 do hold, in particular $R_{\alpha\beta}\, u^\alpha u^\beta = 3/a^2 > 0$, and imply that each timelike geodesic (sufficiently extended) contains conjugate points and (by Theorem 4.1) cut points. Also in this spacetime the GDE (4) is universal (is independent of the geodesic and of the choice of the vector triad along it) and reads

$$\frac{d^2 Z_b}{ds^2} + \frac{1}{a^2} Z_b = 0. \tag{10}$$

The general deviation field vanishing for $s = 0$ is

$$Z^{\mu}(s) = \sum_{b=1}^{3} C_b \, e_b^{\mu}(s) \sin \frac{s}{a} \tag{11}$$

showing that each geodesic has an infinite sequence of conjugate points at $s_n = n\pi a$, $n = 1, 2, \ldots$ .This universal regularity is quite understandable if one takes into account that all timelike geodesics in this spacetime are of the same geometric type. The AdS space is embedded as a 4–dimensional pseudosphere in an auxiliary unphysical five–dimensional space with flat ultrahyperbolic metric, then each timelike geodesic of AdS forms in this ambient space a circle of radius a [15].

Static spherically symmetric (SSS) spacetimes

We assume the standard form of the SSS metric,

$$ds^2 = e^{\nu(r)} \, dt^2 - e^{\lambda(r)} \, dr^2 - r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \tag{12}$$

with arbitrary real functions $\nu(r)$ and $\lambda(r)$. We assume that $\nu(r)$ is monotonic in the interval where it is real. If e^{ν} is decreasing, the gravitation is repulsive (e. g. de Sitter metric), whereas for e^{ν} increasing the gravitational forces are attractive (e. g. Schwarzschild space). The GDE is tractable only for radial and circular geodesics. In the case of radial geodesics the equation consists of three separated equations for Z_a supplemented by three separated first integrals. The equations depend on ν and λ and their derivatives. They were solved in the Schwarzschild and Reissner–Nordström metrics (in the latter case for the mass parameter exceeding the charge, $M^2 > Q^2$). The solution assisted by the use of the GNG coordinates shows that the radial timelike geodesics in the region outside the outer horizon of R–N spacetime have neither conjugate nor cut points, what means that they are globally maximal curves between their points.

A circular timelike geodesic exists for $r = r_0 = \text{const} > 0$ iff $r_0\nu'(r_0) < 2$ and $\nu'(r_0) > 0$, what implies that $g_{00} = e^{\nu}$ is increasing; physically it is clear that circular orbits may exist if gravitation is attractive. The three geodesic deviation equations for circular orbits are universal for all SSS spacetimes admitting them, only numerical coefficients for the given r_0 depend on ν and λ and the same holds for three integrals of motion. This universality of the equations implies universality (up to the values of the constants) of location of conjugate points. In general each circular timelike geodesic contains three infinite sequences of points conjugate to any initial one (in AdS space the three sequences coincide). The existence of one sequence of conjugate points is geometrically obvious and arises by rotations of the 2–surface containing the orbit. Yet the existence of the other two sequences is rather unexpected and shows that even in the case of Schwarzschild manifold, which has been investigated in great detail for almost one century, one can find, employing the method, interesting new geometric results.

Ultrastatic spherically symmetric (USSS) spacetimes

These spacetimes form a special subclass of SSS manifolds for which the unique timelike Killing vector field is a gradient. In the standard coordinates giving rise to the metric (12) this property implies $\nu(r) = 0$ and these coordinates coincide with the GNG ones. The most famous USSS spacetime is the static Einstein universe with $\lambda = -\ln(1 - r^2/a^2)$, where the dimensional constant $a > 0$ and $r \geq 0$; this manifold is unstable. None of other USSS spacetimes is physically significant, nonetheless they are geometrically interesting. Further, they are conceptually important in that they dynamically imitate the inertial frame of SR [13]. This shows that contrary to a widely spread opinion, found in most textbooks on analytic mechanics (and also on SR), the inertial reference frame should not be defined in terms of Newton's first and second law of motion. The notion of the inertial frame makes sense only in Minkowski spacetime and is defined as the frame exhibiting the affine structure of the manifold. In USSS spacetimes timelike geodesics with circular orbits do not exist since $r = \text{const}$ implies the angle $\phi = \text{const}$ and the particle stands still in the space. General timelike geodesics are expressed in terms of some integrals and the equations for the Jacobi scalars are, at least to some extent, solvable.

6 Conclusions

Our experience, acquired by studying a number of possibly simple and symmetric Lorentzian manifolds, clearly indicates that the geodesic structure of most spacetimes is richer and more diverse than it might be expected, e. g. the three maximally symmetric spacetimes are different. This shows that other physically relevant spacetimes are worth investigating in this aspect. The methods of this investigation, presented in this article, in spite of their obvious limitations, are sufficiently effective to support the hope that future work will exhibit many unexpected geometrical features of these spacetimes. At present, the variety of already discovered features prevents us from formulating any general qualitative rule concerning the geodesic structure of spacetimes in general relativity.

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References

[1] Stephani H., Relativity, an introduction to special and general relativity, third edition, par. 3.4, Cambridge University Press, Cambridge, 2004
[2] Hawking S. W., Ellis G. F. R., The large scale structure of space-time, Cambridge University Press, Cambridge, 1973
[3] Wald R. M., General relativity, University of Chicago Press, Chicago, 1984
[4] Podolsky J., Švarc R., Interpreting spacetimes of any dimension using geodesic deviation, Phys. Rev. D85:044057, 2012 [arXiv:1201.4790v2 [gr-qc]]
[5] Fuchs H., Solutions of the equations of geodesic deviation for static spherical symmetric space-times, Ann. d. Physik, 1983, 40, 231-233
[6] Bażański S. L., Hamilton-Jacobi formalism for geodesics and geodesic deviations, J. Math. Phys., 1989, 30, 1018-1029
[7] Bażański S. L., Jaranowski P., Geodesic deviation in the Schwarzschild spacetime, J. Math. Phys., 1989, 30, 1794-1803
[8] Fuchs H., Parallel transport and geodesic deviation in static spherically symmetric space-times, Astron. Nachr., 1990, 311, 219-222
[9] Fuchs H., Deviation of circular geodesics in static spherically symmetric space-times, Astron. Nachr., 1990, 311, 271-276
[10] Beem J. K., Ehrlich P. E., Easley K. L., Global Lorentzian geometry, second edition. Marcel Dekker, New York, 1996
[11] Sokołowski L. M., On the twin paradox in static spacetimes: I. Schwarzschild metric, Gen. Rel. Grav., 2012, 44, 1267-1283 [arXiv:1203.0748 [gr-qc]]
[12] Sokołowski L. M., Golda Z. A., The local and global geometrical aspects of the twin paradox in static spacetimes: I. Three spherically symmetric spacetimes, Acta Phys. Polon. B, 2014, 45, 1051-1075 [arXiv:1402.6511v2 [gr-qc]]
[13] Sokołowski L. M., Golda Z. A., The local and global geometrical aspects of the twin paradox in static spacetimes: II. Reissner-Nordström and ultrastatic metrics, Acta Phys. Polon. B, 2014, 45, 1713-1741 [arXiv:1404.5808 [gr-qc]]
[14] Sokołowski L. M., Golda Z. A., Jacobi fields and conjugate points on timelike geodesics in special spacetimes, Acta Phys. Polon. B, 2015, 46, 773-787
[15] Sokołowski L. M., Golda Z. A., Every timelike geodesic in anti-de Sitter spacetime is a circle of the same radius, Internat. J. Mod. Phys. D, 2016, 25, 1650007 (6 pgs) [arXiv:1602.07111 [gr-qc]]