Demonstr. Math. 2017; 50: 66–71 **DE GRUYTER** OPEN

Demonstratio Mathematica **Open Access**

Research Article

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On the Gauss-Bonnet for the quasi-Dirac operators on the sphere

DOI 10.1515/dema-2017-0007
Received May 17, 2016; accepted July 27, 2016.

Abstract: We investigate examples of Gauss-Bonnet theorem and the scalar curvature for the two-dimensional commutative sphere with quasi-spectral triples obtained by modifying the order-one condition.

Keywords: Gauss-Bonnet, Spectral triples

MSC: 58B34, 46L87, 34L40

Dedicated to Professor Michał Heller on the occasion of His 80th Birthday.

1 Introduction

The noncommutative analogue of the Gauss-Bonnet theorem has drawn attention in recent years [1–5]. Motivated by the explicit computations of the zeta function of the conformally rescaled Laplace operator over the noncommutative torus it became an interesting question of topological invariants and the explicit local formulae for the geometric properties of noncommutative spectral triples.

In a short note [6] we investigated what happens if one relaxes one of the axioms of the spectral triple construction by allowing the order-one condition to be satisfied up to compact operators. As a result we obtained a family of new classical spectral triples, which albeit not being exactly *classical spin geometries* did satisfy most of the axioms of spectral triples [7]. The work was partially motivated by an attempt to make sense of the classical limits of spectral geometries obtained for q-deformed spheres in [8] but could be seen as interesting in its own.

The interesting part is that the constructed Dirac operators represent different classes in K-homology when compared to the standard Dirac operator, yet they do not arise from twisted spinor bundles and twisted Dirac operators. A natural question then is whether the Gauss-Bonnet theorem holds for them as well. This note aims to answer this question for the spheres (which is the only 2-dimensional example explicitly studied, apart from the example of the two-dimensional torus).

2 Quasi-Dirac operators on the sphere

We briefly review here the results of [6], beginning with the equivariant representations and then the construction of Dirac-type operators.

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Let V_l, $l = 0, \frac{1}{2}, 1, \dots$ denote the $2l+1$-dimensional representation of the Lie algebra $\mathfrak{su}(2)$. The orthonormal basis of each V_l shall be denoted by $|l,m\rangle$, $m = -l, -l+1, \dots, l-1, l$. We have:

Proposition 2.1. *For each $N = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots$ there exists an irreducible unitary representation π_N of $\mathcal{A}(S^2)$ on the Hilbert space $\mathcal{H}_N$, which is the completion of $\mathcal{V}_N$:*

$$\mathcal{V}_N = \bigoplus_{l=|N|}^{\infty} V_l,$$

which is equivariant on $\mathcal{V}_N$ with respect to $\mathfrak{su}(2)$ action. The explicit form for π_N on the orthonormal basis $|l,m\rangle \in V_l$, is:

$$\begin{aligned}\pi_N(B)|l,m\rangle = &\sqrt{(l+m+1)(l+m+2)}\,\alpha_l^+\,|l+1,m+1\rangle\\ &+\sqrt{(l+m+1)(l-m)}\,\alpha_l^0\,|l,m+1\rangle\\ &-\sqrt{(l-m)(l-m-1)}\,\alpha_{l-1}^+\,|l-1,m+1\rangle,\end{aligned}\tag{1}$$

$$\begin{aligned}\pi_N(B^*)|l,m\rangle = &-\sqrt{(l-m+2)(l-m+1)}\,\alpha_l^+\,|l+1,m-1\rangle\\ &+\sqrt{(l+m)(l-m+1)}\,\alpha_l^0\,|l,m-1\rangle\\ &+\sqrt{(l+m)(l+m-1)}\,\alpha_{l-1}^+\,|l-1,m-1\rangle,\end{aligned}\tag{2}$$

$$\begin{aligned}\pi_N(A)|l,m\rangle = &-\sqrt{(l-m+1)(l+m+1)}\,\alpha_l^+\,|l+1,m\rangle\\ &+m\,\alpha_l^0\,|l,m\rangle\\ &-\sqrt{(l-m)(l+m)}\,\alpha_{l-1}^+\,|l-1,m\rangle.\end{aligned}\tag{3}$$

where α_l^+, α_l^0 are

$$\begin{aligned}\alpha_l^0 &= \frac{N}{l(l+1)},\\ \alpha_l^+ &= \sqrt{1-\frac{N^2}{(l+1)^2}}\,\frac{1}{\sqrt{(2l+1)(2l+3)}}\end{aligned}\tag{4}$$

2.1 Equivariant spectral triple

The construction of the equivariant spectral triple followed the procedure of [9], we assumed that there the $\mathbb{Z}_2$-grading γ of the Hilbert space and the reality operator J, which maps the algebra to the commutant, are:

$$\gamma|l,m,\pm\rangle = \pm|l,m,\pm\rangle,\tag{5}$$

$$J|l,m,\pm\rangle = i^{2m}|l,-m,\mp\rangle.\tag{6}$$

then, γ commutes with the representation of the algebra and, if the Hilbert space is $\mathcal{H} = \mathcal{H}_N \oplus \mathcal{H}_{-N}$, $N \geq 0$, with the diagonal representation of $\mathcal{A}(S^2)$, then the commutant condition is satisfied:

$$J\pi(x)J^{-1} = \pi(x^*), \quad \forall x \in \mathcal{A}(S^2).\tag{7}$$

Only for a half-integer N we have the signs of a two-dimensional spectral geometry, whereas for an integer value of N, we have the sign relations corresponding formally to a six-dimensional (mod 8) real structure. For $N = \frac{1}{2}$ we recover the classical spinor bundle over the sphere.

2.2 The Equivariant Dirac operator

Proposition 2.2. *The family of quasi-Dirac operators and spectral triples, where the order one condition is satisfied up to compact operators is given by $(\mathcal{A}(S^2), \mathcal{H}_N \oplus \mathcal{H}_{-N}, J, \gamma, D_N)$ where:*

$$D_N|l,m,\pm\rangle = \alpha(l+\beta)|l,m,\mp\rangle.\tag{8}$$

Only if $N = \frac{1}{2}$ the order-one condition is satisfied exactly, otherwise it is satisfied up to compact operators $\mathcal{K}(\mathcal{H})$:

$$[[D_N, \pi(x)], \pi(y)] \in \mathcal{K}(\mathcal{H}), \ \forall x, y \in \mathcal{A}(S^2). \tag{9}$$

For different N the K-homology class of the resulting Fredholm module are different and the pairing with the standard K-theory class of the line bundle over the sphere is $2N$.

For the proof and further details see [6].

3 The zeta function of the quasi-Dirac operator

We shall analyse here the zeta function of the quasi-Dirac operators on defined in the previous section.

Let us recall:

$$\zeta_{D_N}(s) = \mathrm{Tr}\, |D_N|^{-s} = \mathrm{Tr}\, (D_N^2)^{-\frac{s}{2}},$$

Taking the spectrum of the Dirac operator from the formula (8) we have:

$$\zeta_{D_N}(s) = 2 \sum_{l=N}^{\infty} (2l+1)\, (\alpha(l+\beta))^{-s},$$

where we have already counted the multiplicities.

We further compute,

$$\zeta_{D_N}(s) = 2 \sum_{l=N}^{\infty} (2l+1)\, (\alpha(l+\beta))^{-s} = 4\alpha^{-s} \sum_{l=N}^{\infty} (l+\beta)^{-s+1} + 2\alpha^{-s}(1-2\beta) \sum_{l=N}^{\infty} (l+\beta)^{-s}.$$

Using the Hurwitz zeta function for $a > 0$

$$\zeta_H(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s},$$

we obtain the following result.

Proposition 3.1. *The zeta function for the equivariant quasi-Dirac operator with $N \geq \frac{1}{2}$ is:*

$$\zeta_{D_N}(s) = \begin{cases} 4\alpha^{-s}\zeta_H(s-1,\beta) & \\ \quad + 2\alpha^{-s}(1-2\beta)\zeta_H(s,\beta) - F_N(s), & N \in \mathbb{N} \\ 4\alpha^{-s}\zeta_H(s-1,\beta+\frac{1}{2}) & \\ \quad + 2\alpha^{-s}(1-2\beta)\zeta_H(s,\beta+\frac{1}{2}) - G_N(s), & N \in \mathbb{N}+\frac{1}{2} \end{cases} \tag{10}$$

where the functions F_N and G_N are both analytic:

$$F_N(s) = \sum_{k=0}^{N-1} 2\alpha^{-s} \frac{2k+1}{(k+\beta)^s},$$

and

$$G_N(s) = \sum_{k=0}^{N-\frac{3}{2}} 2\alpha^{-s} \frac{2k+2}{(k+\beta+\frac{1}{2})^s}.$$

We have now an immediate corollary:

Corollary 3.2. *The volume of each of the spherical geometries set by the operator D_N is:*

$$\mathit{Vol}(S^2)_{D_N} = 4\pi\alpha^{-2}.$$

This follows directly from the formula linking the heat kernel coefficient of D^2 with the Wodzicki residue of the square of the Dirac operator. Since:

$$\mathrm{Res}_{s=2}\,\zeta_{D_N}(s) = \mathrm{Wres} D^{-2} = \frac{1}{\pi}\mathrm{Vol}(S^2)_{D_N},$$

and the Hurwitz zeta function has only one simple pole at $s = 1$ with the residue:

$$\mathrm{Res}_{s=1}\,\zeta_H(s,a) = 1,$$

then using (10) we have

$$\mathrm{Res}_{s=2}\,\zeta_{D_N}(s) = 4\alpha^{-2},$$

hence

$$\mathrm{Vol}(S^2)_{D_N} = 4\pi\alpha^{-2}.$$

3.1 The zeta function at $s = 0$

As a next step we compute the value of the zeta function at $s = 0$, which, in the usual case gives an invariant proportional to the Euler characteristic.

First we need to state the following facts about the Hurwitz zeta function:

Lemma 3.3.

$$\zeta_H(-1,a) = -\frac{1}{2}B_2(a), \qquad \zeta_H(0,a) = \frac{1}{2} - a.$$

where $B_2(x) = x^2 - x + \frac{1}{6}$ is the Bernoulli polynomial.

Using this we obtain:

Proposition 3.4. *We have:*

$$\zeta_{D_N}(0) = 2(\beta - \tfrac{1}{2})^2 + \frac{1}{6} - 2N^2, \qquad N = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$$

Proof. We compute for $N = 1, 2, 3, \ldots$,

$$\begin{aligned}\zeta_{D_N}(0) &= 4\zeta_H(-1,\beta) + 2\,(1-2\beta)\zeta(0,\beta) - F_N(0)\\ &= -2(\beta^2 - \beta + \frac{1}{6}) + 2\,(1-2\beta)(\frac{1}{2} - \beta) - 2N^2\\ &= 2(\beta - \tfrac{1}{2})^2 + \frac{1}{6} - 2N^2.\end{aligned}$$

For the half-integer N,

$$\begin{aligned}\zeta_{D_N}(0) &= 4\zeta_H(-1,\beta+\tfrac{1}{2}) + 2(1-2\beta)\zeta(0,\beta+\tfrac{1}{2}) - G_N(0)\\ &= -2\left((\beta+\tfrac{1}{2})^2 - (\beta+\tfrac{1}{2}) + \frac{1}{6}\right) + 2(1-2\beta)\left(\frac{1}{2} - (\beta+\tfrac{1}{2})\right) - 2(N^2 - \frac{1}{2})\\ &= 2(\beta - \tfrac{1}{2})^2 + \frac{1}{6} - 2N^2.\end{aligned}$$

□

Remark 3.5. *The value of the ζ function of the quasi-Dirac operator depends continuously on the free parameter β and reaches minimum for $\beta = \frac{1}{2}$. For the minimum the value of $\zeta_{D_N}(0)$ depends on N (that is on the K-homology class of the quasi-Dirac operator).*

In fact, β has the interpretation of a simple bounded perturbation of the Dirac operator, which is similar to torsion terms in higher dimensions [10]. Since the value of the zeta function corresponds to the scale invariant part of the action, we again see that this part of the spectral action is minimal for a particular value of β, which corresponds to the *minimal* Dirac operator for a fixed metric.

In fact, the perturbation of the Dirac is of the type:

$$D_\beta = D_0 + \beta \operatorname{sign}(D_0).$$

It is an interesting coincidence that for $N = \frac{1}{2}$ the value of β determines exactly the Dirac operator on the sphere, which satisfies exactly the order-one condition.

Observe also that for $N = \frac{1}{2}$ we have,

Remark 3.6. *The value of the zeta function $\zeta_{D_{\frac{1}{2}}}(0)$ is,*

$$\zeta_{D_{\frac{1}{2}}}(0) = -\frac{1}{3},$$

however, using the relation between the zeta function and heat-trace coefficients:

$$\zeta_{D_{\frac{1}{2}}}(0) = a_2(D_{\frac{1}{2}}^2),$$

and knowing the local expression for a_2 (that already takes into account the Lichnerowicz formula and the dimension of the fibre of spinor bundle),

$$a_2(D_{\frac{1}{2}}^2) = -\frac{1}{24\pi} \int_{S^2} R(g) dv_g.$$

Using now the Gauss-Bonnet theorem that states,

$$\int_M R(g) dv_g = 4\pi \chi(M),$$

we can read out the Euler characteristic for the sphere:

$$\chi(S^2) = -\frac{1}{4\pi} (24\pi)\, \zeta_{D_{\frac{1}{2}}}(0) = 2.$$

3.2 $N = 0$ quasi-Dirac operator as a pseudodifferential operator.

In this section we shall concentrate on the simplest example of the quasi-Dirac operator for $N = 0$ and discuss whether one can write D_0 as a pseudodifferential operator.

Proposition 3.7. *Let Δ be the usual Laplace operator on the two-dimensional sphere S^2, which in the spherical coordinates ϕ, θ is:*

$$\Delta = \frac{1}{\sin(\theta)} \partial_\theta \left(\sin(\theta) \partial_\theta\right) + \frac{1}{\sin^2(\theta)} \partial_\phi^2.$$

Then the spectral triple for $N = 0$ is of the form:

$$\mathcal{H} = \mathcal{A}(S^2) \otimes \mathbb{C}^2, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_0 = \begin{pmatrix} 0 & d_0 \\ d_0 & 0 \end{pmatrix},$$

where

$$d_0 = \sqrt{-\Delta + \frac{1}{4}},$$

Proof. First of all, observe that all algebraic relations of the spectral triple are satisfied and that the Dirac operator is equivariant with respect to the symmetry of the spinor bundle. Then, identifying the basis of the Hilbert space of square summable sections with the spherical harmonics:

$$|l, m, \pm\rangle = Y_{lm}(\theta, \phi)^{\pm}, l = 0, 1, 2, \ldots, m = -l, -l+1, \ldots, l,$$

where $\pm$ denotes the eigenvalue of γ, we have:

$$d_0 Y_{lm}(\theta,\phi) = \sqrt{\left(l(l+1)+\frac{1}{4}\right)} Y_{lm}(\theta,\phi) = (l+\tfrac{1}{2}) Y_{lm}(\theta,\phi). \qquad \square$$

So, although the quasi-Dirac operator is not a differential operator, its square is. It is easy to verify that the zeta function of the quasi-Dirac operator for $N=0$, which we have found above yields:

$$\zeta_{D_0}(0) = \frac{1}{6}.$$

4 Conclusions

Having demonstrated earlier that within the framework of noncommutative geometry it is possible to construct nonequivalent spectral geometries on the two-dimensional sphere, we have posed a question whether the zeta function establishes a valid topological invariant. First of all, we have shown that the freedom in the choice of the quasi-Dirac operators on the sphere even within the same class in K-homology results in an expression, which depends in a continuous way on one parameter. Therefore, one can discuss any possible invariant only at the minimum of the obtained functional.

Moreover, the expression also depends strongly on the choice of the spinor bundle and hence the K-homology class of the chosen spectral triple and for this reason the value of the zeta function is more related to the geometry than to the topology of the underlying algebra.

Acknowledgement: Research partially supported by NCN grant 2012/06/M/ST1/00169.

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