Demonstr. Math. 2017; 50:94–99 DE GRUYTER OPEN

Demonstratio Mathematica **Open Access**

Research Article

Raymond Mortini* and Rudolf Rupp

The algebras of bounded and essentially bounded Lebesgue measurable functions

DOI 10.1515/dema-2017-0010
Received October 19, 2015; accepted February 10, 2016

Abstract: Let X be a set in $\mathbb{R}^n$ with positive Lebesgue measure. It is well known that the spectrum of the algebra $L^\infty(X)$ of (equivalence classes) of *essentially bounded*, complex-valued, measurable functions on X is an extremely disconnected compact Hausdorff space. We show, by elementary methods, that the spectrum M of the algebra $\mathscr{L}_b(X, \mathbb{C})$ of all *bounded* measurable functions on X is not extremely disconnected, though totally disconnected. Let $\Delta = \{\delta_x : x \in X\}$ be the set of point evaluations and let $\mathfrak{g}$ be the Gelfand topology on M. Then $(\Delta, \mathfrak{g})$ is homeomorphic to $(X, \mathcal{T}_{dis})$, where $\mathcal{T}_{dis}$ is the discrete topology. Moreover, Δ is a dense subset of the spectrum M of $\mathscr{L}_b(X, \mathbb{C})$. Finally, the hull $h(I)$, (which is homeomorphic to $M(L^\infty(X))$), of the ideal of all functions in $\mathscr{L}_b(X, \mathbb{C})$ vanishing almost everywhere on X is a *nowhere dense* and extremely disconnected subset of the Corona $M \setminus \Delta$ of $\mathscr{L}_b(X, \mathbb{C})$.

Keywords: bounded Lebesgue measurable functions; essentially bounded functions; spectra and maximal ideal spaces; extremely disconnected space; totally disconnected space

MSC2010: Primary 46J10, Secondary 54G05; 54C20

Introduction

Given a set $X \subseteq \mathbb{R}^n$ of positive Lebesgue measure $\sigma(X)$, let

$$\mathscr{L}^\infty := \left\{ f : D_f \subseteq X \to \mathbb{C} \;\middle|\; \sigma(X \setminus D_f) = 0,\ f\text{-Lebesgue meas.}, ||f||_{e.s} < \infty \right\},$$

where

$$||f||_{e.s} := \inf \left\{ \eta > 0 : \sigma(\{x \in X : |f(x)| \geq \eta\}) = 0 \right\}$$

is the essential supremum of f on X. $\mathscr{L}^\infty$ is called the set of *essentially bounded functions* on X. It is obvious that if f is a bounded continuous function on X, then $||f||_\infty = ||f||_{e.s}$. Define on $\mathscr{L}^\infty$ the following equivalence relation:

$$f \sim g \subseteq if - g \in \mathscr{N} := \{f \in \mathscr{L}^\infty : f \equiv 0 \text{ a.e.}\}.$$

Let $[f] := f + \mathscr{N}$ denote the equivalence class associated with f, and put

$$||\,[f]\,||_\infty := \inf\{||g||_{e.s} : g \sim f\}.$$

Then

$$||[f]||_\infty = ||f||_{e.s} \tag{1}$$

***Corresponding Author: Raymond Mortini:** Université de Lorraine, Département de Mathématiques et Institut Élie Cartan de Lorraine, UMR 7502, Ile du Saulcy, F-57045 Metz, France, E-mail: raymond.mortini@univ-lorraine.fr

Rudolf Rupp: Fakultät für Angew. Mathematik, Physik und Allgemeinwissenschaften, TH-Nürnberg, Kesslerplatz 12, D-90489 Nürnberg, Germany, E-mail: Rudolf.Rupp@th-nuernberg.de

for any representant $f \in \mathscr{L}^\infty$. Moreover, if $L^\infty = L^\infty(X) = \{[f] : f \in \mathscr{L}^\infty\}$, then

$$(L^\infty, +, \times, \underset{S}{\bullet}, ||[\cdot]||_\infty)$$

is a commutative unital Banach algebra such that $||[f]||_\infty^2 = ||[f]^2||_\infty$ for every $f \in L^\infty$. In particular, L^∞ is a uniform algebra. L^∞ is the classical algebra of (equivalence classes) of essentially bounded functions. It is well known that its spectrum (or maximal ideal space) $M(L^\infty)$ is an extremely (also called extremally) disconnected compact Hausdorff space (see e.g. [2]) and that $(L^\infty, ||\cdot||_{e.s})$ is isomorphic isometric to $\big(C(M(L^\infty)), ||\cdot||_\infty\big)$.

In this note we investigate its cousin algebra, $\mathscr{L}_b(X, \mathbb{C})$, of all bounded Lebesgue measurable functions on X endowed with the supremum norm $||\cdot||_\infty$. Let us emphasize that $\mathscr{L}_b(X, \mathbb{C})$ is strictly contained in $\mathscr{L}^\infty$, and that $\mathscr{L}^\infty$ contains unbounded functions. As far as we are aware of, a description of the maximal ideal space M of $\mathscr{L}_b(X, \mathbb{C})$ has never been given in literature. To our surprise, it will turn out that M is not extremely disconnected, but only totally disconnected. We shall also study the hull

$$h(I) := \{m \in M(\mathscr{L}_b(X), \mathbb{C}) : m(f) = 0 \text{ for every } f \in I\}$$

of the ideal I of all functions in $\mathscr{L}_b(X, \mathbb{C})$ vanishing almost everywhere on X. How big is this set? It will be shown that this set is a nowhere dense subset of the Corona $M \setminus \Delta$ of $\mathscr{L}_b(X, \mathbb{C})$, where $\Delta = \{\delta_x : x \in X\}$ is the set of point functionals on $\mathscr{L}_b(X, \mathbb{C})$. Another result will tell us that the restriction of the Gelfand topology $\mathfrak{g}$ to Δ is the discrete topology and that Δ is dense in M.

Our approach is entirely self-contained and thus accessible to graduate students. The results could also be derived by using the advanced theory of Banach lattices, Boolean algebras and Dedekind complete spaces, necessitating though the knowledge of a large amount of classical results. Usually, these advanced methods are given in monographs on $C^\star$-algebras and operator algebras, as for example [1, 7, 10].

It will finally be observed that the topological stable rank (and hence the Bass stable rank) of $\mathscr{L}_b(X, \mathbb{C})$ is one.

1 Some topological tools

Here we list, for the reader's convenience, several standard results needed for our proofs. As usual, let $\dim X$ denote the covering dimension of a compact space X and $C(X) := C(X, \mathbb{C})$ the space of all continuous, complex-valued functions on X. A proof of the following well-known result is given in [8, p.113].

Theorem (A). *Let X be a compact Hausdorff space. The following assertions are equivalent:*
(1) $\dim X = 0$*;*
(2) X is totally disconnected;
(3) The closed-open sets form a basis for the topology on X.

Let us note that a topological space X is called *extremely disconnected* if the closure of any open set is open, or equivalently, if two disjoint open sets have disjoint closures. In view of Theorem A it is clear that extreme disconnectedness implies total disconnectedness. A function of the form

$$f := \sum_{j=1}^{n} \alpha_j \chi_{E_j},$$

$\alpha_j \in \mathbb{C}$, $E_j \cap E_k = \emptyset$ for $j \neq k$, is called a *step-function* (or a *simple function*). The following standard results can easily be shown:

Lemma (B). *Simple functions have the following properties:*

i) *Let* $s = \sum_{k=1}^{n} \lambda_k \chi_{E_k}$ *be a simple function on a set* Y *(where* $E_j \cap E_k = \emptyset$ *for* $j \neq k$*),* $\varepsilon > 0$*, and let* F *be the union of those* E_k *where* $|\lambda_k| \geq \varepsilon$*. Then*

$$U := \{y \in Y : |s(y)| < \varepsilon\} = \{y \in Y : \chi_{F^c}(y) = 1\},$$

where F^c *denotes the complement of* F *in* X*.*

ii) *If, in the case of a topological space* Y*, the sets* E_j *are closed-open, then* F *is closed-open, too.*

Lemma (C)**.** *Let* Y *be a topological space,* $X \subseteq Y$ *and* U *an open subset of* Y*. Then*
(1) $\overline{X} \cap U \subseteq \overline{X \cap U}$.
(2) *If, additionally,* X *is dense in* Y*, then* $\overline{U} = \overline{X \cap U}$.
(3) *If* X *is dense in* Y*, and if* x_0 *is an isolated point in* X*, then* x_0 *is an isolated point in* Y*, whenever* Y *is Hausdorff.*

A commutative unital Banach algebra A is said to have the *topological stable rank* n if n is the smallest integer such that the set

$$U_n(A) := \left\{(f_1, \ldots, f_n) \in A^n : \exists (x_1, \ldots, x_n) \in A^n : \sum_{j=1}^{n} x_j f_j = \mathbf{1}_A\right\}$$

of invertible n-tuples in A is dense in A^n. We refer to [6] for detailed proofs concerning the relation of this concept, introduced by Rieffel, to other notions of stable ranks.

2 The spectrum of the algebra of bounded Lebesgue measurable functions

Let $\ell^\infty(X, \mathbb{C})$ be the set of all bounded, complex-valued functions on X and let $\mathscr{L}_b(X, \mathbb{C})$ be the subset of all bounded Lebesgue measurable functions. Endowed with the usual pointwise algebraic operations and the supremum norm, $\left(\ell^\infty(X, \mathbb{C}), +, \times, \underset{S}{\bullet}, ||\cdot||_\infty\right)$ becomes a uniform algebra with $\mathscr{L}_b(X, \mathbb{C})$ as a closed subalgebra[1]. Complex conjugation being an involution, we actually see that these algebras are commutative $C^\star$-algebras. Many properties of our two algebras above therefore are known consequences of the general abstract theory and can be found for instance in [7, 10]. Our main object will be $\mathscr{L}_b(X, \mathbb{C})$. Its companion algebra, the algebra of all bounded Borel measurable functions on completely regular spaces, is briefly mentioned in [10, III.1.25]. We shall not consider that algebra here. Let us recall the following fact, which immediately follows from the classical theory of Banach algebras. Observe that $M(A)$ denotes the spectrum of a commutative unital complex Banach algebra A.

Proposition 1. *Let* I *be the ideal of all functions in* $\mathscr{L}_b(X, \mathbb{C})$ *which vanish almost everywhere on* X *and let* $h(I) \subseteq M(\mathscr{L}_b(X, \mathbb{C}))$ *be its hull. Then*
(i) $(L^\infty, || \bullet ||_{e.s})$ *and* $(\mathscr{L}_b(X, \mathbb{C})/I, || \bullet ||_q)$ *are isomorphic isometric.*
(ii) $M(\mathscr{L}_b(X, \mathbb{C})/I)$ *is homeomorphic with* $h(I)$ *and* $M(L^\infty)$.

We now present the new results of our note. Note that those parts dealing with the algebra $\ell^\infty(X, \mathbb{C})$ of all bounded functions on X are well-known (see for example [5, 11]); we only include them to facilitate a comparison of both algebras which appear here and in order to get a better view in which aspects 'our' algebra $\mathscr{L}_b(X, \mathbb{C})$ differs from the standard one, $\ell^\infty(X, \mathbb{C})$.

Theorem 2. *Let* A *denote either* $\ell^\infty(X, \mathbb{C})$ *or* $\mathscr{L}_b(X, \mathbb{C})$*. Then the following assertions hold:*

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1 The closedness follows from the fact that a pointwise (and a fortiori a uniform) convergent sequence of measurable functions is measurable.

(1) $U_n(A) = \left\{(f_1, \dots, f_n) \in A^n : \inf_{x\in X} \sum_{j=1}^n |f_j(x)| \geq \delta > 0\right\}$.
(2) *A is isometric isomorphic to* $C(M(A), \mathbb{C})$.
(3) *Let* $\Delta := \{\delta_x : x \in X\}$ *be the set of evaluation functionals on* A. *Then the topological space* $(\Delta, \mathfrak{g})$ *is a discrete space.*
(4) *Under the discrete topology,* $\tilde{X} := (X, \mathfrak{T}_{\text{dis}})$ *can be continuously embedded into* $(M(A), \mathfrak{g})$.
(5) *The set* Δ *is dense in* $(M(A), \mathfrak{g})$.
(6) *Any point in* Δ *is an isolated point in* $M(A)$ *and* $M(A) \setminus \Delta$ *is closed.*
(7) $M(\ell^\infty(X, \mathbb{C}))$ *is homeomorphic to the Stone-Čech compactification of the discrete space* $(X, \mathfrak{T}_{\text{dis}})$ *and* $M(\ell^\infty(X, \mathbb{C}))$ *is extremely disconnected, hence totally disconnected.*
(8) $M(\mathscr{L}_b(X, \mathbb{C}))$ *is totally disconnected, but not extremely disconnected.*
(9) *The hull of the closed ideal* I *of all functions in* $\mathscr{L}_b(X, \mathbb{C})$ *which vanish almost everywhere on* X *is a nowhere dense subset of the Corona* $\mathfrak{C} := M(\mathscr{L}_b(X, \mathbb{C})) \setminus \Delta$ *of* $\mathscr{L}_b(X, \mathbb{C})$ *and is extremely disconnected.*
(10) $\dim M(A) = 0$.
(11) $\operatorname{bsr} A = \operatorname{tsr} A = 1$.

Proof. (1) If for some $g_j \in A$ the identity $\sum_{j=1}^n g_j f_j = 1$ holds, then

$$1 \leq \sum_{j=1}^n ||g_j||_\infty |f_j| \leq M \sum_{j=1}^n |f_j|,$$

and so $\sum_{j=1}^n |f_j|$ is bounded away from zero. For the converse, just consider the functions

$$g_j := \frac{\overline{f}_j}{\sum_{k=1}^n |f_k|^2}$$

which do belong to A and satisfy $\sum_{j=1}^n g_j f_j = 1$.

(2) As a uniform algebra A is isometric isomorphic to $(\widehat{A}, ||\cdot||_{M(A)})$ (see e.g. [3, p. 185]). Since the spectrum $\sigma(f)$ of a real-valued function in A is a subset of the reals (note that $f - \lambda$ is invertible whenever $\lambda \notin \mathbb{R}$), we deduce from the $\mathbb{C}$-linearity of the Gelfand transform that for every $f = u + iv \in A$ and $x \in M(A)$

$$\widehat{\overline{f}}(x) = \underbrace{\widehat{u}(x)}_{\in\mathbb{R}} - i \underbrace{\widehat{v}(x)}_{\in\mathbb{R}} = \overline{\widehat{f}(x)}.$$

Hence $(\widehat{A}, ||\cdot||_{M(A)})$ is a self-adjoint, uniformly closed, point separating subalgebra of $C(M(A))$. By the Stone-Weierstrass theorem, $\widehat{A} = C(M(A))$.

(3) Given $x_0 \in X$, let f be the characteristic function of $\{x_0\}$. Then $f \in A$. The assertion now follows from the following identities. Let $0 < \varepsilon < 1/2$. Then

$$\begin{aligned} U_{\varepsilon,f}(\delta_{x_0}) \cap \Delta &= \{m \in M(A) : |m(f) - \delta_{x_0}(f)| < \varepsilon\} \cap \Delta \\ &= \{\delta_x, x \in X : |f(x) - f(x_0)| < \varepsilon\} \\ &= \{\delta_{x_0}\}. \end{aligned}$$

Thus $\{\delta_{x_0}\}$ is an isolated point in Δ.

(4) The map

$$\begin{cases} (X, \mathfrak{T}_{\text{dis}}) & \to (M(A), \mathfrak{g}) \\ x & \mapsto \delta_x \end{cases}$$

is clearly an embedding by (3).

(5) That Δ is dense in $M(A)$ follows from (1) using the usual arguments given e.g. in [3, p. 191].

(6) Because by (5), Δ is dense in $M(A)$, we deduce from Lemma (C3), that $\{\delta_{x_0}\}$ is open in $M(A)$, too. Since Δ is a union of open singletons, it is an open subset of $M(A)$. Hence $M(A) \setminus \Delta$ is closed.

(7) Since $\tilde{X}$ carries the discrete topology, we see that $\ell^\infty(X, \mathbb{C}) = C_b(\tilde{X}, \mathbb{C})$. Because $\tilde{X}$ is a completely regular space, we get from the Stone-Čech Theorem that $M(\ell^\infty(X, \mathbb{C})) = \beta\tilde{X}$. Moreover, $\beta\tilde{X}$ is extremely disconnected (see [4] and [1, p.437]).

(8) We first show that $M(\mathscr{L}_b(X,\mathbb{C}))$ is totally disconnected. Let $m_0 \in M(\mathscr{L}_b(X,\mathbb{C}))$ and let U be an open neighborhood of m_0. Choose a basic open set (in the Gelfand topology)

$$V := V_{\varepsilon,f_1,\dots,f_n}(m_0) = \bigcap_{j=1}^{n}\{m \in M(\mathscr{L}_b(X,\mathbb{C})) : |\widehat{f_j}(m) - \widehat{f_j}(m_0)| < \varepsilon\}$$

such that $m_0 \in V \subseteq U$. By [9, p. 15], there are measurable step-functions s_j such that $\left\|s_j - f_j + \widehat{f_j}(m_0)\right\|_\infty < \varepsilon/2$. Hence

$$m_0 \in \bigcap_{j=1}^{n}\left\{m \in M(\mathscr{L}_b(X,\mathbb{C})) : |\widehat{s_j}(m)| < \varepsilon/2\right\} \subseteq V \subseteq U.$$

Now, for fixed j, $s_j = \sum_{k=1}^{N(j)} \lambda_{k,j}\chi_{E_{k,j}}$ for pairwise disjoint measurable subsets $E_{k,j}$ of X, $(k = 1,\dots,N(j))$. Given a measurable set $E \subseteq X$ (this set may have measure zero), we claim that

$$\widehat{\chi_E} = \chi_M \tag{2}$$

for some uniquely determined open-closed subset $M = M_E$ of $M(\mathscr{L}_b(X,\mathbb{C}))$. In fact, $\chi_E^2 = \chi_E$ is an idempotent in $\mathscr{L}_b(X,\mathbb{C})$. Hence $\sigma(\chi_E) \subseteq \{0,1\}$. The obvious cases are $\sigma(\chi_E) = \{0\}$, respectively $\sigma(\chi_E) = \{1\}$, leading readily to $M_E = \emptyset$, respectively $M_E = X$. So assume that $\sigma(\chi_E) = \{0,1\}$. Now $\sigma(f) = \widehat{f}(M(\mathscr{L}_b(X,\mathbb{C})))$ for any $f \in \mathscr{L}_b(X,\mathbb{C})$. If we let

$$M := \{m \in M(\mathscr{L}_b(X,\mathbb{C})) : \widehat{\chi_E}(m) = 1\},$$

we see that $\widehat{\chi_E} = \chi_M$. Since $\widehat{\chi_E}$, and hence χ_M, is continuous, M is open-closed. This yields (2). We conclude that

$$\widehat{s_j} = \sum_{k=1}^{N(j)} \lambda_{k,j}\chi_{M_{k,j}}$$

for pairwise disjoint open-closed subsets $M_{k,j}$ of $M(\mathscr{L}_b(X,\mathbb{C}))$. By Lemma (B),

$$\{m \in M(\mathscr{L}_b(X,\mathbb{C})) : |\widehat{s_j}(m)| < \varepsilon/2\} = \{m \in M(\mathscr{L}_b(X,\mathbb{C})) : \chi_{F_j}(m) = 1\}$$

for some closed-open sets $F_j \subseteq M(\mathscr{L}_b(X,\mathbb{C}))$. Accordingly, if $F := \bigcap_{j=1}^{n} F_j$, then

$$m_0 \in F = \{m \in M(\mathscr{L}_b(X,\mathbb{C})) : \chi_F(m) = 1\} \subseteq U.$$

Since F is closed-open, χ_F is continuous on $M(\mathscr{L}_b(X,\mathbb{C}))$ and it can be written as $F = \{\widehat{\chi_S} = 1\}$ for some measurable subset $S \subseteq X$. In fact, by (2), there is $f \in \mathscr{L}_b(X,\mathbb{C})$ such that $\widehat{f} = \chi_F$. Now $\chi_F^2 = \chi_F$; hence $\widehat{f}^2 = \widehat{f}$ and so $\widehat{f^2 - f} = 0$. Since the uniform algebra $\mathscr{L}_b(X,\mathbb{C})$ is semi-simple, $f^2 = f$. Hence f is a characteristic function of some measurable subset S of X. We conclude that the closed-open sets of the form

$$\{m \in M(\mathscr{L}_b(X,\mathbb{C})) : \widehat{\chi_S}(m) = 1\}, \quad S \subseteq X \text{ measurable}, \tag{3}$$

form a basis for the Gelfand topology. Since $M(\mathscr{L}_b(X,\mathbb{C}))$ is a compact Hausdorff space, the total disconnectedness is just an equivalent property (see Theorem A).

Next we show that $M(\mathscr{L}_b(X,\mathbb{C}))$ is not extremely disconnected. Let E be a non-measurable subset of X and let $F := X \setminus E$. Since $\tilde{X}$ carries the discrete topology, E and F are open sets in $\tilde{X}$. Hence, by (4), $\Delta_E := \{\delta_x : x \in E\} \subseteq \Delta$ and $\Delta_F := \{\delta_x : x \in F\} \subseteq \Delta$ are disjoint open sets in $M(\mathscr{L}_b(X,\mathbb{C}),\mathfrak{g})$. In view of achieving a contradiction, suppose that their closures in $M(\mathscr{L}_b(X,\mathbb{C}),\mathfrak{g})$ are disjoint. Since by (2), $\mathscr{L}_b(X,\mathbb{C})$ is a normal algebra on its spectrum, there is $f \in \mathscr{L}_b(X,\mathbb{C})$ such that $\widehat{f} = 1$ on $\overline{\Delta}_E$ and $\widehat{f} = 0$ on $\overline{\Delta}_F$. Since $\Delta_E \cup \Delta_F = \Delta$, we deduce that

$$\widehat{f}^{-1}(\{1\}) \cap \Delta = \{\delta_x : x \in X, \widehat{f}(\delta_x) = 1\} = \Delta_E.$$

Hence $f^{-1}(\{1\}) = E$ and so E is a measurable set, because f is a measurable function. A contradiction. We conclude that $M(\mathscr{L}_b(X,\mathbb{C}))$ is not extremely disconnected.

(9) Because for any $x_0 \in X$, the characteristic function of $\{x_0\}$ is measurable, we see that $\delta_{x_0} \notin h(I)$, the hull of I. Hence

$$\emptyset \neq h(I) \subseteq M(\mathscr{L}_b(X,\mathbb{C})) \setminus \Delta = \mathfrak{C}.$$

Next we show that $h(I)$ is nowhere dense in $\mathfrak{C}$. Assuming the contrary, there is a (relatively) open set V in $\mathfrak{C}$ such that $V \subseteq h(I)$. Let $m_0 \in V$. Then, by (3), there is a basic closed-open set $U := \{\widehat{\chi}_S = 1\}$ in $M(\mathscr{L}_b(X,\mathbb{C}))$, where $S \subseteq X$ is measurable, such that

$$m_0 \in \{m \in \mathfrak{C} : \widehat{\chi}_S(m) = 1\} \subseteq V \subseteq h(I).$$

Now the denseness of Δ in $M(\mathscr{L}_b(X,\mathbb{C}))$ implies that

$$\overline{U \cap \Delta} = \overline{U} = U.$$

Hence S is infinite (because otherwise $U \cap \Delta = \Delta_S := \{\delta_x : x \in S\}$ is a closed set contained in Δ, but $m_0 \in \mathfrak{C} \cap U$). Let (x_n) be a sequence of distinct points in S and consider the ideal

$$J := \{f \in \mathscr{L}_b(X,\mathbb{C}) : f(x_n) = 0 \text{ for almost every } n\}.$$

Then J is a proper ideal. Any maximal ideal containing J is necessarily contained in the corona $\mathfrak{C}$. Hence $h(J) \subseteq \mathfrak{C}$. Also

$$R := \overline{\{\delta_{x_n} : n \in \mathbb{N}\}} \setminus \Delta \subseteq \overline{U} \setminus \Delta = U \setminus \Delta \subseteq V \subseteq h(I).$$

Now the function u, given by $u(x) = 0$ for $x \in X \setminus \{x_n : n \in \mathbb{N}\}$ and $u(x_n) = 1$ for $n \in \mathbb{N}$, is measurable. Thus u belongs to $\mathscr{L}_b(X,\mathbb{C})$ and u is equal to 0 almost everywhere. Consequently, $u \in I$ and so $\widehat{u} \equiv 0$ on $h(I)$. But $\widehat{u}(\delta_{x_n}) = 1$ implies that $\widehat{u}(m) = 1$ for every m in the weak-*-closure of $\{\delta_{x_n} : n \in \mathbb{N}\}$. In particular, $\widehat{u} = 1$ on R; a contradiction. We conclude that $h(I)$ is nowhere dense in $\mathfrak{C}$.

That $h(I)$ is extremely disconnected is a consequence of Proposition 1 and the fact that $h(I)$ is homeomorphic to the extremely disconnected set $M(L^\infty)$ ([2, p. 18]).

(10) That $\dim M(A) = 0$ follows from Theorem A and the fact that $M(A)$ is totally disconnected.

(11) Let $f \in A$. Given $\varepsilon > 0$, the functions

$$F_\varepsilon := \begin{cases} f & \text{if } |f| \geq \varepsilon \\ \varepsilon & \text{if } |f| < \varepsilon \end{cases}$$

belong to A, are invertible, and uniformly approximate f. Hence tsr $A = 1$. But $1 \leq$ bsr $A \leq$ tsr A. Thus bsr $A = 1$, too. □

Proposition 1 and Theorem 2 (9) show in particular that $M(L^\infty)$ is embedded as a nowhere dense subset into the Corona $\mathfrak{C} := M(\mathscr{L}_b(X,\mathbb{C})) \setminus \Delta$ of $\mathscr{L}_b(X,\mathbb{C})$.

Acknowledgement: We thank the referee for drawing attention to reference [10].

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