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Kadison's antilattice theorem for a synaptic algebra

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Abstract: We prove that if A is a synaptic algebra and the orthomodular lattice P of projections in A is complete, then A is a factor if and only if A is an antilattice. We also generalize several other results of R. Kadison pertaining to infima and suprema in operator algebras.

Keywords: synaptic algebra, order unit space, Jordan algebra, spectral resolution, antilattice, factor

MSC: Primary 46B40; Secondary 46L89

1 Introduction

A synaptic algebra [1–13] is a generalization of the self-adjoint part of several structures based on operator algebras. For instance, although a synaptic algebra A need not be norm complete (i.e., Banach), A is isomorphic to a JC-algebra if and only if it is Banach [6, Theorem 5.3]. Numerous additional examples of synaptic algebras can be found in the references cited above.

In [14], Richard Kadison calls the self-adjoint part $\mathfrak{S}$ of an operator algebra an *antilattice* if and only if, whenever two elements of $\mathfrak{S}$ have an infimum in $\mathfrak{S}$, then the elements are comparable (i.e., one is less than or equal to the other). As Kadison remarks, “A moments thought shows that this is as strongly nonlattice as a partially ordered vector space can be” [14, p. 505]. He shows that, in important cases, the condition that $\mathfrak{S}$ is an antilattice is equivalent to the condition that the operator algebra in question is a factor (i.e., its center consists only of scalars). Our main theorem in this paper (Theorem 7) is a version of Kadison's result for a synaptic algebra in which the projections form a complete lattice.

2 Some properties of a synaptic algebra

Axioms for a synaptic algebra can be found in [1, §1] and will not be repeated here. Rather, we shall briefly sketch some of the important features of a synaptic algebra that we shall need below. Readers who have seen operator algebras will be familiar with many of these features—details can be found in the references given in Section 1, especially in [1]. We use ‘iff’ as an abbreviation for ‘if and only if,’ the notation $:=$ means ‘equals by definition,’ and $\mathbb{R}$ is the ordered field of real numbers.

In what follows, we assume that A is a *synaptic algebra* [1, Definition 1.1]. Thus, associated with A is a real or complex associative unital algebra R with unit 1 called the *enveloping algebra* of A , such that, (1) $1 \in A$, (2) A is a real linear subspace of R , (3) the linear space A is a partially ordered order-unit normed space with order

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unit 1 and positive cone $A^+ := \{a \in A : 0 \leq a\}$, and (4) the order-unit norm of $a \in A$ is denoted and defined by $\|a\| := \inf\{0 < \lambda \in \mathbb{R} : -\lambda 1 \leq a \leq \lambda 1\}$ [15, pp. 67–69]. In important examples, the enveloping algebra R is a unital (concrete or abstract) operator algebra with an adjoint mapping $a \mapsto a^*$, A is the self-adjoint part of R , and $A^+ = \{rr^* : r \in R\}$. We shall assume that A is *nontrivial*, i.e., $A \neq \{0\}$. Then $1 \neq 0$, which enables us, as usual, to identify each scalar $\lambda \in \mathbb{R}$ with the element $\lambda 1 \in A$.

Let $M \subseteq A$. Then M is partially ordered under the restriction of the partial order $\leq$ on A . If $m, n \in M$, then the infimum (greatest lower bound) and the supremum (least upper bound) of m and n in M —if they exist—are written as $m \wedge_M n$ and $m \vee_M n$, respectively. An *involution* on M is a mapping $m \mapsto m'$ that is order reversing (i.e., $m \leq n \Rightarrow n' \leq m'$) and of order two (i.e., $(m')' = m$). Such an involution provides a "duality" between existing infima and suprema in M according to $m \wedge_M n = (m' \vee_M n')'$ and $m \vee_M n = (m' \wedge_M n')'$. Note that the mapping $a \mapsto -a$ is an involution on A itself.

Let $a, b \in A$. Then the product ab is calculated in R and may or may not belong to A ; however, it is assumed that $0 \leq a^2 \in A$. Consequently, the Jordan product

$$a \odot b := \frac{1}{2}(ab + ba) = \frac{1}{2}((a+b)^2 - a^2 - b^2) \in A,$$

so A is a real unital special Jordan algebra under $\odot$ [16]. If a commutes with b , i.e., $ab = ba$ in R , we write aCb . Evidently, if aCb , then $ab = a \odot b \in A$. It can be shown that $ab = 0$ iff $ba = 0$. Also, if $a, b \in A^+$ and aCb , then $ab \in A^+$. As a consequence, if $a, b, c \in A$, $a \leq b$, $0 \leq c$, cCa , and cCb , then $ca \leq cb$. Indeed, by the hypotheses, $0 \leq c$, $b - a$ and $cC(b - a)$, whence $0 \leq cb - ca$, i.e., $ca \leq cb$.

Suppose that $a, b \in A$ and put $c := 2(a \odot b)$. Then $aba = a \odot c - a^2 \odot b \in A$ and the mapping $b \mapsto aba$ is called the *quadratic mapping* on A determined by a . It can be shown that the quadratic mapping $b \mapsto aba$ is both linear and order preserving on A .

An element $e \in A$, such that $0 \leq e \leq 1$ is called an *effect*. It can be shown that $e \in A$ is an effect iff $e^2 \leq e$. The subset $E := \{e \in A : 0 \leq e \leq 1\}$ of A forms a convex effect algebra [17] under the partially defined binary operation obtained by restriction to E of the addition operation $+$ on A . The mapping $e \mapsto e^\perp := 1 - e$, called the *orthosupplementation* on E , is an involution on E .

An idempotent element $p = p^2$ in A is called a *projection*, and the set $P := \{p \in A : p = p^2\}$ of all projections in A is a subset of E . In fact, P is precisely the set of all extreme points of the convex set E . Under the restriction to P of the partial order on A , P forms an orthomodular lattice (OML) [1, §5], [18, 19], and the orthocomplementation $p \mapsto p^\perp := 1 - p$ on the OML P is the restriction to P of the orthosupplementation on E . The orthocomplementation mapping is an involution on P , whence we have an infimum-supremum duality on the lattice P . If $e \in E$ and $p \in P$, it can be shown that $e \leq p \Leftrightarrow e = ep \Leftrightarrow e = pe$ and $p \leq e \Leftrightarrow p = pe \Leftrightarrow p = ep$.

Let $p, q \in P$. Then $p \leq q$ iff $p = pq$ iff $p = qp$. We shall write the infimum and the supremum of p and q in the lattice P as $p \wedge q$ and $p \vee q$, respectively, (without subscripts on $\wedge$ and $\vee$). If pCq , then $p \wedge q = pq = qp$ and $p \vee q = p + q - pq$. The projections p and q are said to be *orthogonal*, denoted $p \perp q$, iff $p \leq q^\perp$, or equivalently iff $q \leq p^\perp$. Note that $p \perp q$ iff $pq = 0$ iff $qp = 0$ iff $p + q = p \vee q$. In particular, for the orthocomplement $p^\perp$ of p in P , we have, $p \perp p^\perp$, $pCp^\perp$, $p \wedge p^\perp = pp^\perp = 0$ and $p \vee p^\perp = p + p^\perp = 1$.

If $M \subseteq A$, then $C(M) := \{a \in A : aCm \text{ for all } m \in M\}$ is called the *commutant* of M and $CC(M) := C(C(M))$ is called the *bicommutant* of M . Evidently, $C(M)$ is a linear subspace (or a vector subspace) of A . The subset M of A is said to be *commutative* iff mCn for all $m, n \in M$, i.e., iff $M \subseteq C(M)$. If M is commutative, then so is $CC(M)$. If $a \in A$, then $C(a) := C(\{a\})$ and $CC(a) := CC(\{a\})$.

If $a \in A^+$, then there exists a unique $a^{1/2} \in A^+$, called the *square root* of a , such that $(a^{1/2})^2 = a$. Moreover, $a^{1/2} \in CC(a)$. The *absolute value* of $a \in A$ is denoted and defined by $|a| := (a^2)^{1/2}$ and the *positive* and *negative* parts of a are denoted and defined by $a^+ := \frac{1}{2}(|a| + a)$ and $a^- := \frac{1}{2}(|a| - a)$, respectively. Then $0 \leq a^+$, $a^- \in CC(a)$, $a = a^+ - a^-$, $|a| = a^+ + a^-$, and $a^+ a^- = 0$. It turns out that an element $a \in A$ has an *inverse* a^{-1} in A , such that $aa^{-1} = a^{-1}a = 1$ iff there exists $0 < \epsilon \in \mathbb{R}$, such that $\epsilon \leq |a|$.

If $a \in A$, there exists a unique projection $a^0 \in P$, called the *carrier* of a , such that, for all $b \in A$, $ab = 0 \Leftrightarrow a^0 b = 0$. (Some authors would refer to a^0 as the *support* of a .) It fact $a^0 \in CC(a)$, a^0 is the smallest projection $p \in P$, such that $a = ap$, and if $e \in E$, then e^0 is the smallest projection $p \in P$, such that $e \leq p$. See [1, Theorem 2.10] for additional properties of the carrier.

Lemma 1. (i) If $p, q \in P$, then $p \wedge q = p \wedge_E q$ and $p \vee q = p \vee_E q$.
(ii) If $p, q \in P$, then $p \wedge q$ is also the infimum of p and q in A^+ .

Proof. (i) We have $p, q, p \wedge q \in P \subseteq E$ and $p \wedge q \leq p, q$. Suppose $e \in E$ with $e \leq p, q$. Then $e^0 \leq p, q$, whence $e \leq e^0 \leq p \wedge q$, and therefore $p \wedge q = p \wedge_E q$. By duality, $p \vee q = p \vee_E q$.

(ii) Clearly, $p \wedge q \in A^+$. Suppose that $a \in A^+$ and $a \leq p, q$. Then $0 \leq a \leq p \leq 1$, so $a \in E$, and since $p \wedge q$ is also the infimum of p and q in E (Lemma 1), we have $a \leq p \wedge q$. Therefore, $p \wedge q$ is the infimum of p and q in A^+ . $\square$

Notice that part (ii) of the above lemma is an analogue of [14, Lemma 2].

In view of Lemma 1, no confusion will result if we use the same notation $e \wedge f$ and $e \vee f$ (without subscripts) for existing infima and suprema of effects as we do for infima and suprema of projections. (Actually, the question of when two effects have an infimum in E is important, but not easy to resolve [21].)

There is a very satisfactory spectral theory for A based on the following notions [1, §8]: Let $a \in A$. The *spectral resolution* of a is the one-parameter family of projections $\{p_\lambda : \lambda \in \mathbb{R}\}$ given by $p_\lambda := ((a - \lambda)^+)^0 \in CC(a)$ for all $\lambda \in \mathbb{R}$. (Recall that λ is identified with $\lambda 1 \in A$.) The *spectral lower and upper bounds* for a are defined by $L := \sup\{\lambda \in \mathbb{R} : \lambda \leq a\}$ and $U := \inf\{\lambda \in \mathbb{R} : a \leq \lambda\}$, respectively. By [20, Theorem 3.1], we have $-\infty < L \leq a \leq U < \infty$ and $\|a\| = \max\{|L|, |U|\}$.

A real number ρ belongs to the *resolvent set* of a iff there is an open interval I in $\mathbb{R}$, such that $\rho \in I$ and $p_\lambda = p_\rho$ for all $\lambda \in I$. The *spectrum* of a , in symbols $\text{spec}(a)$, which is defined to be the complement in $\mathbb{R}$ of the resolvent set of a , is a nonempty closed and bounded subset of $\mathbb{R}$ with all of the expected properties.

Theorem 1. Let $a \in A$ with $0 < a$ and let $\{p_\lambda : \lambda \in \mathbb{R}\}$ be the spectral resolution of a . For $\lambda \in \mathbb{R}$, define $q_\lambda := 1 - p_\lambda = ((a - \lambda)^+)^0$. Then: (i) $q_\lambda \in CC(a) \subseteq C(a)$; (ii) If $\lambda < 0$, then $q_\lambda = 1$; (iii) $0 < q_0 = a^0$; (iv) If $0 < \lambda < \|a\|$, then $0 < q_\lambda$ and $\lambda q_\lambda \leq q_\lambda a = a q_\lambda \leq a$; (v) If $\lambda \geq \|a\|$, then $q_\lambda = 0$.

Proof. Part (i) follows from [1, Theorem 8.4 (i)]. Let L and U be the lower and upper spectral bounds for a . Since $0 < a$, we have $0 \leq L$, whence if $\lambda < 0$, then $\lambda < L$, and (ii) then follows from [1, Theorem 8.4 (vi)]. Since $0 < a$, we have $a = a^+$, whence $q_0 = ((a - 0)^+)^0 = a^0 \geq 0$. Also, $a^0 \neq 0$, else $a = 0$, contradicting $0 < a$, and we have (iii).

As $0 \leq L \leq U$ and $\|a\| = \max\{|L|, |U|\}$, we have $U = \|a\|$. Suppose $0 < \lambda < \|a\| = U$. Then by [1, Theorem 8.4 (ii)],

$$(1 - q_\lambda)(a - \lambda) \leq 0 \leq q_\lambda(a - \lambda). \quad (1)$$

Thus, if $q_\lambda = 0$, then $a \leq \lambda$, so $-\lambda \leq a \leq \lambda$, whence $\|a\| \leq \lambda$, contradicting $\lambda < \|a\|$. Therefore, $0 < q_\lambda$, and by (1), we have $\lambda q_\lambda \leq q_\lambda a = a q_\lambda$. Since $(1 - q_\lambda)Ca$ and $0 \leq 1 - q_\lambda$, a , it follows that $0 \leq (1 - q_\lambda)a$, whence we also have $q_\lambda a \leq a$, and therefore (iv) holds. Finally, (v) follows from [1, Theorem 8.4 (v)]. $\square$

Corollary 1. If $0 < a \in A$, then there exists $0 < \lambda \in \mathbb{R}$ and $0 < p \in P$, such that $\lambda p \leq a$.

Proof. Choose $\lambda \in \mathbb{R}$ with $0 < \lambda < \|a\|$, and in Theorem 1 let $p = q_\lambda$. $\square$

An element $s \in A$, such that $s^2 = 1$ is called a *symmetry*, and two elements $a, b \in A$ are said to be *exchanged* by the symmetry s iff $sas = b$ (or, equivalently, iff $sbs = a$) [4]. An element $t \in A$ is called a *partial symmetry* iff $t^2 = p \in P$, and a and b are *exchanged* by t iff $tat = b$ and $tbt = a$. There is a bijective correspondence $s \leftrightarrow p$ between symmetries $s \in A$ and projections $p \in P$ given by $s = 2p - 1$ and $p = \frac{1}{2}(s + 1)$.

Lemma 2. Let $s \in A$ be a symmetry, $s \neq -1$, and $\lambda, \mu \in \mathbb{R}$. Then $\lambda s \leq \mu \Rightarrow \lambda \leq \mu$.

Proof. Assume the hypotheses. Suppose $\lambda s \leq \mu$, and let $p = \frac{1}{2}(s + 1)$ be the projection corresponding to s . Then $s = 2p - 1 = p - p^\perp$, and since $s \neq -1$, we have $0 < p$. Then $\lambda p - \lambda p^\perp \leq \mu$, so $\lambda p = (\lambda p - \lambda p^\perp)p \leq \mu p$, i.e., $0 \leq (\mu - \lambda)p$. If $\mu - \lambda < 0$, then $(\mu - \lambda)p < 0$, contradicting $0 \leq (\mu - \lambda)p$, whence $0 \leq \mu - \lambda$, so $\lambda \leq \mu$. $\square$

A subset $S \subseteq A$ is called a *sub-synaptic algebra* of A iff S is a linear subspace of A , $1 \in S$, and S is closed under the formation of squares, square roots, carriers, and inverses, i.e., S is a synaptic algebra in its own right. For instance, if $M \subseteq A$, then $C(M)$ is a sub-synaptic algebra of A . In particular, if M is a commutative subset of A , then $CC(M)$ is a commutative sub-synaptic algebra of A .

If $p \in P$, then $pAp := \{pap : a \in A\} = \{a \in A : a = pa = ap\}$ is a linear subspace of A that is closed under the formation of squares, square roots, carriers, and inverses. It is a synaptic algebra in its own right with p as its unit element and with $\{prp : r \in R\}$ as its enveloping algebra. The OML of projections in pAp is the interval $P[0, p] := \{q \in P : q \leq p\}$. If t is a partial symmetry in A with $t^2 = p$, then t is a symmetry in pAp .

The *center* of A is the commutative sub-synaptic algebra $C(A)$. If $C(A) = \mathbb{R}$, then A is called a *factor*. For instance, the self-adjoint part $\mathcal{B}^{sa}(\mathfrak{H})$ of the unital C^* -algebra of all bounded linear operators on a Hilbert space $\mathfrak{H}$ is a factor.

Theorem 2. *The synaptic algebra A is a factor iff the only projections in $C(A)$ are 0 and 1.*

Proof. If $C(A) = \mathbb{R}$, then clearly the only projections in $C(A)$ are 0 and 1. Conversely, suppose that the only projections in $C(A)$ are 0 and 1 and let $a \in C(A)$. By [1, Theorem 8.10], the spectral resolution $\{p_\lambda : \lambda \in \mathbb{R}\}$ of a is contained in $C(A) \cap P$, whence $p_\lambda \in \{0, 1\}$ for all $\lambda \in \mathbb{R}$. Therefore, by [1, Theorem 8.4 (iii) and (vii)], there exists $\alpha \in \mathbb{R}$, such that $p_\lambda = 0$ for $\lambda < \alpha$ and $p_\lambda = 1$ for $\alpha \leq \lambda$, and it follows that $\text{spec}(a) = \{\alpha\}$. Consequently, by [1, Theorem 8.9], $a = \alpha 1 = \alpha$. $\square$

3 Two commuting projections

Lemma 3. (Cf. [14, Proof of Theorem 1]) *Suppose that V is a vector subspace of A , $p, q \in P \cap V$, $p \wedge q \in V$. Suppose further that there exist $w \in P \cap V$ with $p, q \leq w$ and $p \wedge_V q$. Put $p_1 := p - p \wedge_V q$ and $q_1 := q - p \wedge_V q$. Then:*

- (i) $p \wedge_V q = p \wedge q \in P \cap V$,
- (ii) $p_1, q_1 \in P \cap V$ with $p_1 \wedge_V q_1 = 0$,
- (iii) $p_1 q_1 = q_1 p_1 = 0$,
- (iv) p commutes with q .

Proof. Put $g := p \wedge q \in P \cap V$ and $g_V := p \wedge_V q \in V$. By Lemma 1, the infimum of p and q is in A^+ . Hence,

$$\text{if } a \in A^+ \text{ and } a \leq p, q, \text{ then } a \leq g. \quad (1)$$

Since $0 \in V$ and $0 \leq p, q$, it follows that $0 \leq g_V$. Hence, by (1) with $a := g_V$, we have $g_V \leq g$. Also, $g \in V$ and $g \leq p, q$, so $g \leq g_V$. Thus, $g_V = g$, and we have (i).

By (i), $g_V \in P \cap V$, and since $g_V \leq p$, it follows that $p_1 = p - g_V \in P \cap V$. Likewise, $q_1 = q - g_V \in P \cap V$. Evidently, $0 \leq p_1, q_1 \in V$. Suppose that $v \in V$ and $v \leq p_1, q_1$. Then $v + g_V \in V$ with $v + g_V \leq p, q$, whence $v + g_V \leq g_V$. So $v \leq 0$, and we have $p_1 \wedge_V q_1 = 0$. This completes the proof of (ii).

Since $p_1 \leq p \leq w$ and $q_1 \leq q \leq w$, we have $p_1 - w, q_1 - w \leq 0$, whence $p_1 + q_1 - w \leq p_1, q_1$. Therefore, $p_1 + q_1 - w \leq p_1 \wedge_V q_1 = 0$, whence $p_1 \leq w - q_1$. As w and q_1 are projections and $q_1 \leq w$, it follows that $w - q_1$ is a projection, whereupon $p_1 = p_1(w - q_1) = p_1 w - p_1 q_1 = p_1 - p_1 q_1$. So $p_1 q_1 = 0$, and we have (iii).

By (i), we have $p_1 = p - p \wedge q$. So $p \wedge q = p - p_1 \leq 1 - p_1$ and it follows that $(p \wedge q)Cp_1$. Likewise, $(p \wedge q)Cq_1$. By (iii), $p_1 Cq_1$, and therefore $(p_1 + p \wedge q)C(q_1 + p \wedge q)$, i.e., pCq , and we have (iv). $\square$

Theorem 3. (Cf. [14, Theorem 1]) *Two projections p and q in A commute iff there exists a vector subspace V of A , such that $p, q, p \wedge q \in V$, there is a projection $w \in V \cap P$ with $p, q \leq w$, and the infimum $p \wedge_V q$ in V of the two projections exists.*

Proof. By Lemma 3, we only have to prove that if p and q commute, then there is a vector subspace V of A with the indicated properties. It suffices to take $V := CC(\{p, q\})$, noting that V is commutative, and $a \in V \Rightarrow |a|, a^0 \in V$. Therefore, by [9, Theorem 5.11] V is a lattice. $\square$

Corollary 2. *Let $p, q \in P$. Then: (i) If $p \wedge_A q$ exists, then $pq = qp$; (ii) If $p \wedge_A q = 0$, then $pq = qp = 0$.*

Proof. Part (i) is an obvious consequence of Theorem 3. To prove (ii), assume that $p \wedge_A q = 0$. Then by Lemma 3 with $V = A$, we have $p = p_1$ and $q = q_1$, whence $pq = 0$ by part (iii) of the lemma. $\square$

In the following theorem we generalize part (ii) of Corollary 2 to an arbitrary pair of elements $a, b \in A$. In [14, Corollary 8], Kadison obtains this result and its corollary for the special case of A being the self-adjoint part of a W^* -algebra.

Theorem 4. *If $a, b \in A$ and $a \wedge_A b = 0$, then $ab = ba = 0$.*

Proof. Assume the hypotheses. We have $0 = a \wedge_A b \leq a, b$, i.e., $a, b \in A^+$. We shall use the notion of the *generalized infimum* of a and b , defined and denoted by $a \sqcap b := \frac{1}{2}(a + b - |a - b|)$ [9, §4]. By [9, Lemma 4.1 (i)], $a \sqcap b \leq a, b$, whence $a \sqcap b \leq a \wedge_A b = 0$. Thus, since $a, b \in A^+$, $ab = ba = 0$ by [9, Lemma 4.4 (ii)]. $\square$

Corollary 3. *If $a, b \in A$, $c := a \wedge_A b$ exists, cCa , and cCb , then aCb .*

Proof. Assume the hypotheses. Then $0 \leq a - c, b - c$, and if $k \in A$ with $k \leq a - c, b - c$, it follows that $k + c \leq a, b$, whence $k + c \leq c$, so $k \leq 0$. Therefore, $(a - c) \wedge_A (b - c) = 0$. By Theorem 4, $(a - c)(b - c) = (b - c)(a - c) = 0$. Consequently, $ab = cb + ac - c^2 = bc + ca - c^2 = ba$. $\square$

4 The antilattice theorem

In this section we prove our main theorem (Theorem 7) giving necessary and sufficient conditions for a synaptic algebra A with a complete projection lattice P to be an antilattice. In particular, Theorem 7 shows that a synaptic algebra with a complete projection lattice is an antilattice iff it is a factor. To begin with, we have the following.

Theorem 5. *If A is an antilattice, then A is a factor.*

Proof. Suppose that A is an antilattice. It will be sufficient to show that the only projections in $C(A)$ are 0 and 1 (Theorem 2). Let $p \in C(A) \cap P$. We claim that $p \wedge_A p^\perp$ exists and equals 0. Obviously, $0 \leq p, p^\perp$. Suppose $a \in A$ and $a \leq p, p^\perp$. As $p \in C(A)$, it follows that a commutes with both p and $p^\perp$, whence $pa \leq pp^\perp = 0$ and $p^\perp a \leq p^\perp p = 0$, and so $a = (p + p^\perp)a = pa + p^\perp a \leq 0$. Therefore, $p \wedge_A p^\perp = 0$. Consequently, if A is an antilattice and $p \in C(A) \cap P$, then either $p \leq p^\perp$ or $p^\perp \leq p$, i.e., $p = 0$, or $p = 1$. $\square$

Lemma 4. *If P is a complete OML, A is a factor, and $0 < p, q \in P$ with $p \perp q$, then there exists a symmetry $t \in A$, such that $tpt \leq q$ or $tqt \leq p$. Moreover, $t \neq \pm 1$.*

Proof. Assume the hypotheses. By [4, Lemma 8.5], there is a central projection $h \in C(A) \cap P$ and a symmetry $t \in A$, such that ph and a subprojection of q are exchanged by t and $q(1 - h)$ and a subprojection of p are exchanged by t . Since A is a factor, the central projection h is either 0 or 1 (Theorem 2), whence $tpt \leq q$ or $tqt \leq p$. If $t = \pm 1$, then $tpt = p$ and $tqt = q$, so $p \leq q$ or $q \leq p$. Since $p \perp q$, $p = 0$ or $q = 0$ contradicting $0 < p, q$. $\square$

Lemma 5. *Let $a, b \in A$ with $a \wedge_A b = 0$ and suppose that $p, q \in P$, $0 < \lambda, \mu \in \mathbb{R}$, $\lambda p \leq a$ and $\mu q \leq b$. Then $p \wedge_A q = pq = qp = 0$.*

Proof. Assume the hypotheses and let $\kappa := \min\{\lambda, \mu\}$. Suppose that $g \in A$ and $g \leq p, q$. Then $\kappa g \leq \kappa p \leq a$ and $\kappa g \leq \kappa q \leq b$, whence $\kappa g \leq 0$. Therefore, $g \leq 0$. Thus, $p \wedge_A q = 0$, and by Corollary 2 (ii) $pq = qp = 0$. $\square$

Lemma 6. *Suppose that A is not an antilattice. Then there are projections $p, q \in P$ with $p \perp q$, $0 < p, q$, and $p \wedge_A q = pq = qp = 0$.*

Proof. Since A is not an antilattice, there exist $c, d \in A$, such that $c \not\leq d$, $d \not\leq c$, and $c \wedge_A d$ exists in A . Put $a := c - c \wedge_A d$ and $b := d - c \wedge_A d$. Obviously, $0 \leq a, b$, and since $c \not\leq d$ and $d \not\leq c$, we have $0 < a, b$. Suppose $k \in A$ with $k \leq a, b$. Then $k + c \wedge_A d \leq c, d$, whence $k + c \wedge_A d \leq c \wedge_A d$, and it follows that $k \leq 0$. Therefore, $a \wedge_A b = 0$.

By Corollary 1, there exist projections $0 < p, q$ and real numbers $0 < \lambda, \mu$, such that $\lambda p \leq a$ and $\mu q \leq b$. Then by Lemma 5, $p \wedge_A q = pq = qp = 0$, and since $pq = 0$, we have $p \perp q$. $\square$

Lemma 7. *Suppose that $p \in P$, $p \neq 0, 1$, and there exists a symmetry $s \in A$ that exchanges p and $p^\perp$. Then there exists $k \in A$ with $k \leq p, p^\perp$ but $k \neq 0$.*

Proof. By the hypotheses, we have $0 < p, p^\perp \in P$ and there is a symmetry $s \in A$, such that $sps = p^\perp$. Clearly, $s \neq \pm 1$. Put

$$\alpha := -\frac{5}{4}, \beta := -\frac{3}{4}, \gamma := 1, \text{ and } k = 2s - \alpha^2 \in A. \quad (1)$$

Note that $k = 2s - \frac{25}{16}$, $sks = k$, and

$$\alpha^2 - \beta^2 = 1, (\alpha + \beta)\gamma = -2, \text{ and } \beta^2 + \gamma^2 = \frac{25}{16}. \quad (2)$$

We have

$$sp = p^\perp s, ps = sp^\perp, sp + ps = sp + sp^\perp = s(p + p^\perp) = s1 = s$$

and

$$sp^\perp + p^\perp s = ps + sp = s. \quad (3)$$

Put $d := ap + \gamma s + \beta p^\perp$. Then by (2) and (3),

$$\begin{aligned} 0 \leq d^2 &= \alpha^2 p + \alpha \gamma (ps + sp) + \gamma^2 + \beta \gamma (sp^\perp + p^\perp s) + \beta^2 p^\perp \\ &= \alpha^2 p + \beta^2 (1 - p) + \alpha \gamma s + \gamma^2 + \beta \gamma s = (\alpha^2 - \beta^2)p + (\alpha + \beta)\gamma s + \beta^2 + \gamma^2 \\ &= p - 2s + \frac{25}{16} = p - k, \end{aligned}$$

whence $0 \leq p - k$. So $k \leq p$. Therefore, $0 \leq s(p - k)s = sps - k = p^\perp - k$, so $k \leq p^\perp$. But $2 \not\leq \frac{25}{16}$, whence by Lemma 2 $2s \not\leq \frac{25}{16}$ and therefore $k = 2s - \frac{25}{16} \neq 0$. $\square$

Theorem 6. *Suppose that whenever $0 < p, q \in P$ with $p \perp q$, there exists a symmetry t in A , such that $tpt \leq q$ or $tqt \leq p$. Then A is an antilattice.*

Proof. Assume the hypothesis. Aiming for a contradiction, we assume that A is not an antilattice. By Lemma 6, there are projections $0 < p, q \in P$ with $p \perp q$ and $p \wedge_A q = pq = qp = 0$. Thus, by hypothesis, there exists a symmetry t in A , such that $tpt \leq q$ or $tqt \leq p$. By relabeling if necessary, we can assume that $tpt \leq q$. Thus, $0 < tpt \in P$ and $tpt \leq q \leq p^\perp$, so $tpt \perp p$. Therefore, $p(tpt) = (tpt)p = 0$ and $p \vee tpt = p + tpt$. If $a \in A$ and $a \leq p, tpt$, then $a \leq p, q$. It follows that $a \leq 0$ and hence $p \wedge_A tpt = 0$.

Now we are going to drop down to the synaptic algebra $A_1 \subseteq A$ defined by $A_1 := (p + tpt)A(p + tpt)$ with unit element $u := p + tpt$. The projection lattice of A_1 is the interval $P[0, u] = \{q \in P : q \leq u\}$ in P . We have $0 < p, tpt \in P[0, u]$ with $p(tpt) = (tpt)p = 0$ and $p + tpt = p \vee tpt = u$, whence tpt is the orthocomplement of p in A_1 . Clearly, $p \wedge_{A_1} tpt = 0$. Put $s := tp + pt$. Then s is a partial symmetry in A with $s^2 = p + tpt$, and $s(p + tpt) = (p + tpt)s = s$, so s is a symmetry in A_1 . Moreover, $sps = tpt$. Applying Lemma 7 to the synaptic algebra A_1 , we find that there exists $k \in A_1$ with $k \leq p, tpt$ but $k \neq 0$, contradicting $p \wedge_{A_1} tpt = 0$. $\square$

Theorem 7. Suppose that the OML P of projections in A is complete. Then the following conditions are equivalent:

- (i) A is an antilattice.
- (ii) A is a factor.
- (iii) If $0 < p, q \in P$ with $p \perp q$, then there exists a symmetry t in A , such that $tpt \leq q$ or $tqt \leq p$.

Proof. Theorem 5 shows that (i) $\Rightarrow$ (ii), Lemma 4 shows that (ii) $\Rightarrow$ (iii), and Theorem 6 shows that (iii) $\Rightarrow$ (i). □

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References

- [1] Foulis D. J., [Synaptic algebras](#), Math. Slovaca, 2010, 60, 631-654
- [2] Foulis D. J., Pulmannová S., [Projections in a synaptic algebra](#), Order, 2010, 27, 235-257
- [3] Foulis D. J., Pulmannová S., [Type-decomposition of a synaptic algebra](#), Found. Phys., 2013, 43, 948-968
- [4] Foulis D. J., Pulmannová S., [Symmetries in synaptic algebras](#), Math. Slovaca, 2014, 64, 751-776
- [5] Foulis D. J., Pulmannová S., [Commutativity in a synaptic algebra](#), Math. Slovaca, 2016, 66, 469-482
- [6] Foulis D. J., Pulmannová S., Banach synaptic algebras, Internat. J. Theoret. Phys. (in press), DOI: 10.1007/s10773-017-3641-y
- [7] Foulis D. J., Jenčová A., Pulmannová S., Two projections in a synaptic algebra, Linear Algebra Appl., 2015, 478, 162-187
- [8] Foulis D. J., Jenčová A., Pulmannová S., A projection and an effect in a synaptic algebra, Linear Algebra Appl., 2015, 485, 417-441
- [9] Foulis D. J., Jenčová A., Pulmannová S., [Vector lattices in synaptic algebras](#), Math. Slovaca, 2017, 57, 1509-1524
- [10] Foulis D. J., Jenčová A., Pulmannová S., [States and synaptic algebras](#), Rep. Math. Phys., 2017, 79, 13-32
- [11] Foulis D. J., Jenčová A., Pulmannová S., [Every synaptic algebra has the monotone square root property](#), Positivity, 2017, 21, 919-930
- [12] Foulis D. J., Jenčová A., Pulmannová S., A Loomis-Sikorski theorem and functional calculus for a generalized Hermitian algebra, Rep. Math. Phys., 2017, 80, 255-275
- [13] Pulmannová S., [A note on ideals in synaptic algebras](#), Math. Slovaca, 2012, 62, 1091-1104
- [14] Kadison R. V., [Order properties of bounded self-adjoint operators](#), Proc. Amer. Math. Soc., 1951, 2, 505-510
- [15] Alfsen E. M., Compact Convex Sets and Boundary Integrals, 1st ed., Springer-Verlag, New York, 1971
- [16] McCrimmon K., A taste of Jordan algebras, 1st ed., Springer-Verlag, New York, 2004
- [17] Gudder S., Pulmannová S., Bugajski S., Beltrametti E., Convex and linear effect algebras, Rep. Math. Phys., 1999, 44, 359-379
- [18] Beran L., Orthomodular Lattices, An Algebraic Approach, 1st ed., D. Reidel Publishing Company, Dordrecht, 1985
- [19] Kalmbach G., Orthomodular Lattices, Academic Press, 1st ed., London, New York, 1983
- [20] Foulis D. J., Pulmannová S., Spectral resolution in an order unit space, Rep. Math. Phys., 2008, 62, 323-344
- [21] Gheondea A., Gudder S., Jonas P., On the infimum of quantum effects, J. Math. Phys., 2005, 46