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New bounds for Shannon, Relative and Mandelbrot entropies via Hermite interpolating polynomial

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Abstract: To procure inequalities for divergences between probability distributions, Jensen's inequality is the key to success. Shannon, Relative and Zipf-Mandelbrot entropies have many applications in many applied sciences, such as, in information theory, biology and economics, etc. We consider discrete and continuous cyclic refinements of Jensen's inequality and extend them from convex function to higher order convex function by means of different new Green functions by employing Hermite interpolating polynomial whose error term is approximated by Peano's kernel. As an application of our obtained results, we give new bounds for Shannon, Relative and Zipf-Mandelbrot entropies.

Keywords: n -convex function, Hermite interpolating polynomial, new Green functions, Shannon entropy, Relative entropy, Zipf-Mandelbrot entropy

MSC: 26A51, 26D15, 26E60, 94A17, 94A15

1 Introduction

Made mathematically rigorous around 1940s, information theory is a comparatively new branch of mathematics. It deals with information processing, storing, retrieval and decision making. Narrowly speaking it studies all theoretical problems concerned with transformation of information over the communication channels. The subject received a considerable boost when Shannon published an extraordinary paper in 1948 stating the properties of information sources and the properties of the communication channels which are used to transmit the output of these sources.

Jensen's inequality for differentiable convex functions has significant applications in information theory. It is used to obtain upper bounds for several quantitative measures arising from information theory for continuous random variables. It also helps computing several useful bounds for joint entropy, conditional entropy and mutual information. It provides different counterpart inequalities of Shannon entropy which is one of the major tools used in information theory and hence helps solving many problems in accounting, economics, psychology, statistics, ecology, computer science etc.

In what follows, we shall use the following conventions for the sake of simplicity Conditions(C), Hermite(H), Lagrange(L) and Taylor(T). Let $\varphi \in C^n[\alpha_1, \alpha_2]$ and consider the interval with points $-\infty < \alpha_1 = b_1 < b_2 \cdots < b_t = \alpha_2 < \infty$, ($t \geq 2$). Then there exists a unique polynomial $\Lambda_H(\cdot)$ of degree $(n - 1)$ satisfying any of

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the following axioms:

$$\mathbf{HC}: (1 \leq \omega \leq t, \sum_{\omega=1}^t s_{\omega} + t = n)$$

$$\Lambda_H^{(\sigma)}(b_{\omega}) = \varphi^{(\sigma)}(b_{\omega}); \quad 0 \leq \sigma \leq s_{\omega}. \quad (1)$$

Further particular cases are :

$$\mathbf{LC}: (t = n, s_{\omega} = 0 \text{ for all } \omega)$$

$$\Lambda_L(b_{\omega}) = \varphi(b_{\omega}), \quad 1 \leq \omega \leq n,$$

$$\mathbf{Type}(\eta, n - \eta)\mathbf{C}: (t = 2, 1 \leq \eta \leq n - 1, s_1 = \eta - 1, s_2 = n - \eta - 1)$$

$$\Lambda_{(\eta, n)}^{(\sigma)}(\alpha_1) = \varphi^{(\sigma)}(\alpha_1), \quad 0 \leq \sigma \leq \eta - 1,$$

$$\Lambda_{(\eta, n)}^{(\sigma)}(\alpha_2) = \varphi^{(\sigma)}(\alpha_2), \quad 0 \leq \sigma \leq n - \eta - 1,$$

$$\mathbf{Two-point TC}: (n = 2\eta, t = 2, s_1 = s_2 = \eta - 1)$$

$$\Lambda_{2T}^{(\sigma)}(\alpha_1) = \varphi^{(\sigma)}(\alpha_1), \quad \Lambda_{2T}^{(\sigma)}(\alpha_2) = \varphi^{(\sigma)}(\alpha_2), \quad 0 \leq \sigma \leq \eta - 1.$$

The associated error $|E_H(z)|$ can be approximated by Green's function (Peano's kernel) concerning boundary value problem for multiple points

$$\chi^{(n)}(z) = 0, \quad \chi^{(\sigma)}(b_{\omega}) = 0, \quad 0 \leq \sigma \leq s_{\omega}, \quad 1 \leq \omega \leq t,$$

which is stated in the coming theorem:

Theorem 1.1. [1] Let $-\infty < \alpha_1 < \alpha_2 < \infty$ with $\alpha_1 \leq b_1 < b_2 < \dots < b_t \leq \alpha_2$, ($t \geq 2$) be the given points, and $\varphi \in C^n([\alpha_1, \alpha_2])$. Then we have

$$\varphi(z) = \Lambda_H(z) + R_H(\varphi, z) \quad (2)$$

where $\Lambda_H(z)$ represents a Hermite polynomial, i.e.

$$\Lambda_H(z) = \sum_{\omega=1}^t \sum_{\sigma=0}^{s_{\omega}} H_{\sigma\omega}(z) \varphi^{(\sigma)}(b_{\omega});$$

where $H_{\sigma\omega}$ are Hermite basis given as

$$H_{\sigma\omega}(z) = \frac{1}{\sigma!} \frac{\vartheta(z)}{(z - b_{\omega})^{s_{\omega}+1-\sigma}} \sum_{s=0}^{s_{\omega}-\sigma} \frac{1}{s!} \frac{d^s}{dz^s} \left(\frac{(z - b_{\omega})^{s_{\omega}+1}}{\vartheta(z)} \right) \Bigg|_{z=b_{\omega}} (z - b_{\omega})^s, \quad (3)$$

with

$$\vartheta(z) = \prod_{\omega=1}^t (z - b_{\omega})^{s_{\omega}+1},$$

and remainder

$$R_H(\varphi, z) = \int_{\alpha_1}^{\alpha_2} G_{H,n}(z, r) \varphi^{(n)}(r) dr,$$

where $G_{H,n}(z, r)$ is defined by

$$G_{H,n}(z, r) = \begin{cases} \sum_{\omega=1}^l \sum_{\sigma=0}^{s_{\omega}} \frac{(b_{\omega}-r)^{n-\sigma-1}}{(n-\sigma-1)!} H_{\sigma\omega}(z), & r \leq z; \\ - \sum_{\omega=l+1}^t \sum_{\sigma=0}^{s_{\omega}} \frac{(b_{\omega}-r)^{n-\sigma-1}}{(n-\sigma-1)!} H_{\sigma\omega}(z), & r \geq z; \end{cases} \quad (4)$$

for all $b_l \leq r \leq b_{l+1}$; $l = 0, \dots, t$ with $b_0 = \alpha_1$ and $b_{t+1} = \alpha_2$.

Considering the particular cases for the Hermite conditions **(HC)**, we have the following corollary:

Corollary 1.2. For Lagrange conditions **(LC)**, we have

$$\varphi(z) = \Lambda_L(z) + R_L(\varphi, z) \quad (5)$$

where $\Lambda_L(z)$ indicates Lagrange interpolating polynomial i.e.,

$$\Lambda_L(z) = \sum_{\omega=1}^n \prod_{\substack{s=1 \\ s \neq \omega}}^n \left(\frac{z - b_s}{b_\omega - b_s} \right) \varphi(b_\omega)$$

with remainder $R_L(\varphi, z)$

$$R_L(\varphi, z) = \int_{\alpha_1}^{\alpha_2} G_L(z, r) \varphi^{(n)}(r) dr$$

with

$$G_L(z, r) = \frac{1}{(n-1)!} \begin{cases} \sum_{\omega=1}^l (b_\omega - r)^{n-1} \prod_{\substack{s=1 \\ s \neq \omega}}^n \left(\frac{z - b_s}{b_\omega - b_s} \right), & r \leq z; \\ - \sum_{\omega=l+1}^n (b_\omega - r)^{n-1} \prod_{\substack{s=1 \\ s \neq \omega}}^n \left(\frac{z - b_s}{b_\omega - b_s} \right), & r \geq z; \end{cases} \quad (6)$$

where $b_l \leq r \leq b_{l+1}$, $l = 1, 2, \dots, n-1$ with $b_1 = \alpha_1$ and $b_n = \alpha_2$.

Considering type $(\eta, n - \eta)$ conditions **(Type $(\eta, n - \eta)$ C)**, we get

$$\varphi(z) = \Lambda_{(\eta, n)}(z) + R_{(\eta, n)}(\varphi, z) \quad (7)$$

where $\Lambda_{(\eta, n)}(z)$ is $(\eta, n - \eta)$ interpolating polynomial, that is

$$\Lambda_{(\eta, n)}(z) = \sum_{\sigma=0}^{\eta-1} \tau_\sigma(z) \varphi^{(\sigma)}(\alpha_1) + \sum_{\sigma=0}^{n-\eta-1} \zeta_\sigma(z) \varphi^{(\sigma)}(\alpha_2),$$

with

$$\tau_\sigma(z) = \frac{1}{\sigma!} (z - \alpha_1)^\sigma \left(\frac{z - \alpha_2}{\alpha_1 - \alpha_2} \right)^{n-\eta} \sum_{s=0}^{\eta-1-\sigma} \binom{n-\eta+s-1}{s} \left(\frac{z - \alpha_1}{\alpha_2 - \alpha_1} \right)^s \quad (8)$$

and

$$\zeta_\sigma(z) = \frac{1}{\sigma!} (z - \alpha_2)^\sigma \left(\frac{z - \alpha_1}{\alpha_2 - \alpha_1} \right)^\eta \sum_{s=0}^{n-\eta-1-\sigma} \binom{\eta+s-1}{s} \left(\frac{z - \alpha_2}{\alpha_1 - \alpha_2} \right)^s \quad (9)$$

along with the remainder $R_{(\eta, n)}(\varphi, z)$, given as

$$R_{(\eta, n)}(\varphi, z) = \int_{\alpha_1}^{\alpha_2} G_{(\eta, n)}(z, r) \varphi^{(n)}(r) dr$$

with

$$G_{(\eta, n)}(z, r) = \begin{cases} \sum_{\omega=0}^{\eta-1} \left[\sum_{p=0}^{\eta-1-\omega} \binom{n-\eta+p-1}{p} \left(\frac{z - \alpha_1}{\alpha_2 - \alpha_1} \right)^p \right] \frac{(z - \alpha_1)^\omega (\alpha_1 - r)^{n-\omega-1}}{\omega! (n - \omega - 1)!} \left(\frac{\alpha_2 - z}{\alpha_2 - \alpha_1} \right)^{n-\eta}, & \alpha_1 \leq r \leq z \leq \alpha_2; \\ - \sum_{\sigma=0}^{n-\eta-1} \left[\sum_{q=0}^{n-\eta-\sigma-1} \binom{\eta+q-1}{q} \left(\frac{\alpha_2 - z}{\alpha_2 - \alpha_1} \right)^q \right] \frac{(z - \alpha_2)^\sigma (\alpha_2 - r)^{n-\sigma-1}}{\sigma! (n - \sigma - 1)!} \left(\frac{z - \alpha_1}{\alpha_2 - \alpha_1} \right)^\eta, & \alpha_1 \leq z \leq r \leq \alpha_2. \end{cases} \quad (10)$$

For Type Two-point Taylor conditions **(Two-point TC)**, from Theorem 1.1 we have

$$\varphi(z) = \Lambda_{2T}(z) + R_{2T}(\varphi, z) \quad (11)$$

where $\Lambda_{2T}(z)$ is the two-point Taylor interpolating polynomial i.e.,

$$\Lambda_{2T}(z) = \sum_{\sigma=0}^{\eta-1} \sum_{s=0}^{\eta-1-\sigma} \binom{\eta+s-1}{s} \left[\frac{(z-\alpha_1)^\sigma}{\sigma!} \left(\frac{z-\alpha_2}{\alpha_1-\alpha_2} \right)^\eta \left(\frac{z-\alpha_1}{\alpha_2-\alpha_1} \right)^s \varphi^{(\sigma)}(\alpha_1) \right. \\ \left. + \frac{(z-\alpha_2)^\sigma}{\sigma!} \left(\frac{z-\alpha_1}{\alpha_2-\alpha_1} \right)^\eta \left(\frac{z-\alpha_2}{\alpha_1-\alpha_2} \right)^s \varphi^{(\sigma)}(\alpha_2) \right] \quad (12)$$

and the remainder $R_{2T}(\varphi, z)$ is given by

$$R_{2T}(\varphi, z) = \int_{\alpha_1}^{\alpha_2} G_{2T}(z, r) \varphi^{(n)}(r) dr$$

with

$$G_{2T}(z, r) = \begin{cases} \frac{(-1)^\eta}{(2\eta-1)!} p^\eta(z, r) \sum_{\omega=0}^{\eta-1} \binom{\eta-1+\omega}{\omega} (z-r)^{\eta-1-\omega} q^\omega(z, r), & r \leq z; \\ \frac{(-1)^\eta}{(2\eta-1)!} q^\eta(z, r) \sum_{\omega=0}^{\eta-1} \binom{\eta-1+\omega}{\omega} (r-z)^{\eta-1-\omega} p^\omega(z, r), & r \geq z; \end{cases} \quad (13)$$

where $p(z, r) = \frac{(r-\alpha_1)(\alpha_2-z)}{\alpha_2-\alpha_1}$, $q(z, r) = p(z, r)$, $\forall z, r \in [\alpha_1, \alpha_2]$.

(Beesack [2], Levin [3]) characterize the non-negativity of Green's function $G_{H,n}(z, r)$:

Lemma 1.3. (i) $\frac{G_{H,n}(z, r)}{\vartheta(z)} > 0$, $b_1 \leq z \leq b_t, b_1 \leq r \leq b_t$.

(ii) $G_{H,n}(z, r) \leq \frac{1}{(n-1)!(\alpha_2-\alpha_1)} |\vartheta(z)|$.

(iii) $\int_{\alpha_1}^{\alpha_2} G_{H,n}(z, r) dr = \frac{\vartheta(z)}{n!}$.

For $j = 1, \dots, 5$, consider the well known Lagrange-Green function along with new Green functions $G_j : [\alpha_1, \alpha_2] \times [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ defined as

$$G_1(z, r) = \begin{cases} \frac{(\alpha_2-z)(\alpha_1-r)}{\alpha_2-\alpha_1}, & \alpha_1 \leq r \leq z; \\ \frac{(\alpha_2-r)(\alpha_1-z)}{\alpha_2-\alpha_1}, & z \leq r \leq \alpha_2. \end{cases} \quad (14)$$

$$G_2(z, r) = \begin{cases} \alpha_1 - r, & \alpha_1 \leq r \leq z; \\ \alpha_1 - z, & z \leq r \leq \alpha_2; \end{cases} \quad (15)$$

$$G_3(z, r) = \begin{cases} z - \alpha_2, & \alpha_1 \leq r \leq z; \\ r - \alpha_2, & z \leq r \leq \alpha_2; \end{cases} \quad (16)$$

$$G_4(z, r) = \begin{cases} z - \alpha_1, & \alpha_1 \leq r \leq z; \\ r - \alpha_1, & z \leq r \leq \alpha_2; \end{cases} \quad (17)$$

$$G_5(z, r) = \begin{cases} \alpha_2 - r, & \alpha_1 \leq r \leq z; \\ \alpha_2 - z, & z \leq r \leq \alpha_2. \end{cases} \quad (18)$$

All these functions are convex and continuous w.r.t both z and r , and the following Lemma holds.

Lemma 1.4. Let $\varphi \in C^2[\alpha_1, \alpha_2]$. Then the following identities hold:

$$\varphi(z) = \frac{\alpha_2 - z}{\alpha_2 - \alpha_1} \varphi(\alpha_1) + \frac{z - \alpha_1}{\alpha_2 - \alpha_1} \varphi(\alpha_2) + \int_{\alpha_1}^{\alpha_2} G_1(z, r) \varphi''(r) dr, \quad (19)$$

$$\varphi(z) = \varphi(\alpha_1) + (z - \alpha_1)\varphi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} G_2(z, r)\varphi''(r)dr, \quad (20)$$

$$\varphi(z) = \varphi(\alpha_2) + (\alpha_2 - z)\varphi'(\alpha_1) + \int_{\alpha_1}^{\alpha_2} G_3(z, r)\varphi''(r)dr, \quad (21)$$

$$\varphi(z) = \varphi(\alpha_2) - (\alpha_2 - \alpha_1)\varphi'(\alpha_2) + (z - \alpha_1)\varphi'(\alpha_1) + \int_{\alpha_1}^{\alpha_2} G_4(z, r)\varphi''(r)dr, \quad (22)$$

$$\varphi(z) = \varphi(\alpha_1) + (\alpha_2 - \alpha_1)\varphi'(\alpha_1) - (\alpha_2 - z)\varphi'(\alpha_2) + \int_{\alpha_1}^{\alpha_2} G_5(z, r)\varphi''(r)dr. \quad (23)$$

Proof. Consider the integral

$$\int_{\alpha_1}^{\alpha_2} G_j(z, r)\varphi''(r)dr = \int_{\alpha_1}^z G_j(z, r)\varphi''(r)dr + \int_z^{\alpha_2} G_j(z, r)\varphi''(r)dr.$$

Fix $j = 1, \dots, 5$ and perform the integration for the specific value of the Green's function, we will obtain the identities (19)–(23) for $j = 1, \dots, 5$. $\square$

Remark 1.5. The Green's function $G_1(\cdot, \cdot)$ is called Lagrange Green's function (see [4]). The new Green functions $G_j(\cdot, \cdot)$, ($j = 2, 3, 4, 5$), were introduced by Pečarić et al. in [5]. The result (20) given in the previous Lemma represents a special case of the representation of the function using the so-called 'two-point right focal' interpolating polynomial in case when $n = 2$ and $p = 0$ (see [1]).

The most important inequality concerning convex functions is the classical Jensen's inequality (see [6]). We present some recent work on cyclic refinements of classical and discrete Jensen's inequalities (see [7]). To make statements of that work simple, we need the following hypotheses:

(H₁) Let $2 \leq k \leq m$ be integers, and let $p_1, \dots, p_m$ and $\lambda_1, \dots, \lambda_k$ represent positive probability distributions.

(H₂) Let C be a convex subset of a real vector space Z and φ be a real valued convex function defined on C .

Theorem 1.6. Assume (H₁) and (H₂). If $z_1, \dots, z_m \in C$, then

$$\begin{aligned} \varphi\left(\sum_{u=1}^m p_u z_u\right) &\leq C_{dis} = C_{dis}(\varphi, \mathbf{z}, \mathbf{p}, \lambda) \\ &:= \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}\right) \varphi\left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} z_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}}\right) \leq \sum_{u=1}^m p_u \varphi(z_u), \end{aligned} \quad (24)$$

where $u + v$ means $u + v - n$ in case of $u + v > m$.

Theorem 1.6 can be considered as the weighted form of Theorem 2.1 in [8]. To refine the classical Jensen's inequality, we first introduce some hypotheses and notations.

(H₃) Let $(Z, \mathcal{B}, \delta)$ be a probability space.

Let $l \geq 2$ be a fixed integer. For $j = 1, \dots, l$, the σ -algebra in Z^l generated by the projection mappings

$pr_j : Z^l \rightarrow Z$ defined by

$$pr_j(z_1, \dots, z_l) := z_j$$

is denoted by $\mathcal{B}^l$ where δ^l is the product measure on $\mathcal{B}^l$. This measure is uniquely (δ is σ -finite) specified by

$$\delta^l(B_1 \times \dots \times B_l) := \delta(B_1) \dots \delta(B_l), \quad B_j \in \mathcal{B}, \quad j = 1, \dots, l.$$

(H₄) Let g be a δ -integrable function on Z having values in an interval $I \subset \mathbb{R}$.

(H₅) Let f be a convex function on I such that $f \circ g$ is δ -integrable on Z .

Under the conditions (H₁) and (H₃-H₅) we define

$$C_{int} = C_{int}(\varphi, f, \delta, \mathbf{p}, \lambda) := \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \right) \int_{Z^m} \varphi \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} f(z_{u+v})}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \right) d\delta^m(z_1, \dots, z_m), \quad (25)$$

and for $t \in [0, 1]$

$$C_{par}(t) = C_{par}(t, \varphi, f, \delta, \mathbf{p}, \lambda) := \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \right) \times \int_{Z^m} \varphi \left(t \frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} f(z_{u+v})}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} + (1-t) \int_Z f d\delta \right) d\delta^m(z_1, \dots, z_m), \quad (26)$$

where $u+v$ means $u+v-m$ in case of $u+v > m$.

Remark 1.7. Lemma 2.1 (b) in [9] assures that the integrals in (25) and (26) exist and are finite.

Theorem 1.8. Assume (H₁) and (H₃-H₅). Then

$$\varphi \left(\int_Z f d\delta \right) \leq C_{par}(t) \leq C_{int} \leq \int_Z \varphi \circ f d\delta, \quad t \in [0, 1].$$

In order to achieve our goals, we consider the following hypotheses for next sections.

(M₁) Let $I \subset \mathbb{R}$ be an interval, $\mathbf{z} := (z_1, \dots, z_m) \in I^m$ and let $p_1, \dots, p_m$ and $\lambda_1, \dots, \lambda_k$ represent positive probability distributions for $2 \leq k \leq n$.

Remark 1.9. Under the conditions (M₁), we define

$$J_1(\varphi) = J_1(\mathbf{z}, \mathbf{p}, \lambda; \varphi) := \sum_{u=1}^m p_u \varphi(z_u) - C_{dis}(\varphi, \mathbf{z}, \mathbf{p}, \lambda)$$

$$J_2(\varphi) = J_2(\mathbf{z}, \mathbf{p}, \lambda; \varphi) := C_{dis}(\varphi, \mathbf{z}, \mathbf{p}, \lambda) - \varphi \left(\sum_{u=1}^m p_u z_u \right)$$

where $\varphi : I \rightarrow \mathbb{R}$ is a function. The functionals $\varphi \rightarrow J_i(\varphi)$ are linear and Theorem 1.6 implies that

$$J_i(\varphi) \geq 0, \quad i = 1, 2$$

provided that φ is a convex function.

Assume (H₁) and (H₃-H₅). Then we have the following additional linear functionals

$$J_3(\varphi) = J_3(\varphi, f, \delta, \mathbf{p}, \lambda) := \int_Z \varphi \circ f d\delta - C_{int}(\varphi, f, \delta, \mathbf{p}, \lambda) \geq 0,$$

$$J_4(\varphi) = J_4(t, \varphi, f, \delta, \mathbf{p}, \lambda) := \int_Z \varphi \circ f d\delta - C_{par}(t, \varphi, f, \delta, \mathbf{p}, \lambda) \geq 0; \quad t \in [0, 1],$$

$$J_5(\varphi) = J_5(t, \varphi, f, \delta, \mathbf{p}, \lambda) := C_{int}(\varphi, f, \delta, \mathbf{p}, \lambda) - C_{par}(t, \varphi, f, \delta, \mathbf{p}, \lambda) \geq 0; \quad t \in [0, 1],$$

$$J_6(\varphi) = J_6(t, \varphi, f, \delta, \mathbf{p}, \lambda) := C_{par}(t, \varphi, f, \delta, \mathbf{p}, \lambda) - \varphi \left(\int_Z f d\delta \right) \geq 0; \quad t \in [0, 1].$$

2 Extensions of cyclic refinements of Jensen's inequality by Hermite interpolation

We start this section by considering the discrete as well as continuous version of cyclic refinements of Jensen's inequality and construct the generalized new identities having real weights utilizing Hermite interpolating polynomial.

Theorem 2.1. Let $m, k \in \mathbb{N}$, $p_1, \dots, p_m$ and $\lambda_1, \dots, \lambda_k$ be real tuples for $2 \leq k \leq m$, such that $\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \neq 0$ for $u = 1, \dots, m$ with $\sum_{u=1}^m p_u = 1$ and $\sum_{v=1}^k \lambda_v = 1$. Also let $z \in [\alpha_1, \alpha_2] \subset \mathbb{R}$ and $\mathbf{z} \in [\alpha_1, \alpha_2]^m$. Assume $\varphi \in C^n[\alpha_1, \alpha_2]$ and consider the interval with points $-\infty < \alpha_1 = b_1 < b_2 \dots < b_t = \alpha_2 < \infty$, ($t \geq 2$) such that $\varphi(\alpha_1) = \varphi(\alpha_2)$, $\varphi'(\alpha_1) = 0 = \varphi'(\alpha_2)$, $H_{\sigma\omega}$ in (3) are Hermite basis, $G_{H,n}(z, r)$ in (4) be the Hermite-Green function and G_j , ($j = 1, \dots, 5$) be the Green functions defined in (14)–(18), respectively.

For the linear functionals $J_i(\cdot)$ ($i = 1, 2$), assume further that $\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} z_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \in [\alpha_1, \alpha_2]$ for $u = 1, \dots, m$.

For the linear functionals $J_i(\cdot)$ ($i = 3, \dots, 6$), assume further that $\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} f(z_{u+v})}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \in [\alpha_1, \alpha_2]$ for $u = 1, \dots, m$.

Then for $i = 1, \dots, 6$, we have the following generalized identities:

(a)

$$J_i(\varphi(z)) = \sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \varphi^{(\sigma)}(b_\omega) J_i(H_{\sigma\omega}(z)) + \int_{\alpha_1}^{\alpha_2} J_i(G_{H,n}(z, r)) \varphi^{(n)}(r) dr. \quad (27)$$

(b)

$$J_i(\varphi(z)) = \int_{\alpha_1}^{\alpha_2} J_i(G_j(z, r)) \sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \varphi^{(\sigma+2)}(b_\omega) H_{\sigma\omega}(r) dr + \int_{\alpha_1}^{\alpha_2} \int_{\alpha_1}^{\alpha_2} J_i(G_j(z, r)) G_{H,n-2}(r, \xi) \varphi^{(n)}(\xi) d\xi dr. \quad (28)$$

Proof. Fix $i = 1, \dots, 6$.

- (a) Applying cyclic Jensen's type linear functionals $J_i(\cdot)$ on (2) and practicing properties of the functional, we get (27).
 (b) For fixed $j = 2$, testing (20) in cyclic Jensen's type functional $J_i(\cdot)$ and employing the properties of $J_i(\cdot)$ along with the assumed conditions, we have

$$\begin{aligned} J_i(\varphi) &= J_i(\varphi(\alpha_1)) + J_i((z - \alpha_1)\varphi'(\alpha_2)) + \int_{\alpha_1}^{\alpha_2} J_i(G_2(z, r)) \varphi''(r) dr \\ &= \varphi'(\alpha_2) J_i(z) + \int_{\alpha_1}^{\alpha_2} J_i(G_2(z, r)) \varphi''(r) dr = \int_{\alpha_1}^{\alpha_2} J_i(G_2(z, r)) \varphi''(r) dr. \end{aligned} \quad (29)$$

By Theorem 1.1, $\varphi''(r)$ can be expressed as:

$$\varphi''(r) = \sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} H_{\sigma\omega}(r) \varphi^{(\sigma+2)}(b_\omega) + \int_{\alpha_1}^{\alpha_2} G_{H,n-2}(r, \xi) \varphi^{(n)}(\xi) d\xi. \quad (30)$$

Putting (30) in (29), we get (28) respectively for $j = 2$ and $i = 1, \dots, 6$. The cases for $j = 1, 3, 4, 5$, are treated analogously. $\square$

Now we obtain extensions and improvements of discrete and integral cyclic Jensen type linear functionals, with real weights.

Theorem 2.2. Consider φ be an n -convex function along with the suppositions of Theorem 2.1. Then we conclude the following results:

(a) If for all $i = 1, \dots, 6$,

$$J_i\left(G_{H,n}(z, r)\right) \geq 0, \quad r \in [\alpha_1, \alpha_2] \quad (31)$$

holds, then we have

$$J_i(\varphi(z)) \geq \sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \varphi^{(\sigma)}(b_\omega) J_i\left(H_{\sigma\omega}(z)\right) \quad (32)$$

for $i = 1, \dots, 6$.

(b) If for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$

$$J_i\left(G_j(z, r)\right) \geq 0, \quad r \in [\alpha_1, \alpha_2] \quad (33)$$

holds, provided that s_ω is odd for each $\omega = 2, 3, 4, \dots, t$, then

$$J_i(\varphi(z)) \geq \int_{\alpha_1}^{\alpha_2} J_i\left(G_j(z, r)\right) \sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \varphi^{(\sigma+2)}(b_\omega) H_{\sigma\omega}(r) dr \quad (34)$$

for $i = 1, \dots, 6$.

(c) If (33) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$, provided that s_ω is odd for each $\omega = 2, 3, 4, \dots, t-1$ and s_t is even, then (34) holds in reverse direction for $i = 1, \dots, 6$.

Proof. (a) Fix $i = 1, \dots, 6$.

As the function $\varphi \in C^n[\alpha_1, \alpha_2]$ and assumed to be n -convex, we use the characterization of n -convex function $\varphi^{(n)}(z) \geq 0$ for all $z \in [\alpha_1, \alpha_2]$ (see [10], p. 16). Hence we can apply Theorem 2.1(a) to obtain (32).

(b) Fix $i = 1, \dots, 6$ and $j = 1, \dots, 5$.

As we have discussed in part(a) $\varphi^{(n)}(z) \geq 0$ for all $z \in [\alpha_1, \alpha_2]$. Also as it is given that s_ω is odd for each $\omega = 2, 3, 4, \dots, t$, we have $\vartheta(r) = \prod_{\omega=1}^t (r - b_\omega)^{s_\omega+1} \geq 0$ for any $r \in [\alpha_1, \alpha_2]$. Therefore taking into account Lemma 1.3 (i) we have $G_{H,n-2}(r, \xi) \geq 0$. Thus by applying Theorem 2.1(b) result (34) is obtained.

(c) If s_t is even then $(r - b_t)^{s_t+1} \leq 0$ for any $r \in [\alpha_1, \alpha_2]$. Also clearly $(r - b_1)^{s_1+1} \geq 0$ for any $r \in [\alpha_1, \alpha_2]$ and $\prod_{\omega=2}^{t-1} (r - b_\omega)^{s_\omega+1} \geq 0$ for $r \in [\alpha_1, \alpha_2]$ if s_ω is odd for each $\omega = 2, 3, 4, \dots, t-1$. Therefore summing this

information $\vartheta(r) = \prod_{\omega=1}^t (r - b_\omega)^{s_\omega+1} \leq 0$ for any $r \in [\alpha_1, \alpha_2]$ and taking into account Lemma 1.3 (i) we have $G_{H,n-2}(r, \xi) \leq 0$. Thus applying Theorem 2.1(b) gives (34) in reverse direction. $\square$

By using **Type**($\eta, n - \eta$)**C**: ($t = 2, 1 \leq \eta \leq n - 1, s_1 = \eta - 1, s_2 = n - \eta - 1$) we give the following result:

Corollary 2.3. Consider φ be n -convex function along with the suppositions of Theorem 2.1. Let $G_{(\eta,n)}$ be the Green's function defined in (10) and $\tau_\sigma, \zeta_\sigma$ be as defined (8), (9) respectively. Then we conclude the following results:

(a) If for all $i = 1, \dots, 6$,

$$J_i\left(G_{(\eta,n)}(z, r)\right) \geq 0, \quad r \in [\alpha_1, \alpha_2] \quad (35)$$

holds, then we have

$$J_i(\varphi(z)) \geq \sum_{\sigma=0}^{\eta-1} \varphi^{(\sigma)}(\alpha_1) J_i\left(\tau_\sigma(z)\right) + \sum_{\sigma=0}^{n-\eta-1} \varphi^{(\sigma)}(\alpha_2) J_i\left(\zeta_\sigma(z)\right) \quad (36)$$

for $i = 1, \dots, 6$.

(b) If (33) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$, provided that $n - \eta$ is even, then

$$J_i(\varphi(z)) \geq \int_{\alpha_1}^{\alpha_2} J_i(G_j(z, r)) \left(\sum_{\sigma=0}^{\eta-1} \varphi^{(\sigma+2)}(\alpha_1) \tau_\sigma(r) + \sum_{\sigma=0}^{n-\eta-1} \varphi^{(\sigma+2)}(\alpha_2) \zeta_\sigma(r) \right) dr \quad (37)$$

for $i = 1, \dots, 6$.

(c) If (33) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$, provided that $n - \eta$ is odd then (37) holds in reverse direction for $i = 1, \dots, 6$.

By using **Two-point TC**: ($n = 2\eta$, $t = 2$, $s_1 = s_2 = \eta - 1$) we get the following result:

Corollary 2.4. Consider φ be n -convex function along with the suppositions of Theorem 2.1. Let G_{2T} be the Green's function defined in (13), then we conclude the following results:

(a) If for all $i = 1, \dots, 6$,

$$J_i(G_{2T}(z, r)) \geq 0, \quad r \in [\alpha_1, \alpha_2] \quad (38)$$

holds, then we have

$$J_i(\varphi(z)) \geq \sum_{\sigma=0}^{\eta-1} \sum_{s=0}^{\eta-1-\sigma} \binom{\eta+s-1}{s} \left[\varphi^{(\sigma)}(\alpha_1) J_i \left(\frac{(z-\alpha_1)^\sigma}{\sigma!} \left(\frac{z-\alpha_2}{\alpha_1-\alpha_2} \right)^\eta \left(\frac{z-\alpha_1}{\alpha_2-\alpha_1} \right)^s \right) + \varphi^{(\sigma)}(\alpha_2) J_i \left(\frac{(z-\alpha_2)^\sigma}{\sigma!} \left(\frac{z-\alpha_1}{\alpha_2-\alpha_1} \right)^\eta \left(\frac{z-\alpha_2}{\alpha_1-\alpha_2} \right)^s \right) \right] \quad (39)$$

for $i = 1, \dots, 6$.

(b) If (33) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$, provided that η is even, then

$$J_i(\varphi(z)) \geq \int_{\alpha_1}^{\alpha_2} J_i(G_j(z, r)) \sum_{\sigma=0}^{\eta-1} \sum_{s=0}^{\eta-1-\sigma} \binom{\eta+s-1}{s} \left[\frac{(r-\alpha_1)^\sigma}{\sigma!} \left(\frac{r-\alpha_2}{\alpha_1-\alpha_2} \right)^\eta \left(\frac{r-\alpha_1}{\alpha_2-\alpha_1} \right)^s \varphi^{(\sigma)}(\alpha_1) + \frac{(r-\alpha_2)^\sigma}{\sigma!} \left(\frac{r-\alpha_1}{\alpha_2-\alpha_1} \right)^\eta \left(\frac{r-\alpha_2}{\alpha_1-\alpha_2} \right)^s \varphi^{(\sigma)}(\alpha_2) \right] dr. \quad (40)$$

for $i = 1, \dots, 6$.

(c) If (33) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$, provided that η is odd then (40) holds in reverse direction for $i = 1, \dots, 6$.

We will finish the present section by the following generalizations of cyclic refinements of Jensen inequalities:

Theorem 2.5. If the assumptions of Theorem 2.1 are fulfilled with additional conditions that $p_1, \dots, p_m$ and $\lambda_1, \dots, \lambda_k$ are nonnegative tuples for $2 \leq k \leq m$, such that $\sum_{u=1}^m p_u = 1$ and $\sum_{v=1}^k \lambda_v = 1$. Then for $\varphi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ being n -convex function, we conclude the following results:

(a) If (32) is valid and the function

$$\Gamma(z) := \sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} H_{\sigma\omega}(z) \varphi^{(\sigma)}(b_\omega). \quad (41)$$

is convex, the right side of (32) is nonnegative, implying

$$J_i(\varphi) \geq 0, \quad i = 1, \dots, 6. \quad (42)$$

(b) If s_ω is odd for each $\omega = 2, 3, 4, \dots, t$, then (34) holds. Further

$$\sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} H_{\sigma\omega}(r) \varphi^{(\sigma+2)}(b_\omega) \geq 0. \quad (43)$$

the right side of (34) is nonnegative, particularly (42) is established for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$.

(c) Inequality (34) holds reversely if s_ω is odd for each $\omega = 2, 3, 4, \dots, t-1$ and s_t is even. Moreover, let (43) hold in reverse direction, then the reverse of (42) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$.

Proof. (a) Fix $i = 1, \dots, 6$.

As (32) is valid, the R.H.S. of (32) can be written in the functional form $J_i(\Gamma)$ for all $(i = 1, \dots, 6)$, after reorganizing this side. Employing Remark 1.9 the nonnegativity of the R.H.S. of (32) is secure, especially (42) is established.

(b) Fix $i = 1, \dots, 6$.

We have assumed positive weights and for all $j = 1, \dots, 5$, $G_j(z, r)$ is convex. Thus by practicing Remark 1.9, $J_i(G_j(z, r)) \geq 0$. As φ is n -convex and s_ω odd for each $\omega = 2, 3, 4, \dots, t$, hence by following Theorem 2.2 (b), we obtain (34). Now taking into account the positivity of $J_i(G_j(z, r))$ and (41), we get (42).

(c) Similar to the proof of (b) □

Using **Type**($\eta, n - \eta$)**C**., we have the following result:

Corollary 2.6. Under the assumptions of Theorem 2.5, if $\varphi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ being an n -convex function, we conclude the following results:

(a) If (36) is valid along with the function

$$\Theta(z) := \sum_{\sigma=0}^{\eta-1} \tau_\sigma(z) \varphi^{(\sigma)}(\alpha_1) + \sum_{\sigma=0}^{n-\eta-1} \zeta_\sigma(z) \varphi^{(\sigma)}(\alpha_2), \quad (44)$$

to be convex, the right side of (36) is nonnegative, implying that (42) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$.

(b) If $n - \eta$ is even, (37) holds. Further

$$\sum_{\sigma=0}^{\eta-1} \varphi^{(\sigma+2)}(\alpha_1) \tau_\sigma(r) + \sum_{\sigma=0}^{n-\eta-1} \varphi^{(\sigma+2)}(\alpha_2) \zeta_\sigma(r) \geq 0, \quad (45)$$

the right side of (37) is nonnegative, particularly (42) is established for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$.

(c) Inequality (37) holds reversely if $n - \eta$ is odd. Moreover, let (45) hold in reverse direction. Then the reverse of (42) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$.

Proof. Similar to Theorem 2.5. □

Using **Two-point TC**, we have the following result:

Corollary 2.7. Under the assumptions of Theorem 2.5, if $\varphi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ being n -convex function, we conclude the following results:

(a) If (39) is valid along with the function

$$Y(z) = \sum_{\sigma=0}^{\eta-1} \sum_{s=0}^{\eta-1-\sigma} \binom{\eta+s-1}{s} \left[\frac{(z-\alpha_1)^\sigma}{\sigma!} \left(\frac{z-\alpha_2}{\alpha_1-\alpha_2} \right)^\eta \left(\frac{z-\alpha_1}{\alpha_2-\alpha_1} \right)^s \varphi^{(\sigma)}(\alpha_1) \right. \\ \left. + \frac{(z-\alpha_2)^\sigma}{\sigma!} \left(\frac{z-\alpha_1}{\alpha_2-\alpha_1} \right)^\eta \left(\frac{z-\alpha_2}{\alpha_1-\alpha_2} \right)^s \varphi^{(\sigma)}(\alpha_2) \right] \quad (46)$$

to be convex, the right side of (39) is nonnegative, implying that (42) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$.

(b) If η is even, (40) holds. Further

$$\sum_{\sigma=0}^{\eta-1} \sum_{s=0}^{\eta-1-\sigma} \binom{\eta+s-1}{s} \left[\frac{(r-\alpha_1)^\sigma}{\sigma!} \left(\frac{r-\alpha_2}{\alpha_1-\alpha_2} \right)^\eta \left(\frac{r-\alpha_1}{\alpha_2-\alpha_1} \right)^s \varphi^{(\sigma)}(\alpha_1) + \frac{(r-\alpha_2)^\sigma}{\sigma!} \left(\frac{r-\alpha_1}{\alpha_2-\alpha_1} \right)^\eta \left(\frac{r-\alpha_2}{\alpha_1-\alpha_2} \right)^s \varphi^{(\sigma)}(\alpha_2) \right] \geq 0. \quad (47)$$

the right side of (40) is nonnegative, particularly (42) is established for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$.

(c) Inequality (40) holds reversely if η is odd. Moreover, let (47) hold in reverse direction. Then the reverse of (42) holds for all $i = 1, \dots, 6$ and $j = 1, \dots, 5$.

Proof. Similar to Theorem 2.5. □

3 Monotonicity of functionals for n -convex functions at a point

Pečarić et al. in [11] introduced a new class of functions that are n -convex at a point.

Definition 1. Let $\varphi : I \rightarrow \mathbb{R}$ be a function and d be any point in the interior of I . φ is said to be $(n+1)$ -convex at point d if there exists a constant C in such a way that the function

$$\Phi(z) = \varphi(z) - \frac{C}{n!} z^n \quad (48)$$

is n -concave on $I \cap (-\infty, d]$ and n -convex on $I \cap [d, \infty)$. A function φ is said to be $(n+1)$ -concave at point d if the function $-\varphi$ is $(n+1)$ -convex at point d .

Witkowski et al. in [11] deduced the necessary and sufficient conditions on two linear functionals

$$\Gamma : C([\alpha_1, d]) \rightarrow \mathbb{R}$$

and

$$Y : C([d, \alpha_2]) \rightarrow \mathbb{R},$$

such that the inequality $\Gamma(\varphi) \leq Y(\varphi)$ holds for every function φ that is n -convex at point d . In the present section we will give inequalities of such type for the particular linear functionals obtained from the inequalities in the previous section. Let ζ^n denote the monomials $\zeta^n(x) = x^n$, $n \in \mathbb{N}_0$. For the rest of this section, $\Gamma_i(\varphi_{[\alpha_1, d]})$ and $Y_i(\varphi_{[d, \alpha_2]})$ will denote the linear functionals obtained as the difference of the L. H. S. and R. H. S. of inequality (32), where $i = 1, \dots, 6$, applied to the intervals $[\alpha_1, d]$ and $[d, \alpha_2]$ respectively, i. e., for interval $\alpha_1 = b_1 < b_2 \cdots < b_t = d$, ($t \geq 2$) let $\mathbf{z} \in [\alpha_1, d]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^k$ and for interval $d = \tilde{b}_1 < \tilde{b}_2 \cdots < \tilde{b}_t = \alpha_2$, ($t \geq 2$) let $\mathbf{y} \in [d, \alpha_2]^{\tilde{m}}$, $\mathbf{q} \in \mathbb{R}^{\tilde{m}}$, $\tilde{\lambda} \in \mathbb{R}^{\tilde{k}}$ let

$$\Gamma_i(\varphi_{[\alpha_1, d]}) := J_i(\varphi(z)) - \sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \varphi^{(\sigma)}(b_\omega) J_i \left(H_{\sigma\omega}(z) \right), \quad (49)$$

$$Y_i(\varphi_{[d, \alpha_2]}) := J_i(\varphi(y)) - \sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \varphi^{(\sigma)}(\tilde{b}_\omega) J_i \left(H_{\sigma\omega}(y) \right). \quad (50)$$

It is significant to reveal that by constructing new linear functionals $\Gamma_i(\varphi_{[\alpha_1, d]})$ and $Y_i(\varphi_{[d, \alpha_2]})$, identity (27) for $i = 1, \dots, 6$, enforced to the corresponding intervals $[\alpha_1, d]$ and $[d, \alpha_2]$ takes the shape:

$$\Gamma_i(\varphi_{[\alpha_1, d]}) = \int_{\alpha_1}^d J_i \left(G_{H,n}(z, r) \right) \varphi^{(n)}(r) dr, \quad (51)$$

$$Y_i(\varphi_{[d, \alpha_2]}) = \int_d^{\alpha_2} J_i \left(G_{H,n}(y, r) \right) \varphi^{(n)}(r) dr. \quad (52)$$

Promptly, we are in a position to state the following theorem :

Theorem 3.1. For the interval $\alpha_1 = b_1 < b_2 \cdots < b_t = d$, ($t \geq 2$) consider $\mathbf{z} \in [\alpha_1, d]^m$, $\mathbf{p} \in \mathbb{R}^m$, $\lambda \in \mathbb{R}^k$ and for interval $d = \tilde{b}_1 < \tilde{b}_2 \cdots < \tilde{b}_t = \alpha_2$, ($t \geq 2$) consider $\mathbf{y} \in [d, \alpha_2]^m$, $\mathbf{q} \in \mathbb{R}^m$, $\tilde{\lambda} \in \mathbb{R}^k$ be such that

For $i = 1, \dots, 6$, assume

$$J_i \left(G_{H,n}(z, r) \right) \geq 0, \quad r \in [\alpha_1, d], \quad (53)$$

$$J_i \left(G_{H,n}(y, r) \right) \geq 0, \quad r \in [d, \alpha_2], \quad (54)$$

$$\int_{\alpha_1}^d J_i \left(G_{H,n}(z, r) \right) dr = \int_d^{\alpha_2} J_i \left(G_{H,n}(y, r) \right) dr \quad (55)$$

where $G_{H,n}(z, r)$ is the Hermite-Green function given in (4) and let $\Gamma_i(\varphi_{[\alpha_1, d]})$ and $Y_i(\varphi_{[d, \alpha_2]})$ be the linear functionals introduced in (49) and (50). If $\varphi : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}$ is $(n+1)$ -convex at point d , then

$$\Gamma_i(\varphi_{[\alpha_1, d]}) \leq Y_i(\varphi_{[d, \alpha_2]}) \quad \text{for } i = 1, \dots, 6. \quad (56)$$

Proof. Fix $i = 1, \dots, 6$. Employing Definition 1, set up function $\Phi = \varphi - \frac{C}{n!} \zeta^n$ in such a way that the function Φ is n -concave on $[\alpha_1, d]$ and n -convex on $[d, \alpha_2]$. Executing Theorem 2.2 to Φ on the interval $[\alpha_1, d]$, we have

$$0 \geq \Gamma_i(\Phi) = \Gamma_i(\varphi_{[\alpha_1, d]}) - \frac{C}{(n)!} \Gamma_i(\zeta_{[\alpha_1, d]}^n). \quad (57)$$

Identically practicing Theorem 2.2 to Φ on the interval $[d, \alpha_2]$, we get

$$0 \leq Y_i(\Phi) = Y_i(\varphi_{[d, \alpha_2]}) - \frac{C}{(n)!} Y_i(\zeta_{[d, \alpha_2]}^n). \quad (58)$$

Furthermore, using monomials $\zeta^n(\cdot)$ in identities (51) and (52) give

$$\Gamma_i(\zeta_{[\alpha_1, d]}^n) = n! \int_{\alpha_1}^d J_i \left(G_{H,n}(z, r) \right) dr, \quad (59)$$

$$Y_i(\zeta_{[d, \alpha_2]}^n) = n! \int_d^{\alpha_2} J_i \left(G_{H,n}(y, r) \right) dr. \quad (60)$$

Therefore assumption (55) is equivalent to

$$\Gamma_i(\zeta_{[\alpha_1, d]}^n) = Y_i(\zeta_{[d, \alpha_2]}^n).$$

So from (57) and (58), one can get

$$\Gamma_i(\varphi_{[\alpha_1, d]}) \leq \frac{C}{(n)!} \Gamma_i(\zeta_{[\alpha_1, d]}^n) = \frac{C}{(n)!} Y_i(\zeta_{[d, \alpha_2]}^n) \leq Y_i(\varphi_{[d, \alpha_2]}). \quad (61)$$

So (56) is established for $i = 1, \dots, 6$. □

We conclude the following significant remarks:

Remark 3.2. In fact, inequality (56) still holds if we replace assumption (55) with the weaker assumption that $C \left(Y_i(\zeta_{[d, \alpha_2]}^n) - \Gamma_i(\zeta_{[\alpha_1, d]}^n) \right) \geq 0$ for $i = 1, \dots, 6$.

Remark 3.3. We can also establish the monotonicity of the linear functionals coming from differences of inequalities (34) by constructing their respective conditions and following Theorem 3.1.

Remark 3.4. In a similar fashion one can also construct related inequalities for n -convex function at a point of cyclic refinements of Jensen inequalities by considering inequalities obtained for **Type**(η , $n-\eta$)**C** and **Two-point TC** in Corollaries 2.3 and 2.4, respectively.

4 Applications to information theory

Let $\varphi : (0, \infty) \rightarrow (0, \infty)$ be a convex function with $p := (p_1, \dots, p_m)$ and $q := (q_1, \dots, q_m)$ be positive probability distributions. Then φ -divergence functional is defined in [12, 13] as follows

$$I_\varphi(\mathbf{p}, \mathbf{q}) = \sum_{u=1}^m q_u \varphi \left(\frac{p_u}{q_u} \right).$$

Surveying the classical Csiszár divergence functional, we propose a new functional:

Definition 2. Let $\varphi : I \rightarrow \mathbb{R}$ be a function with I an interval in $\mathbb{R}$. Let $\mathbf{p} := (p_1, \dots, p_m) \in \mathbb{R}^m$, and $\mathbf{q} := (q_1, \dots, q_m) \in]0, \infty[^m$ such that

$$\frac{p_u}{q_u} \in I, \quad u = 1, \dots, m.$$

Then let

$$\tilde{I}_\varphi(\mathbf{p}, \mathbf{q}) = \sum_{u=1}^m q_u \varphi \left(\frac{p_u}{q_u} \right). \quad (62)$$

Remark 4.1. Now as a consequence of Theorem 2.2 we consider the discrete extensions of cyclic refinements of Jensen's inequalities for ($i = 1$), from (32) with respect to n -convex function φ in the explicit form:

$$\begin{aligned} \sum_{u=1}^m p_u \varphi(z_u) - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \right) \varphi \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} z_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \right) &\geq \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \varphi^{(\sigma)}(b_\omega) \right) \\ &\times \left(\sum_{u=1}^m p_u H_{\sigma\omega}(z_u) - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} \right) H_{\sigma\omega} \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v} z_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \right) \right), \end{aligned} \quad (63)$$

where $H_{\sigma\omega}$ are Hermite basis defined in (3).

Theorem 4.2. Let $m, k \in \mathbb{N}$ ($2 \leq k \leq m$), $\lambda_1, \dots, \lambda_k$ be positive probability distributions. Let $\mathbf{p} := (p_1, \dots, p_m) \in \mathbb{R}^m$, and $\mathbf{q} := (q_1, \dots, q_m) \in (0, \infty)^m$ such that

$$\frac{p_u}{q_u} \in [\alpha_1, \alpha_2], \quad u = 1, \dots, m.$$

Also let $\varphi \in C^n[\alpha_1, \alpha_2]$ and consider the interval with points $-\infty < \alpha_1 = b_1 < b_2 \cdots < b_t = \alpha_2 < \infty$, ($t \geq 2$) such that φ is an n -convex function. Then the following inequality holds:

$$\begin{aligned} \tilde{I}_\varphi(\mathbf{p}, \mathbf{q}) \geq & \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \varphi \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) + \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \varphi^{(\sigma)}(b_\omega) \right) \\ & \times \left(\sum_{u=1}^m q_u H_{\sigma\omega} \left(\frac{p_u}{q_u} \right) - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) H_{\sigma\omega} \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) \right). \end{aligned} \quad (64)$$

Proof. Replacing p_u with q_u and z_u with $\frac{p_u}{q_u}$ for $(u = 1, \dots, m)$ in (63), we get (65). $\square$

Theorem 4.3. Let $m, k \in \mathbb{N}$ ($2 \leq k \leq m$), $\lambda_1, \dots, \lambda_k$ be positive probability distributions. Let $\mathbf{p} := (p_1, \dots, p_m) \in \mathbb{R}^m$, and $\mathbf{q} := (q_1, \dots, q_m) \in (0, \infty)^m$ such that

$$\frac{p_u}{q_u} \in [\alpha_1, \alpha_2], \quad u = 1, \dots, m.$$

Also let $\varphi \in C^n[\alpha_1, \alpha_2]$ and consider interval with points $-\infty < \alpha_1 = b_1 < b_2 \cdots < b_t = \alpha_2 < \infty$, ($t \geq 2$) such that φ is n -convex function. Then the following inequality holds:

$$\begin{aligned} \tilde{I}_\varphi(\mathbf{p}, \mathbf{q}) \geq & \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \varphi \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) + \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \varphi^{(\sigma)}(b_\omega) \right) \\ & \times \left(\sum_{u=1}^m q_u H_{\sigma\omega} \left(\frac{p_u}{q_u} \right) - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) H_{\sigma\omega} \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) \right). \end{aligned} \quad (65)$$

Proof. Replacing p_u with q_u and z_u with $\frac{p_u}{q_u}$ for $(u = 1, \dots, m)$ in (63), we get (65). $\square$

We now explore two exceptional cases of the previous result.

One corresponds to the entropy of a discrete probability distribution.

Shannon entropy and related measures are increasingly used in molecular ecology and population genetics, information theory, dynamical systems and statistical physics (see [14, 15]. For positive m -tuple $\mathbf{q} = (q_1, \dots, q_m)$ such that $\sum_{u=1}^m q_u = 1$, the **Shannon entropy** is defined by

$$S(\mathbf{q}) = - \sum_{u=1}^m q_u \ln q_u. \quad (66)$$

Corollary 4.4. Let $m, k \in \mathbb{N}$ ($2 \leq k \leq m$), $\lambda_1, \dots, \lambda_k$ be positive probability distributions.

(a) If $\mathbf{q} := (q_1, \dots, q_m) \in (0, \infty)^m$ and ($n = \text{even}$), then

$$\begin{aligned} \sum_{u=1}^m q_u \ln q_u \geq & \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \ln \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) + \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \frac{(-1)^\sigma (\sigma-1)!}{(b_\omega)^\sigma} \right) \\ & \times \left(\sum_{u=1}^m q_u H_{\sigma\omega} \left(\frac{1}{q_u} \right) - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) H_{\sigma\omega} \left(\frac{1}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) \right). \end{aligned} \quad (67)$$

(b) If $\mathbf{q} := (q_1, \dots, q_m)$ is a positive probability distribution and ($n = \text{even}$), then we get the bounds for the Shannon entropy of $\mathbf{q}$.

$$S(\mathbf{q}) \leq - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \ln \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) - \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_{\omega}} \frac{(-1)^{\sigma} (\sigma-1)!}{(b_{\omega})^{\sigma}} \right) \times \left(\sum_{u=1}^m q_u H_{\sigma\omega} \left(\frac{1}{q_u} \right) - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) H_{\sigma\omega} \left(\frac{1}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) \right). \quad (68)$$

If ($n = \text{odd}$), then (67) and (68) hold in reverse directions.

Proof. (a) Using $\varphi(x) := -\ln x$, and $\mathbf{p} := (1, 1, \dots, 1)$ in Theorem 4.3, we get the required results.

(b) It is a specific case of (a). $\square$

The second case corresponds to the relative entropy or Kullback–Leibler divergence between two probability distributions. One of the best known distance function used in mathematical statistics [16–18], information theory [19, 20] and signal processing [21, 22] is Kullback-Leibler distance. The **Kullback-Leibler** distance [23, 24] between the positive probability distributions $\mathbf{p} = (p_1, \dots, p_m)$ and $\mathbf{q} = (q_1, \dots, q_m)$ is defined by

$$D(\mathbf{q} \parallel \mathbf{p}) = \sum_{u=1}^m q_u \ln \left(\frac{q_u}{p_u} \right). \quad (69)$$

Corollary 4.5. Let $m, k \in \mathbb{N}$ ($2 \leq k \leq m$), $\lambda_1, \dots, \lambda_k$ be positive probability distributions.

(a) If $\mathbf{q} := (q_1, \dots, q_m)$, $\mathbf{p} := (p_1, \dots, p_m) \in (0, \infty)^m$ and ($n = \text{even}$), then

$$\sum_{u=1}^m q_u \ln \left(\frac{q_u}{p_u} \right) \geq \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \ln \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \right) + \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_{\omega}} \frac{(-1)^{\sigma} (\sigma-1)!}{(b_{\omega})^{\sigma}} \right) \times \left(\sum_{u=1}^m q_u H_{\sigma\omega} \left(\frac{p_u}{q_u} \right) - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) H_{\sigma\omega} \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) \right). \quad (70)$$

(b) If $\mathbf{q} := (q_1, \dots, q_m)$, $\mathbf{p} := (p_1, \dots, p_m)$ are positive probability distributions and ($n = \text{even}$), then we have

$$D(\mathbf{q} \parallel \mathbf{p}) \geq \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) \ln \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}} \right) + \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_{\omega}} \frac{(-1)^{\sigma} (\sigma-1)!}{(b_{\omega})^{\sigma}} \right) \times \left(\sum_{u=1}^m q_u H_{\sigma\omega} \left(\frac{p_u}{q_u} \right) - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v} \right) H_{\sigma\omega} \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} p_{u+v}}{\sum_{v=0}^{k-1} \lambda_{v+1} q_{u+v}} \right) \right). \quad (71)$$

If ($n = \text{odd}$), then (70) and (71) hold in reverse directions.

Proof. (a) Using $\varphi(x) := -\ln x$ in Theorem 4.3, we get the desired results.

(b) It is particular case of (a). $\square$

Zipf's law is one of the basic laws in information science and is extensively applied in linguistics. It was named by George Zipf (1932), who discovered the counting problem of each word appearing in the text. If we

ranked (r) word according to the frequency of word occurrence (φ), the product of these two numbers would be a constant (C):

$$C = r\varphi.$$

Apart from the use of this law in information science and linguistics, Zipf's law has a mythical impact in economics. This distribution in economics is known as Pareto's law which analyzes the distribution of the wealthiest members of the community [[25], p. 125]. These two laws are the same in the mathematical sense, but they are applied in a different context [[26], p. 294].

For $m \in \{1, 2, \dots\}$, $c \geq 0$ and $d > 0$ the **Zipf-Mandelbrot** law (probability mass function) is stated as

$$\varphi(u; m, c, d) = \frac{1}{((u+c)^d H_{m,c,d})}, \quad u = (1, 2, \dots, m) \quad (72)$$

where

$$H_{m,c,d} = \sum_{k=1}^m \frac{1}{(k+c)^d}.$$

The probability mass function can be given as in the (72) and $H_{m,c,d}$ which may be thought of as a generalization of a harmonic number. In the formula, u is the rank of the data, and c and d are parameters of the distribution. In the limit as n approaches infinity, this becomes the Hurwitz zeta function $\zeta(s, t)$. For finite m and $c = 0$ the Zipf-Mandelbrot law becomes Zipf's law. For infinite m and $c = 0$ it becomes a Zeta distribution.

Let $m \in \{1, 2, \dots\}$, $c \geq 0$, $d > 0$, then **Zipf-Mandelbrot entropy** can be given as :

$$Z(H, c, d) = \frac{d}{H_{m,c,d}} \sum_{u=1}^m \frac{\ln(u+c)}{(u+c)^d} + \ln(H_{m,c,d}). \quad (73)$$

Consider

$$q_u = \varphi(u; m, c, d) = \frac{1}{((u+c)^d H_{m,c,d})}. \quad (74)$$

Application of Zipf-Mandelbrot law can be found in linguistics [27–29], information sciences [30] and also is often applicable in ecological field studies [29, 31]. Now we state our results involving entropy introduced by Mandelbrot Law:

Theorem 4.6. Let $m, k \in \mathbb{N}$ ($2 \leq k \leq m$), $\lambda_1, \dots, \lambda_k$ be positive probability distributions and $\mathbf{q}$ be as defined in (74) by Zipf-Mandelbrot law with parameters $m \in \{1, 2, \dots\}$, $c \geq 0$, $d > 0$. For ($n = \text{even}$), the following holds

$$\begin{aligned} S(\mathbf{q}) = Z(H, c, d) \leq & - \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{((u+v+c)^d H_{m,c,d})} \right) \ln \left(\frac{1}{H_{m,c,d}} \sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{((u+v+t)^d)} \right) \\ & - \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \frac{(-1)^\sigma (\sigma-1)!}{(b_\omega)^\sigma} \right) \left(\sum_{u=1}^m \frac{1}{((u+c)^d H_{m,c,d})} H_{\sigma\omega} \left(((u+c)^d H_{m,c,d}) \right) \right) \\ & + \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \frac{(-1)^\sigma (\sigma-1)!}{(b_\omega)^\sigma} \right) \left(\sum_{u=1}^m \left(\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{((u+v+c)^d H_{m,c,d})} \right) H_{\sigma\omega} \left(\frac{1}{\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{((u+v+c)^d H_{m,c,d})}} \right) \right). \quad (75) \end{aligned}$$

If ($n = \text{odd}$), then (75) holds in reverse direction.

Proof. Substituting this $q_u = \frac{1}{((u+c)^d H_{m,c,d})}$ in Corollary 4.4(b), we get the desired result. It is interesting to see that $\sum_{u=1}^m q_u = 1$. Moreover, using the above q_u in Shannon entropy (66), we get Mandelbrot entropy (73)

$$\begin{aligned} S(q) &= -q_u \ln q_u = -\sum_{u=1}^m \frac{1}{((u+c)^d H_{m,c,d})} \ln \frac{1}{((u+c)^d H_{m,c,d})} \\ &= \frac{-1}{(H_{m,c,d})} \sum_{u=1}^m \frac{1}{(u+c)^d} \ln \frac{1}{(u+c)^d H_{m,c,d}} \\ &= \frac{-1}{(H_{m,c,d})} \sum_{u=1}^m \frac{1}{(u+c)^d} \left(\ln(1) - s \ln(u+c) - \ln(H_{m,c,d}) \right) \\ &= \frac{1}{(H_{m,c,d})} \sum_{u=1}^m \frac{1}{(u+c)^d} \left(s \ln(u+c) + \ln(H_{m,c,d}) \right) \\ &= \frac{s}{(H_{m,c,d})} \sum_{u=1}^m \frac{\ln(u+c)}{(u+c)^d} + \ln(H_{m,c,d}). \end{aligned} \quad (76)$$

□

Corollary 4.7. Let $m, k \in \mathbb{N}$ ($2 \leq k \leq m$), $\lambda_1, \dots, \lambda_k$ be positive probability distributions and for $c_1, c_2 \in [0, \infty)$, $d_1, d_2 > 0$, let $H_{m,c_1,d_1} = \frac{1}{(u+c_1)^{d_1}}$ and $H_{m,c_2,d_2} = \frac{1}{(u+c_2)^{d_2}}$. Now using $q_u = \frac{1}{(u+c_1)^{d_1} H_{m,c_1,d_1}}$ and $p_u = \frac{1}{(u+c_2)^{d_2} H_{m,c_2,d_2}}$ in Corollary 4.5(b), with ($n = \text{even}$), then the following holds

$$\begin{aligned} D(\mathbf{q} \parallel \mathbf{p}) &= \sum_{u=1}^m \frac{1}{(u+c_1)^{d_1} H_{m,c_1,d_1}} \ln \left(\frac{(u+c_2)^{d_2} H_{m,c_2,d_2}}{(u+c_1)^{d_1} H_{m,c_1,d_1}} \right) \\ &\geq \sum_{u=1}^m \left(\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_1)^{d_1} H_{m,c_1,d_1}} \right) \ln \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} \frac{1}{(u+v+c_1)^{d_1} H_{m,c_1,d_1}}}{\sum_{v=0}^{k-1} \lambda_{v+1} \frac{1}{(u+v+c_2)^{d_2} H_{m,c_2,d_2}}} \right) \\ &\quad + \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \frac{(-1)^\sigma (\sigma-1)!}{(b_\omega)^\sigma} \right) \left(\sum_{u=1}^m \frac{1}{((u+c_1)^{d_1} H_{m,c_1,d_1})} H_{\sigma\omega} \left(\frac{((u+c_2)^{d_2} H_{m,c_2,d_2})}{((u+c_1)^{d_1} H_{m,c_1,d_1})} \right) \right) \\ &\quad - \left(\sum_{\omega=1}^t \sum_{\sigma=0}^{s_\omega} \frac{(-1)^\sigma (\sigma-1)!}{(b_\omega)^\sigma} \right) \left(\sum_{u=1}^m \left(\sum_{v=0}^{k-1} \frac{\lambda_{v+1}}{(u+v+c_1)^{d_1} H_{m,c_1,d_1}} \right) H_{\sigma\omega} \left(\frac{\sum_{v=0}^{k-1} \lambda_{v+1} \frac{1}{(u+v+c_1)^{d_1} H_{m,c_1,d_1}}}{\sum_{v=0}^{k-1} \lambda_{v+1} \frac{1}{(u+v+c_2)^{d_2} H_{m,c_2,d_2}}} \right) \right) \end{aligned} \quad (77)$$

If ($n = \text{odd}$), then (77) holds in reverse direction.

Remark 4.8. It is interesting to note that, in a similar fashion we are able to construct different estimations of φ -divergences along with their applications to Shannon and Mandelbrot entropies using the other inequalities for n -convex functions constructed in Theorem 2.2 for discrete case of cyclic refinements of Jensen's inequality.

Remark 4.9. Now as a consequences of Corollary 2.3 for **Type**($\eta, n - \eta$)**C** and Corollary 2.4 for **Two-point TC**, we can also consider the discrete extensions of cyclic refinements of Jensen's inequalities for ($i = 1$), from (36) and (39) with respect to n -convex function φ to construct upper bounds for Shannon, Relative and Mandelbrot entropies.

5 Concluding remarks

It is refreshing to note that the obtained inequalities for n -convex functions in the first section are worth more as they enable us to give a variety of new and sharp upper bounds for Grüss and Ostrowski type inequalities (see [32]) as an application of the results obtained by Dragomir et al. in [33]. Furthermore, we can construct a variety of functionals from the inequalities introduced in the Theorem 2.2 and present Cauchy and Lagrange type mean value theorems for n -convex functions. More than that, taking into account n -exponentially convex approach in [34, 35] (see also [36, 37]), a new collection of nontrivial examples of n -exponentially and exponentially convex functions can be established. Finally we are also able to construct monotonic Cauchy means.

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