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Differential equations with random Gamma distributed moments of non-instantaneous impulses and p-moment exponential stability

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Abstract: Nonlinear differential equations with impulses occurring at random time and acting non-instantaneously on finite intervals are studied. We consider the case when the time where the impulses occur is Gamma distributed. Lyapunov functions are applied to obtain sufficient conditions for the p-moment exponential stability of the trivial solution of the given system.

Keywords: fractional differential equations, random moments of impulses, non-instantaneous impulses, p-moment exponential stability

MSC: 34A08, 34A37, 34D20, 34F05

1 Introduction

Impulsive effects exist in many evolution processes when the states change abruptly at certain moments of time. These equations arise naturally in models in medicine, biology, economics, mechanics and electronics. In the literature many authors consider impulsive differential equations with determined impulsive moments and there are two popular types of deterministic impulses:

- instantaneous impulses: the duration of the changes is relatively short compared to the overall duration of the whole process. The model is given by impulsive differential equations (see for example, [1–6], the monographs [7, 8] and the references cited therein);

- non-instantaneous impulses - an impulsive action, which starts at an arbitrary fixed point and remains active on a finite time interval. E. Hernandez and D. O'Regan [9] introduced a new class of abstract differential equations for which the impulses are not instantaneous and they investigated the existence of mild and classical solutions. As a motivation for the study of this type we consider the following simplified situation concerning the hemodynamical equilibrium of a person. In the case of a decompensation (for example, high or low levels of glucose) one can prescribe some intravenous drugs (insulin). Since the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous processes, we can interpret the situation as an impulsive action which starts abruptly and stays active on a finite time interval. Recently results concerning non-instantaneous impulses were obtained for differential

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equations [10–12], delay integro-differential equations [13], abstract differential equations [9], and fractional differential equations [14–16].

In some real world phenomena the process changes at uncertain moments. Modeling such real world phenomena where the state of the process under investigation changes instantaneously at uncertain moments will lead to combining ideas in differential equations and probability theory. When there is uncertainty in the behavior of the state of the investigated process an appropriate model is usually a stochastic differential equation where one or more of the terms in the differential equation are stochastic processes, and this usually results with the solution being a stochastic process [17–20]. Differential equations with instantaneously acting impulses at random times were studied in [21, 22] but there are some inaccuracies in the mixing properties of deterministic variables and random variables, and inaccuracies in the convergence of a sequence of real numbers to a random variable.

In this paper we study nonlinear differential equations subject to impulses starting abruptly at some random points and their action continue on intervals with a given finite length. We study the case of Gamma distributed random variables between two consecutive moments of impulses and the interval where the impulses act is of a constant length. The p-moment exponential stability of the solution is studied using Lyapunov functions.

2 Non-instantaneous impulses in differential equations

Initially, we will give a brief overview of differential equations with deterministic non-instantaneous impulses.

Let $T_0 \geq 0$ be a given point and the increasing sequence of positive points $\{T_k\}_{k=1}^{\infty}$ and the sequence of non-negative numbers $\{d_i\}_{i=1}^{\infty}$ be given, such that $\lim_{k \rightarrow \infty} T_k = \infty$. Denote $d_0 = 0$.

Consider the initial value problem (IVP) for the system of *non-instantaneous impulsive differential equations* (NIDE) with fixed points of impulses

$$\begin{aligned} x' &= f(t, x(t)) \quad \text{for } t \in [T_k + d_k, T_{k+1}], \quad k = 0, 1, 2, \dots, \\ x(t) &= I_k(t, x(T_k - 0)) \quad \text{for } t \in (T_k, T_k + d_k], \quad k = 1, 2, \dots, \\ x(T_0) &= x_0, \end{aligned} \quad (1)$$

where $x, x_0 \in \mathbb{R}^n$, $f : [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $I_i : [T_i, T_i + d_i] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(i = 1, 2, 3, \dots)$.

The solution of NIDE (1) depends not only on the initial condition (T_0, x_0) but on the moments of impulses $T_k, k = 1, 2, \dots$ and we will denote it by $x(t; T_0, x_0, \{T_k\})$. We will assume that

$$x(T_k; T_0, x_0, \{T_k\}) = \lim_{t \rightarrow T_k - 0} x(t; T_0, x_0, \{T_k\})$$

for any $k = 1, 2, \dots$ and

$$\lim_{t \rightarrow T_k + d_k - 0} x(t; T_0, x_0, \{T_k\}) = \lim_{t \rightarrow T_k + d_k + 0} x(t; T_0, x_0, \{T_k\}) \quad \text{for any } k = 1, 2, \dots$$

We will assume the following conditions are satisfied

H1. $f(t, 0) = 0$ and $I_k(t, 0) = 0$ for $t \geq 0, k = 1, 2, \dots$

H2. For any initial value (T_0, x_0) the ODE $x' = f(t, x)$ with $x(T_0) = x_0$ has a unique solution $x(t) = x(t; T_0, x_0)$ defined for $t \geq T_0$.

Example 1. (differential equations with fixed points of non-instantaneous impulses). Consider the following IVP for the scalar NIDE

$$\begin{aligned} x' &= a_k x \quad \text{for } t \in [T_k + d_k, T_{k+1}], \quad k = 0, 1, 2, \dots, \\ x(t) &= I_k(t, x(T_k - 0)) \quad \text{for } t \in (T_k, T_k + d_k], \quad k = 1, 2, \dots, \\ x(T_0) &= x_0, \end{aligned} \quad (2)$$

where $x, x_0 \in \mathbb{R}$, $a_k, k = 0, 1, 2, \dots$, are constants.

The solution $x(t) = x(t; T_0, x_0, \{T_k\})$ of (2) is

$$x(t) = \begin{cases} x_0 e^{a_0(t-T_0)} & \text{for } t \in [T_0, T_1], \\ I_k(t, x(T_k - 0)) & \text{for } t \in (T_k, T_k + d_k], \quad k = 1, 2, \dots, \\ I_k(T_k + d_k, x(T_k - 0)) e^{a(t-T_k-d_k)} & \text{for } t \in [T_k + d_k, T_{k+1}], \quad k = 1, 2, 3, \dots \end{cases}$$

If $I_k(t, x) = b_k(t)x$, $b_k : [T_k, T_k + d_k] \rightarrow \mathbb{R}/\{1\}$, $k = 1, 2, 3, \dots$, then the solution of NIDE (2) is given by (note the product is 1 if $k = 1$)

$$x(t) = \begin{cases} x_0 e^{a_0(t-T_0)} & \text{for } t \in [T_0, T_1], \\ x_0 e^{\sum_{i=0}^{k-1} a_i(T_{i+1}-T_i-d_i)} \left(\prod_{i=1}^{k-1} b_i(T_i + d_i) \right) b_k(t) & \text{for } t \in (T_k, T_k + d_k], \\ x_0 e^{\sum_{i=0}^{k-1} a_i(T_{i+1}-T_i-d_i)} \left(\prod_{i=1}^k b_i(T_i + d_i) \right) e^{a_k(t-T_k-d_k)} & \text{for } t \in [T_k + d_k, T_{k+1}]. \end{cases}$$

If $b_k(t) = e^{a_k t}$, $k = 1, 2, 3, \dots$ the solution of NIDE (2) is given by

$$x(t) = \begin{cases} x_0 e^{a_0(t-T_0)} & \text{for } t \in [T_0, T_1], \\ x_0 e^{a_k t - a_0 T_0 + \sum_{i=0}^{k-1} a_i T_{i+1}} & \text{for } t \in (T_k, T_k + d_k], \quad k = 1, 2, \dots, \\ x_0 e^{a_k t - a_0 T_0 + \sum_{i=0}^{k-1} a_i T_{i+1}} & \text{for } t \in [T_k + d_k, T_{k+1}], \quad k = 1, 2, \dots \end{cases} \quad (3)$$

If $a_k = a$, $k = 0, 1, \dots$ and $I_k(t, x) = bx$ with $b \neq 1$, $k = 1, 2, 3, \dots$ is a constant, then the solution of NIDE (2) is

$$x(t) = \begin{cases} x_0 e^{a(t-T_0)} & \text{for } t \in [T_0, T_1], \\ x_0 b^k e^{a \sum_{i=0}^{k-1} (T_{i+1}-s_k)} & \text{for } t \in (T_k, T_k + d_k], \quad k = 1, 2, \dots, \\ x_0 b^k e^{a \sum_{i=0}^{k-1} (T_{i+1}-T_k-d_k)} e^{a(t-T_k-d_k)} & \text{for } t \in [T_k + d_k, T_{k+1}], \quad k = 1, 2, \dots \end{cases} \quad (4)$$

If $a_k = 0$, $k = 0, 1, \dots$ and $I_k(t, x) = b_k(t)x$, $b_k : [t_k, s_k] \rightarrow \mathbb{R}/\{1\}$, $k = 1, 2, 3, \dots$, then the solution of NIDE (2) is

$$x(t) = \begin{cases} x_0 & \text{for } t \in [T_0, T_1], \\ x_0 \left(\prod_{i=1}^{k-1} b_i(s_i) \right) b_k(t) & \text{for } t \in (T_k, s_k], \quad k = 1, 2, \dots, \\ x_0 \left(\prod_{i=1}^k b_i(s_i) \right) & \text{for } t \in [s_k, T_{k+1}], \quad k = 1, 2, 3, \dots \end{cases} \quad (5)$$

□

In Section 6 we will need the following result for the initial value problem for a scalar linear differential inequality with non-instantaneous fixed moments of impulses:

$$\begin{aligned} u' &\leq -m_k u \quad \text{for } T_k + d_k \leq t \leq T_{k+1}, \quad k = 0, 1, 2, \dots, \\ u(t) &\leq b_k u(T_k - 0) \quad \text{for } T_k < t \leq T_k + d_k, \quad k = 1, 2, \dots, \\ u(T_0) &\leq 0. \end{aligned} \quad (6)$$

Proposition 1. Let $m_k > 0$ ($k = 0, 1, 2, \dots$) and $b_k > 0$ ($k = 1, 2, \dots$) be real constants. Then $m(t) \leq 0$ for $t \geq T_0$.

Proof: Let $t \in [T_0, T_1]$. Then the function $u(t)$ is continuous on $[T_0, T_1]$ and $u(t) \leq u(T_0)e^{-m_0(t-T_0)} \leq 0$.

Let $t \in (T_1, T_1 + d_1]$. Then the function $u(t) \leq b_1 u(T_1 - 0) \leq 0$.

Let $t \in [T_1 + d_1, T_2]$. Then the function $u(t)$ is continuous on $[T_1 + d_1, T_2]$ and $u(t) \leq u(T_1 + d_1)e^{-m_1(t-T_1-d_1)} \leq 0$.

Continue this process. □

3 Random non-instantaneous impulses in differential equations

Let the probability space $(\Omega, \mathcal{F}, P)$ be given. Let $\{\tau_k\}_{k=1}^\infty$ be a sequence of random variables defined on the sample space Ω .

Assume $\sum_{k=1}^\infty \tau_k = \infty$ with probability 1.

Remark 1. The random variables τ_k will define the time between two consecutive impulsive moments of the impulsive differential equation with random impulses.

We will assume the following condition is satisfied:

H3. The random variables $\{\tau_k\}_{k=1}^\infty$ are independent Gamma distributed random variables with the same rate parameter λ and shape parameter α_k respectively, i.e., $\tau_k \in \Gamma(\alpha_k, \lambda)$ with $\lambda > 0$, $\alpha_k > 0$.

We recall some properties of the Gamma distribution:

- (i) If $X \in \text{Exp}(\lambda)$, then $X \in \Gamma(1, \lambda)$;
- (ii) If $X \in \Gamma(\alpha_1, c)$ and $Y \in \Gamma(\alpha_2, c)$ are independent, then $X + Y \in \Gamma(\alpha_1 + \alpha_2, c)$;
- (iii) The cumulative distribution function (CDF) of $X \in \Gamma(\alpha, c)$ is

$$F(x; \alpha, c) = P(X \leq x) = \frac{\gamma(\alpha, cx)}{\Gamma(\alpha)} \quad \text{for } x > 0$$

and the density function (DF) is

$$f(x; \alpha, c) = \frac{c^\alpha x^{\alpha-1} e^{-cx}}{\Gamma(\alpha)} \quad \text{for } x > 0,$$

where $\gamma(A, t) = \int_0^t x^{A-1} e^{-x} dx$, $A, t > 0$, is the lower incomplete Gamma function. Note that there is an equivalent presentation of the lower incomplete Gamma function

$$\gamma(A, t) = t^A e^{-t} \Gamma(A) \sum_{j=0}^{\infty} \frac{t^j}{\Gamma(A + j + 1)}. \quad (7)$$

Using the properties of the Gamma distribution we obtain the following result:

Proposition 2. Let condition (H3) be satisfied and the sequence of random variables $\{\Xi_k\}_{k=1}^\infty$ be such that $\Xi_n = \sum_{i=1}^n \tau_i$, $n = 1, 2, \dots$. Then $\Xi_n \in \Gamma(\sum_{i=1}^n \alpha_i, \lambda)$, i.e., the cdf of Ξ_n is

$$F(t; \sum_{i=1}^n \alpha_i, \lambda) = P(\Xi_n < t) = \begin{cases} \frac{\gamma(\sum_{i=1}^n \alpha_i, t\lambda)}{\Gamma(\sum_{i=1}^n \alpha_i)} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

Corollary 1. If condition (H3) is satisfied for $\alpha_k = 1$, $k = 1, 2, \dots$, i.e., $\tau_k \in \text{Exp}(\lambda)$, then the cdf of Ξ_n is

$$F(t; \lambda) = P(\Xi_n < t) = \begin{cases} 1 - e^{-\lambda t} \sum_{j=0}^{n-1} \frac{(\lambda t)^j}{j!} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases}$$

Let the sequence of positive numbers $\{d_k\}_{k=1}^\infty$ be given. Denote $d_0 = 0$.

Consider the following conditions:

H4. The positive numbers $\{d_k\}_{k=1}^\infty$ are such that $\lim_{n \rightarrow \infty} \sum_{k=1}^n d_k = \infty$.

H5. The positive numbers $\{d_k\}_{k=1}^\infty$ are such that $\lim_{n \rightarrow \infty} \sum_{k=1}^n d_k = B < \infty$.

Remark 2. Note that if there exist constants $c > 0$, $s \leq 1$ and for any $k = 1, 2, \dots$ at least one of inequalities $d_k \geq \frac{c}{s^k}$ or $d_k \geq \frac{c}{k^s}$ holds, then condition (H4) is satisfied.

If $c = d > 0$, $s = 1$ and $d_k = d > 0$, or $c = 1$, $s = 1$, $d_k = \frac{1}{k}$, $k = 1, 2, \dots$, then condition (H4) is satisfied.

If there exist constants $c > 0$, $s > 1$ and for any $k = 1, 2, \dots$ at least one of inequalities $d_k \leq \frac{c}{s^k}$ or $d_k \leq \frac{c}{k^s}$ holds, then condition (H5) is satisfied.

If $c = 1$, $s = 2$ and $d_k = \frac{1}{2^k}$, $k = 1, 2, \dots$, then condition (H5) is satisfied.

Define the sequence of random variables $\{\xi_k\}_{k=0}^\infty$ by

$$\xi_k = T_0 + \sum_{i=1}^k \tau_i + \sum_{i=1}^{k-1} d_i, \quad k = 0, 1, 2, \dots, \quad (8)$$

where $T_0 \geq 0$ is a fixed point.

The random variable τ_k measures the waiting time of the k -th impulse after the $(k-1)$ -st impulse occurs and stops its action, d_k is the length of the acting time interval of the k -th impulse and the random variable ξ_k denotes the length of time until k impulses occur for $t \geq T_0$.

We note that $\{\xi_k\}_{k=0}^\infty$ is a sequence of random variables defined by the recurrence formula $\xi_k = \xi_{k-1} + \tau_k + d_{k-1}$, $k = 1, 2, \dots$ with $d_0 = 0$.

Remark 3. The random variable ξ_n will be called the waiting time and it gives the arrival time of n -th impulses in the impulsive differential equation with random impulses.

Also, $\Xi_k = \sum_{i=1}^k \tau_i = \xi_k - T_0 - \sum_{i=1}^{k-1} d_i$, $k = 1, 2, \dots$

Let the points t_k be arbitrary values of the corresponding random variables τ_k , $k = 1, 2, \dots$. Define two increasing sequences of points

$$T_k = T_0 + \sum_{i=1}^k t_i + \sum_{i=1}^{k-1} d_i, \quad k = 1, 2, 3, \dots$$

and

$$s_k = T_k + d_k, \quad k = 1, 2, 3, \dots$$

Note that T_k and s_k are values of the corresponding random variables ξ_k and $\xi_k + d_k$ and $s_{k-1} \leq T_k \leq s_k$. Then the initial value problem for the system of NIDE with fixed points of impulses (1) reduces to

$$\begin{aligned} x' &= f(t, x(t)) \quad \text{for } t \in (T_k + d_k, T_{k+1}], \quad k = 0, 1, 2, \dots, \\ x(t) &= I_k(t, x(T_k - 0)) \quad \text{for } t \in (T_k, T_k + d_k], \quad k = 1, 2, \dots, \\ x(T_0) &= x_0, \end{aligned} \quad (9)$$

where $x, x_0 \in \mathbb{R}^n$, $f: [0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $I_i: [t_i, s_i] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(i = 1, 2, 3, \dots)$.

The set of all solutions $x(t; T_0, x_0, \{T_k\})$ of NIDE (9) for any values t_k of the random variables τ_k , $k = 1, 2, \dots$ generates a specific stochastic process with state space $\mathbb{R}^n$. We denote it by $x(t; T_0, x_0, \{\tau_k\})$ and we will say that it is a solution of the following initial value problem for non-instantaneous impulsive differential equations with random moments of impulses (RNIDE)

$$\begin{aligned} x'(t) &= f(t, x(t)) \quad \text{for } t \geq T_0, \quad \xi_k + d_k < t < \xi_{k+1}, \quad k = 0, 1, \dots, \\ x(t) &= I_k(t, x(\xi_k)), \quad \text{for } \xi_k < t < \xi_k + d_k, \quad k = 1, 2, \dots, \\ x(T_0) &= x_0, \end{aligned} \quad (10)$$

where $x_0 \in \mathbb{R}^n$.

Definition 1. Let t_k be a value of the random variable τ_k , $k = 1, 2, 3, \dots$ and $T_k = T_0 + \sum_{i=1}^k t_i$, $k = 1, 2, \dots$. Then the solution $x(t; T_0, x_0, \{T_k\})$ of the IVP for the IDE with fixed points of impulses (1) is called a sample path solution of the IVP for the RIDE (10).

Any sample path solution

$$x(t; T_0, x_0, \{T_k\}) \in C^1((T_k, T_{k+1}], \mathbb{R}^n),$$

for $k = 0, 1, 2, \dots$

Definition 2. A stochastic process $x(t; T_0, x_0, \{\tau_k\})$ with an uncountable state space $\mathbb{R}^n$ is said to be a solution of the IVP for the system of RIDE (10) if for any values t_k of the random variable τ_k , $k = 1, 2, 3, \dots$ and $T_k = T_0 + \sum_{i=1}^k t_i$, $k = 1, 2, \dots$ the corresponding function $x(t; T_0, x_0, \{T_k\})$ is a sample path solution of the IVP for RIDE (10).

4 Preliminary results for Gamma distributed moments of impulses

We will use the following probability result:

Lemma 4.1. *Let the continuous random variables ξ, η be such that $\xi \geq \eta$. Then for any $t \geq 0$ the equality*

$$P(\eta < t < \xi) = F_\eta(t) - F_\xi(t)$$

holds.

Proof: Using continuity of the random variables η and ξ we obtain

$$P(\eta < t < \xi) = P(\eta < t \cap \xi > t) = P(\eta < t \setminus (\eta < t \cap \xi < t)) = P(\eta < t \setminus \xi < t) = P(\eta < t) - P(\xi < t). \quad (11)$$

□

For any $t \geq T_0$ consider the events

$$S_0(t) = \{\omega \in \Omega : t - T_0 < \tau_1(\omega)\}$$

and

$$S_k(t) = \{\omega \in \Omega : \xi_k(\omega) + d_k < t < \xi_{k+1}(\omega)\}, \quad k = 1, 2, \dots,$$

where the random variables ξ_k , $k = 1, 2, \dots$ are defined by (8).

In the case of Gamma distributed random variables τ_k , $k = 1, 2, \dots$ we obtain the following results:

Proposition 3. *For any $t \geq T_0$ the equality $P(S_0(t)) = 1 - \frac{\gamma(\alpha_1, \lambda(t-T_0))}{\Gamma(\alpha_1)}$ holds.*

Lemma 4.2. *Let conditions (H3), (H4) be satisfied and $t \geq T_0$ be a given fixed number. Then the probability that there will be exactly k impulses until time t is*

$$P(S_k(t)) = \begin{cases} 0 & \text{for } k > C_t, \\ \frac{\gamma(\sum_{i=1}^k \alpha_i, (t-T_0-\sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^k \alpha_i)} - \frac{\gamma(\sum_{i=1}^{k+1} \alpha_i, (t-T_0-\sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^{k+1} \alpha_i)} & \text{for } k \leq C_t, \end{cases}$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + \sum_{i=1}^c d_i\}$.

Proof: Let $t > T_0$ be fixed. According to condition (H4) there exists a natural number C_t , such that $T_0 + \sum_{i=1}^{C_t-1} d_i < t \leq T_0 + \sum_{i=1}^{C_t} d_i$. Consider the sequence of random variables Ξ_k , $k \geq 1$, defined as above. For any natural number k , by Lemma 4.1, we obtain

$$\begin{aligned} P(S_k(t)) &= P(T_0 + \sum_{i=1}^k \tau_i + \sum_{i=1}^k d_i < t < T_0 + \sum_{i=1}^{k+1} \tau_i + \sum_{i=1}^k d_i) \\ &= P(\Xi_k < t - T_0 - \sum_{i=1}^k d_i) - P(\Xi_{k+1} < t - T_0 - \sum_{i=1}^k d_i). \end{aligned} \quad (12)$$

If $k > C_t$, then $t < T_0 + \sum_{i=1}^k d_i$ and $P(S_k(t)) = 0$.

Now if $k \leq C_t$, then $t - T_0 - \sum_{i=1}^k d_i \geq 0$ and from (12) and Proposition 2 we get

$$\begin{aligned} P(S_k(t)) &= F_{\Xi_k}(t - T_0 - \sum_{i=1}^k d_i; \sum_{i=1}^k \alpha_i, \lambda) - F_{\Xi_{k+1}}(t - T_0 - \sum_{i=1}^k d_i; \sum_{i=1}^{k+1} \alpha_i, \lambda) \\ &= \frac{\gamma(\sum_{i=1}^k \alpha_i, (t - T_0 - \sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^k \alpha_i)} - \frac{\gamma(\sum_{i=1}^{k+1} \alpha_i, (t - T_0 - \sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^{k+1} \alpha_i)}. \end{aligned}$$

□

Corollary 2. Let condition (H3), (H4) be satisfied with $\alpha_i = \alpha$, $i = 1, 2, \dots$ and $t \geq T_0$ be a given fixed number. Then

$$P(S_k(t)) = \begin{cases} 0 & \text{for } k > C_t, \\ \int_0^{(t-T_0-\sum_{i=1}^k d_i)\lambda} \left(\frac{x^{k\alpha-1}}{\Gamma(k\alpha)} - \frac{x^{(k+1)\alpha-1}}{\Gamma((k+1)\alpha)} \right) e^{-x} dx & \text{for } k \leq C_t, \end{cases}$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + \sum_{i=1}^c d_i\}$.

Corollary 3. If the condition (H3) is satisfied with $\alpha_k = 1$, i.e., $\tau_k \in \text{Exp}(\lambda)$, and $d_k = d > 0$, $k = 1, 2, \dots$, then

$$P(S_k(t)) = \begin{cases} 0 & \text{for } k > C_t, \\ e^{-\lambda(t-T_0)} \frac{(\lambda e^{\lambda d}(t-T_0-kd))^k}{k!} \leq e^{-\lambda(t-T_0)} \frac{(\lambda e^{\lambda d}(t-T_0))^k}{k!} & \text{for } k \leq C_t. \end{cases}$$

The proof of Corollary 3 follows from Lemma 4.2 and equality $\gamma(k, t) - \frac{\gamma(k+1, t)}{k} = \frac{t^k e^{-t}}{k}$.

Lemma 4.3. (Upper bound of $S_k(t)$) Let conditions (H3), (H4) be satisfied. Then for any $k \leq C_t$ there exists a polynomial $R_k(x)$ (as described in the proof) with positive coefficients defined for $x \geq 0$, such that

$$P(S_k(t)) \leq e^{-(t-T_0-\sum_{i=1}^k d_i)\lambda} R_k((t-T_0-\sum_{i=1}^k d_i)\lambda) \quad \text{for } t > T_0 + \sum_{i=1}^k d_i. \quad (13)$$

Proof: Let $k \leq C_t$ be a natural fixed number and $t > T_0 + \sum_{i=1}^k d_i$. Denote $A_k = \sum_{i=1}^k d_i$ and $z = (t - T_0 - \sum_{i=1}^k d_i)\lambda$. From Lemma 4.2 and the presentation (7) of the lower incomplete Gamma function we obtain

$$\begin{aligned} P(S_k(t)) &= z^{A_k} e^{-z} \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(A_k + j + 1)} - z^{A_{k+1}} e^{-z} \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(A_{k+1} + j + 1)} \\ &= e^{-z} \left[\sum_{j=0}^{\infty} \left(\frac{z^{A_k+j}}{\Gamma(A_k + j + 1)} - \frac{z^{A_{k+1}+j}}{\Gamma(A_{k+1} + j + 1)} \right) \right] \\ &= e^{-z} \sum_{j=0}^{N-1} \frac{z^{A_k+j}}{\Gamma(A_k + j + 1)} + e^{-z} \left[\sum_{j=0}^{\infty} \left(\frac{z^{A_k+N+j}}{\Gamma(A_k + N + j + 1)} - \frac{z^{A_{k+1}+j}}{\Gamma(A_{k+1} + j + 1)} \right) \right] \\ &= e^{-z} \sum_{j=0}^{N-1} \frac{z^{A_k+j}}{\Gamma(A_k + j + 1)} + e^{-z} \sum_{j=0}^{\infty} \frac{z^{A_k+N+j}}{\Gamma(A_k + N + j + 1)} - \frac{\gamma(A_{k+1}, z)}{\Gamma(A_{k+1})}, \end{aligned} \quad (14)$$

where N is an arbitrary natural number.

For any natural number N we define two functions $f_N, g_N : [0, \infty) \rightarrow [0, 1]$ by the equalities

$$f_N(z) = e^{-z} \sum_{j=0}^{N-1} \frac{z^{A_k+j}}{\Gamma(A_k + j + 1)}$$

and

$$g_N(z) = e^{-z} \sum_{j=0}^{\infty} \frac{z^{A_k+N+j}}{\Gamma(A_k + N + j + 1)} - \frac{\gamma(A_{k+1}, z)}{\Gamma(A_{k+1})}.$$

Then from (14) we get

$$P(S_k(t)) = f_N(z) + g_N(z), \quad (15)$$

where N is an arbitrary natural number.

For any fixed $z \geq 0$ the sequence $\{f_N(z)\}_{N=1}^\infty$ is strictly increasing and for any natural N we have the inequality $\sup_{z \geq 0} f_N(z) \leq 1$ and the function $f_N(z)$ is continuous in $[0, \infty)$. Therefore, the sequence $\{f_N(z)\}_{N=1}^\infty$ is convergent and $\lim_{N \rightarrow \infty} f_N(z) = F(z)$, where $F(z) = e^{-z} \sum_{j=0}^\infty \frac{z^{A_k+j}}{\Gamma(A_k+j+1)}$. Note that the function $F(z)$ is continuous on $[0, \infty)$ (see [23]).

Denote $A_N = \sup_{z \geq 0} |F(z) - f_N(z)|$. The sequence $\{A_N\}_{N=1}^\infty$ is decreasing and therefore it is convergent. From the definition of the function $F(z)$ the equality $\lim_{N \rightarrow \infty} A_N = 0$ follows, i.e., $F(z)$ is uniformly continuous on $[0, \infty)$.

Take the limit as $N \rightarrow \infty$ in (15) and we obtain

$$P(S_k(t)) = e^{-z} \sum_{j=0}^\infty \frac{z^{A_k+j}}{\Gamma(A_k+j+1)} + \lim_{N \rightarrow \infty} g_N(z) \quad \text{for } z \geq 0. \quad (16)$$

From (14) and (16) we get

$$\lim_{N \rightarrow \infty} g_N(z) = -z^{A_{k+1}} e^{-z} \sum_{j=0}^\infty \frac{z^j}{\Gamma(A_{k+1}+j+1)} < 0 \quad \text{for } z > 0.$$

Then, there exists a natural number $N_0 > 0$, such that for any $N \geq N_0$ the inequality $g_N(z) < 0$ holds. Then from (15) and $z^a \leq z^{[a]} + z^{[a]+1}$, $z \geq 0$, the inequality

$$P(S_k(t)) = f_N(z) + g_N(z) < f_N(z) = e^{-z} \sum_{j=0}^{N-1} \frac{z^{A_k+j}}{\Gamma(A_k+j+1)} \leq e^{-z} \sum_{j=0}^{N-1} \frac{z^{[A_k]+j} + z^{[A_k]+j+1}}{\Gamma(A_k+j+1)} \quad \text{for } N \geq N_0 \quad (17)$$

holds, where $[a]$ is the integer part of the real number $a > 0$.

If $N = N_0$, then from the inequality (17) we obtain

$$P(S_k(t)) \leq e^{-z} \sum_{j=0}^{N_0-1} \frac{z^{[A_k]+j} + z^{[A_k]+j+1}}{\Gamma(A_k+j+1)}.$$

Since the coefficients in the power series are positive numbers, there exists a polynomial $R_k(z)$ with a degree not more than $N_0 + [A_k] + 1$, such that

$$\sum_{j=0}^{N_0-1} \frac{z^{[A_k]+j} + z^{[A_k]+j+1}}{\Gamma(A_k+j+1)} \leq R_k(z)$$

and

$$P(S_k(t)) \leq e^{-z} R_k(z).$$

□

Corollary 4. Let conditions (H3), (H4) be satisfied. Then for any $k = 1, 2, \dots$ and any positive number $c > 0$ there exist numbers $M_k > 0$ and $K_k > 0$ depending on both k and c , such that

$$P(S_k(t)) \leq e^{-\lambda(t-T_0)} K_k (t - T_0)^{M_k} \quad \text{for } t \geq T_0 + c. \quad (18)$$

Proof: Let k be a natural number and $c > 0$ be a number. From Lemma 4.3 there exists a polynomial $R_k(x)$ (as described in the proof of Lemma 4.3) with positive coefficients defined for $x \geq 0$, such that (13) is satisfied. Consider the polynomial $R_k((t - T_0 - \sum_{i=1}^k d_i)\lambda)$ which could be presented as a polynomial depending on $t - T_0$, i.e., there exists a polynomial $P_k(x)$ of degree not more than the degree of $R_k(x)$, such that $P_k(t - T_0) = R_k((t - T_0 - \sum_{i=1}^k d_i)\lambda)$. Also for $t > T_0 + c$ there exist numbers $M_k = M(k, c) > 0$ and $K_k = K(k, c) > 0$, such that $P_k(t - T_0) \leq K_k (t - T_0)^{M_k}$. Then

$$P(S_k(t)) \leq e^{-\lambda(t-T_0)} e^{\lambda \sum_{i=1}^k d_i} K_k (t - T_0)^{M_k}$$

for $t > T_0 + c$.

□

Lemma 4.4. *Let conditions (H3), (H5) be satisfied and $t \geq T_0$ be a given fixed number. Then the probability that there will be exactly k impulses until time t is:*

– for $t \geq B$:

$$P(S_k(t)) = \frac{\gamma(\sum_{i=1}^k \alpha_i, (t - T_0 - \sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^k \alpha_i)} - \frac{\gamma(\sum_{i=1}^{k+1} \alpha_i, (t - T_0 - \sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^{k+1} \alpha_i)};$$

– for $t < B$:

$$P(S_k(t)) = \begin{cases} 0 & \text{for } k > C_t, \\ \frac{\gamma(\sum_{i=1}^k \alpha_i, (t - T_0 - \sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^k \alpha_i)} - \frac{\gamma(\sum_{i=1}^{k+1} \alpha_i, (t - T_0 - \sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^{k+1} \alpha_i)} & \text{for } k \leq C_t, \end{cases}$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + \sum_{i=1}^c d_i\}$.

The proof of Lemma 4.4 is similar to the one of Lemma 4.2, so we omit it.

Remark 4. *The claim of Lemma 4.3 and Corollary 4 are true if condition (H4) is replaced by (H5). In the proof we use Lemma 4.4 instead of Lemma 4.2.*

For any $t \geq T_0$ consider the events

$$W_k(t) = \{\omega \in \Omega : \xi_k(\omega) < t < \xi_k(\omega) + d_k\}, \quad k = 1, 2, \dots,$$

where the random variables ξ_k , $k = 1, 2, \dots$ are defined as above.

In the case of Gamma distributed random variables τ_k , $k = 1, 2, \dots$ we obtain the following result:

Lemma 4.5. *Let conditions (H3), (H4) hold and $t \geq T_0$ be a given fixed number. Then the probability the time t is immediately after the k -th random impulse but not further away than d_k from it is given by*

$$P(W_k(t)) = \begin{cases} 0 & \text{for } k > C_t, \\ \frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t - T_0 - \sum_{i=1}^{k-1} d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)} - \frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t - T_0 - \sum_{i=1}^k d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)} & \text{for } k \leq C_t, \end{cases}$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + \sum_{i=1}^c d_i\}$.

Proof: Let $t \geq T_0$ be fixed. According to condition (H4) there exists a natural number C_t , such that $T_0 + \sum_{i=1}^{C_t-1} d_i < t \leq T_0 + \sum_{i=1}^{C_t} d_i$. Consider the sequence of random variables Ξ_k , $k \geq 1$, defined as above. For any natural k , by Lemma 4.1, we obtain

$$\begin{aligned} P(W_k(t)) &= P(\xi_k < t < \xi_k + d_k) = P(t - T_0 - \sum_{i=1}^k d_i < \sum_{i=1}^k \tau_i < t - T_0 - \sum_{i=1}^{k-1} d_i) \\ &= F_{\Xi_k}(t - T_0 - \sum_{i=1}^{k-1} d_i) - F_{\Xi_k}(t - T_0 - \sum_{i=1}^k d_i). \end{aligned} \quad (19)$$

If $k > C_t$, then $t < T_0 + \sum_{i=1}^k d_i$ and $P(W_k(t)) = 0$.

Now if $k \leq C_t$, then $t - T_0 - \sum_{i=1}^k d_i \geq 0$ and from (19) and Proposition 1 we get

$$P(W_k(t)) = \frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t - T_0 - \sum_{i=1}^{k-1} d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)} - \frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t - T_0 - \sum_{i=1}^k d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)}. \quad (20)$$

□

Remark 5. Note that in the case $d_k = 0$, $k = 1, 2, \dots$ from (20) it follows that $P(W_k(t)) = 0$ for any $t \geq t_0$ and $k = 1, 2, \dots$, i.e., we obtain the case of instantaneous random impulses.

Corollary 5. (Upper bound of $W_k(t)$). Let conditions (H3), (H4) hold and $t > T_0$ be a given fixed number. Then for $k : 1 \leq k \leq C_t$

$$P(W_k(t)) \leq d_k e^{-\lambda(t-T_0-\sum_{i=1}^k d_i)} \frac{(\lambda(t-T_0))^{\sum_{i=1}^k \alpha_i}}{\Gamma(\sum_{i=1}^k \alpha_i)(t-T_0-\sum_{i=1}^{k-1} d_i)},$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + \sum_{i=1}^c d_i\}$.

Proof: Applying the Integral Mean Value Theorem and the inequalities $t - T_0 - \sum_{i=1}^{k-1} d_i < t - T_0$ we obtain

$$\begin{aligned} & \gamma\left(\sum_{i=1}^k \alpha_i, \lambda(t-T_0-\sum_{i=1}^{k-1} d_i)\right) - \gamma\left(\sum_{i=1}^k \alpha_i, \lambda(t-T_0-\sum_{i=1}^k d_i)\right) \\ &= \int_0^{\lambda(t-T_0-\sum_{i=1}^{k-1} d_i)} x^{\sum_{i=1}^k \alpha_i - 1} e^{-x} dx - \int_0^{\lambda(t-T_0-\sum_{i=1}^k d_i)} x^{\sum_{i=1}^k \alpha_i - 1} e^{-x} dx \\ &= \int_{\lambda(t-T_0-\sum_{i=1}^k d_i)}^{\lambda(t-T_0-\sum_{i=1}^{k-1} d_i)} x^{\sum_{i=1}^k \alpha_i - 1} e^{-x} dx \\ &\leq \lambda d_k \left(\lambda(t-T_0-\sum_{i=1}^{k-1} d_i)\right)^{\sum_{i=1}^k \alpha_i - 1} e^{-\lambda(t-T_0-\sum_{i=1}^k d_i)} \\ &\leq d_k e^{-\lambda(t-T_0-\sum_{i=1}^k d_i)} \frac{(\lambda(t-T_0))^{\sum_{i=1}^k \alpha_i}}{(t-T_0-\sum_{i=1}^{k-1} d_i)}. \end{aligned}$$

□

Corollary 6. Let conditions (H3), (H4) be satisfied with $\alpha_i = \alpha$, $i = 1, 2, \dots$ and $t \geq T_0$ be a given fixed number. Then the probability that the time t is immediately after the k -th random impulse but not further away than d_k from it is given by

$$P(W_k(t)) = \begin{cases} 0 & \text{for } k > C_t, \\ \int_{\lambda(t-T_0-\sum_{i=1}^k d_i)}^{\lambda(t-T_0-\sum_{i=1}^{k-1} d_i)} \frac{x^{k\alpha-1}}{\Gamma(k\alpha)} e^{-x} dx & \text{for } k \leq C_t, \end{cases}$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + \sum_{i=1}^c d_i\}$.

Corollary 7. If the condition (H3) is satisfied with $\alpha_k = 1$, i.e., $\tau_k \in \text{Exp}(\lambda)$ and $d_k = d > 0$, $k = 1, 2, \dots$, $t \geq T_0$ be a given fixed number, then

$$P(W_k(t)) = \begin{cases} 0 & \text{for } k > C_t, \\ \lambda d \frac{(\lambda(t-T_0-(k-1)d))^{k-1}}{k!} e^{-\lambda(t-T_0-kd)} & \text{for } k \leq C_t, \end{cases}$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + cd\}$.

Lemma 4.6. Let conditions (H3), (H5) be satisfied and $t \geq T_0$ be a given fixed number. Then the probability the time t is immediately after the k -th random impulse but not further away than d_k from it is given by

– for $t \geq B$:

$$P(W_k(t)) = \frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t - T_0 - \sum_{i=1}^{k-1} d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)} - \frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t - T_0 - \sum_{i=1}^k d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)};$$

– for $t < B$:

$$P(W_k(t)) = \begin{cases} 0 & \text{for } k > C_t, \\ \frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t - T_0 - \sum_{i=1}^{k-1} d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)} - \frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t - T_0 - \sum_{i=1}^k d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)} & \text{for } k \leq C_t, \end{cases}$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + \sum_{i=1}^c d_i\}$.

The proof of Lemma 4.6 is similar to the one of Lemma 4.4, so we omit it.

Corollary 8. (Upper bound of $W_k(t)$). Let conditions (H3), (H5) hold and $t > T_0$ be a given fixed number. Then

$$P(W_k(t)) \leq e^{\lambda B} \lambda d_k \frac{\left(\lambda(t - T_0 - \sum_{i=1}^{k-1} d_i)\right)^{\sum_{i=1}^k \alpha_i - 1}}{\Gamma(\sum_{i=1}^k \alpha_i)} e^{-\lambda(t - T_0)}.$$

5 Linear differential equations with random moments of non-instantaneous impulses

We now obtain a formula for the solution of the initial value problem for a scalar linear differential equation with random non-instantaneous moments of impulses:

$$\begin{aligned} u' &= -m_k u \quad \text{for } \xi_k + d_k < t < \xi_{k+1}, \quad k = 0, 1, 2, \dots, \\ u(t) &= b_k u(\xi_k) \quad \text{for } \xi_k < t < \xi_k + d_k, \quad k = 1, 2, \dots, \\ u(T_0) &= u_0, \end{aligned} \tag{21}$$

where $u_0 \in \mathbb{R}$, $m_k > 0$, ($k = 0, 1, 2, \dots$) and $b_k \neq 1$, ($k = 1, 2, \dots$) are real constants.

Lemma 5.1. Let the following conditions be satisfied:

1. Condition (H3) is fulfilled.
2. The inequalities $m_i + \lambda \geq m_k$ hold for all natural i, k , such that $i < k$.

Then the solution of the IVP for the scalar linear differential equation with random non-instantaneous moments of impulses (21) is:

$$u(t; T_0, u_0, \{\tau_k\}) = \begin{cases} u_0 e^{-\sum_{i=0}^{k-1} m_i \tau_{i+1}} \left(\prod_{i=1}^k b_i \right) & \text{for } \xi_k < t \leq \xi_k + d_k, \quad k = 1, 2, \dots, \\ u_0 e^{-\sum_{i=0}^{k-1} m_i \tau_{i+1}} \left(\prod_{i=1}^k b_i \right) e^{-m_k(t - \xi_k - d_k)} & \text{for } \xi_k + d_k < t < \xi_{k+1}, \quad k = 0, 1, 2, \dots, \end{cases} \tag{22}$$

and the expected value of the solution is:

$$\begin{aligned} E(|u(t; T_0, u_0, \{\tau_k\})|) &= |u_0| e^{-m_0(t - T_0)} P(S_0(t)) + \sum_{k=1}^{\infty} |u_0| \left(\prod_{i=1}^k |b_i| \right) \left(\prod_{i=1}^k \left(\frac{\lambda}{m_{i-1} + \lambda} \right)^{\alpha_i} \right) P(W_k(t)) \\ &\quad + \sum_{k=1}^{\infty} |u_0| \left(\prod_{i=1}^k |b_i| \right) e^{-m_k(t - T_0 - \sum_{i=1}^k d_i)} \prod_{i=1}^k \left(\frac{\lambda}{m_{i-1} - m_k + \lambda} \right)^{\alpha_i} P(S_k(t)). \end{aligned}$$

Proof: Let the point t_k be an arbitrary value of τ_k , $k = 1, 2, \dots$. Define as above the points $T_k = T_0 + \sum_{i=1}^k t_i + \sum_{i=1}^{k-1} d_i$ which are values of the random variables ξ_k defined by (8). Consider the linear scalar NIDE with fixed points of impulses:

$$\begin{aligned} u' &= -m_k u \quad \text{for } T_k + d_k < t < T_{k+1}, \quad k = 0, 1, 2, \dots, \\ u(t) &= b_k u(T_k - 0) \quad \text{for } T_k < t < T_k + d_k, \quad k = 1, 2, \dots, \\ u(T_0) &= u_0. \end{aligned} \quad (23)$$

The solution of (23) is a sample path solution of (21). The solution of NIDE (23) is given by

$$u(t; T_0, u_0, \{T_k\}) = \begin{cases} u_0 e^{-\sum_{i=0}^{k-1} m_i(T_{i+1}-T_i-d_i)} \left(\prod_{i=1}^k b_i \right) & \text{for } T_k < t \leq T_k + d_k, \quad k = 1, 2, \dots, \\ u_0 e^{-\sum_{i=0}^{k-1} m_i(T_{i+1}-T_i-d_i)} \left(\prod_{i=1}^k b_i \right) e^{-m_k(t-T_k-d_k)} & \text{for } T_k + d_k < t < T_{k+1}, \quad k = 0, 1, 2, \dots \end{cases} \quad (24)$$

Formula (24) and Definition 2 establishes (22).

From formula (22) and the independence of the random variables τ_k we obtain

$$\begin{aligned} E(|u(t; T_0, u_0, \{\tau_k\})|) &= |u_0| e^{-m_0(t-T_0)} P(S_0(t)) + \sum_{k=1}^{\infty} E(|u(t; T_0, u_0, \{\tau_k\})| | S_k(t)) P(S_k(t)) \\ &\quad + \sum_{k=1}^{\infty} E(|u(t; T_0, u_0, \{\tau_k\})| | W_k(t)) P(W_k(t)) \\ &= |u_0| e^{-m_0(t-T_0)} (\tau_1 > t - T_0) P(S_0(t)) + \sum_{k=1}^{\infty} |u_0| \left(\prod_{i=1}^k |b_i| \right) E\left(e^{-\sum_{i=1}^k m_{i-1} \tau_i}\right) P(W_k(t)) \\ &\quad + \sum_{k=1}^{\infty} |u_0| \left(\prod_{i=1}^k |b_i| \right) E\left(e^{-\sum_{i=0}^{k-1} m_i \tau_{i+1}} e^{-m_k(t-\xi_k-d_k)}\right) P(S_k(t)) \\ &= |u_0| e^{-m_0(t-T_0)} P(S_0(t)) + \sum_{k=1}^{\infty} |u_0| \left(\prod_{i=1}^k |b_i| \right) E\left(e^{-\sum_{i=0}^{k-1} m_i \tau_{i+1}}\right) P(W_k(t)) \\ &\quad + \sum_{k=1}^{\infty} |u_0| \left(\prod_{i=1}^k |b_i| \right) e^{-m_k(t-T_0-\sum_{i=1}^k d_i)} E\left(e^{\sum_{i=1}^k (m_k-m_{i-1})\tau_i}\right) P(S_k(t)). \end{aligned} \quad (25)$$

Using the definition of the density function of the Gamma distribution and substituting $(m_{i-1} + \lambda)x = s$ we get

$$\begin{aligned} E e^{-m_{i-1} \tau_i} &= \int_0^{\infty} e^{-m_{i-1} x} \frac{\lambda^{\alpha_i} x^{\alpha_i-1} e^{-\lambda x}}{\Gamma(\alpha_i)} dx \\ &= \frac{\lambda^{\alpha_i}}{\Gamma(\alpha_i)} \int_0^{\infty} e^{-(m_{i-1}+\lambda)x} x^{\alpha_i-1} dx \\ &= \frac{1}{(m_{i-1} + \lambda)^{\alpha_i}} \frac{(\lambda)^{\alpha_i}}{(\alpha_i - 1)!} \int_0^{\infty} e^{-s} s^{\alpha_i-1} ds \\ &= \left(\frac{\lambda}{m_{i-1} + \lambda} \right)^{\alpha_i} \end{aligned} \quad (26)$$

and substituting $(m_{i-1} - m_k + \lambda)x = s$ we get

$$\begin{aligned}
 E e^{(m_k - m_{i-1})\tau_i} &= \int_0^\infty e^{(m_k - m_{i-1})x} \frac{\lambda^{\alpha_i} x^{\alpha_i - 1} e^{-\lambda x}}{\Gamma(\alpha_i)} dx \\
 &= \frac{\lambda^{\alpha_i}}{\Gamma(\alpha_i)} \int_0^\infty e^{-(m_{i-1} - m_k + \lambda)x} x^{\alpha_i - 1} dx \\
 &= \frac{1}{(m_{i-1} - m_k + \lambda)^{\alpha_i}} \frac{(\lambda)^{\alpha_i}}{(\alpha_i - 1)!} \int_0^\infty e^{-s} s^{\alpha_i - 1} ds \\
 &= \left(\frac{\lambda}{m_{i-1} - m_k + \lambda} \right)^{\alpha_i}.
 \end{aligned} \tag{27}$$

From (27) we obtain

$$E\left(e^{\sum_{i=1}^k (m_k - m_{i-1})\tau_i}\right) = \prod_{i=1}^k \left(\frac{\lambda}{m_{i-1} - m_k + \lambda} \right)^{\alpha_i}. \tag{28}$$

Substitute (26) and (28) in (25) and obtain (23). □

Corollary 9. *Let the conditions of Lemma 5.1 be satisfied. Then for any $t \geq T_0$*

$$\begin{aligned}
 E(|u(t; T_0, u_0, \{\tau_k\})|) &= |u_0| e^{-m_0(t-T_0)} \left(1 - \frac{\gamma(\alpha_1, (t-T_0)\lambda)}{\Gamma(\alpha_1)} \right) \\
 &+ \sum_{k=1}^{C_t-1} |u_0| \prod_{i=1}^k \left(|b_i| \left(\frac{\lambda}{m_{i-1} + \lambda} \right)^{\alpha_i} \right) \left(\frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t-T_0 - \sum_{i=1}^{k-1} d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)} \right. \\
 &- \left. \frac{\gamma(\sum_{i=1}^k \alpha_i, \lambda(t-T_0 - \sum_{i=1}^k d_i))}{\Gamma(\sum_{i=1}^k \alpha_i)} \right) \\
 &+ \sum_{k=1}^{C_t-1} |u_0| e^{-m_k(t-T_0 - \sum_{i=1}^k d_i)} \prod_{i=1}^k \left(|b_i| \left(\frac{\lambda}{m_{i-1} - m_k + \lambda} \right)^{\alpha_i} \right) \left(\frac{\gamma(\sum_{i=1}^k \alpha_i, (t-T_0 - \sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^k \alpha_i)} \right. \\
 &- \left. \frac{\gamma(\sum_{i=1}^{k+1} \alpha_i, (t-T_0 - \sum_{i=1}^k d_i)\lambda)}{\Gamma(\sum_{i=1}^{k+1} \alpha_i)} \right),
 \end{aligned}$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + \sum_{i=1}^c d_i\}$.

The proof follows from Proposition 3, Lemma 4.2, Lemma 4.5 and Lemma 5.1.

Corollary 10. *Let the conditions of Lemma 5.1 be satisfied with $m_k = m$, $k = 1, 2, \dots$. Then for any $t \geq T_0$*

$$\begin{aligned}
 E(|u(t; T_0, u_0, \{\tau_k\})|) &= |u_0| e^{-m(t-T_0)} P(S_0(t)) + |u_0| \sum_{k=1}^\infty \left(\prod_{i=1}^k |b_i| \left(\frac{\lambda}{m + \lambda} \right)^{\alpha_i} \right) P(W_k(t)) \\
 &+ |u_0| e^{-m(t-T_0)} \sum_{k=1}^\infty \left(\prod_{i=1}^k |b_i| e^{m d_i} \right) P(S_k(t)).
 \end{aligned} \tag{29}$$

Lemma 5.2. (upper bound of the expected value). *Let the conditions of Lemma 5.1 be satisfied. Then*

$$\begin{aligned}
 E(|u(t; T_0, u_0, \{\tau_k\})|) &= |u_0| e^{-m_0(t-T_0)} P(S_0(t)) + |u_0| e^{-\lambda(t-T_0)} \sum_{k=1}^{C_t-1} d_k \frac{\left(\prod_{i=1}^k |b_i| e^{\lambda d_i} \left(\frac{\lambda^2(t-T_0)}{m_{i-1} + \lambda} \right)^{\alpha_i} \right)}{\Gamma(\sum_{i=1}^k \alpha_i)(t-T_0 - \sum_{i=1}^{k-1} d_i)} \\
 &+ |u_0| \sum_{k=1}^\infty e^{-m_k(t-T_0)} \left(\prod_{i=1}^k |b_i| e^{m_k d_i} \left(\frac{\lambda}{m_{i-1} - m_k + \lambda} \right)^{\alpha_i} \right) P(S_k(t)),
 \end{aligned}$$

where $C_t = \min\{c \in \mathbb{Z} : t \leq T_0 + \sum_{i=1}^c d_i\}$.

The proof follows from Lemma 5.1 and Corollary 5.

Corollary 11. (upper bound of the expected value) Let the conditions of Lemma 5.1 be satisfied.

If there exists a constant $M \geq 1$, such that

$$\sum_{k=1}^{C_t-1} d_k \frac{\prod_{i=1}^k \left(e^{\lambda d_i} |b_i| \left(\frac{\lambda^2(t-T_0)}{m_{i-1}+\lambda} \right)^{\alpha_i} \right)}{(t-T_0 - \sum_{i=1}^{k-1} d_i) \Gamma(\sum_{i=1}^k \alpha_i)} \leq M, \quad (30)$$

$$t > T_0$$

and

$$\prod_{i=1}^k \left(e^{m_k d_i} |b_i| \left(\frac{\lambda}{m_{i-1} - m_k + \lambda} \right)^{\alpha_i} \right) \leq M, \quad (31)$$

$$k = 1, 2, \dots$$

then

$$E(|u(t; T_0, u_0, \{\tau_k\})|) \leq |u_0| e^{-m_0(t-T_0)} P(S_0(t)) + |u_0| e^{-\lambda(t-T_0)} M + |u_0| \sum_{k=1}^{C_t-1} e^{-m_k(t-T_0)} MP(S_k(t)). \quad (32)$$

If additionally there exists a constant $m^* > 0$, such that $\lambda \geq m^*$, $m_i \geq m^*$, $i = 1, 2, \dots, k$, then

$$E(|u(t; T_0, u_0, \{\tau_k\})|) \leq |u_0| M e^{-m^*(t-T_0)} \left(1 + \sum_{k=1}^{C_t-1} P(S_k(t)) \right) \leq 2|u_0| M e^{-m^*(t-T_0)}. \quad (33)$$

6 p-moment exponential stability for RNIDE

The main goal of the paper is to define the exponential stability of the zero solution of RNIDE (10) (with $x_0 = 0$) and to obtain some sufficient conditions for it.

Definition 3. Let $p > 0$. Then the trivial solution ($x_0 = 0$) of the RNIDE (10) is said to be p -moment exponentially stable if for any initial point $x_0 \in \mathbb{R}^n$ there exist constants $\alpha, \mu > 0$, such that $E[||x(t; T_0, x_0, \{\tau_k\})||^p] < \alpha ||x_0||^p e^{-\mu(t-T_0)}$ for all $t > T_0$, where $x(t; T_0, x_0, \{\tau_k\})$ is the solution of the IVP for the RNIDE (10).

Remark 6. We note that the two-moment exponential stability for stochastic equations is known as exponential stability in mean square.

In this section we will use Lyapunov functions to obtain sufficient conditions for the p -moment exponential stability of the trivial solution of nonlinear system with impulses occurring at random moments and acting non-instantaneously.

Definition 4. Let $J \subset \mathbb{R}_+$ be a given interval and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$ be a given set. We will say that the function $V(t, x) : J \times \Delta \rightarrow \mathbb{R}_+$, $V(t, 0) \equiv 0$ belongs to the class $\Lambda(J, \Delta)$ if it is continuous on $J \times \Delta$ and locally Lipschitzian with respect to its second argument.

For functions $V(t, x) \in \Lambda(J, \Delta)$ we will use Dini derivatives defined by :

$$D_+ V(t, x) = \limsup_{h \rightarrow 0^+} \frac{1}{h} \left\{ V(t, x) - V(t-h, x-hf(t, x)) \right\} \text{ for } t \in J, x \in \Delta,$$

where there exists $h_1 > 0$, such that $t-h \in J$, $x-hf(t, x) \in \Delta$ for $0 < h \leq h_1$.

Theorem 6.1. *Let the following conditions be satisfied:*

1. *Conditions (H1), (H2), (H3), (H4) hold.*
2. *The function $V \in \Lambda([T_0, \infty), \mathbb{R}^n)$ and there exist positive constants a, b , such that*
 - (i) $a||x||^p \leq V(t, x) \leq b||x||^p$ for $t \geq T_0, x \in \mathbb{R}^n$;
 - (ii) *there exists a constant m satisfying $0 < m \leq \lambda$, such that the inequality*

$$D_+ V(t, x) \leq -mV(t, x), \quad \text{for } t > T_0, x \in \mathbb{R}^n$$

holds;

- (iii) *for any $k = 1, 2, \dots$ there exist functions $w_k \in C(\mathbb{R}_+, \mathbb{R}_+)$, such that*

$$V(t, I_k(x)) \leq w_k(t)V(t, x) \quad \text{for } t \geq T_0, x \in \mathbb{R}^n. \quad (34)$$

3. *There exist positive constants $M \geq 1$ and $C_k < 1, k = 1, 2, \dots$, such that $w_k(t) \leq C_k$ for $t \geq T_0$ and for $t > T_0$ the following hold*

$$2\lambda \sum_{k=1}^{\infty} d_k e^{\lambda \sum_{i=1}^k d_i} (\lambda(t - T_0))^{\sum_{i=1}^k \alpha_i - 1} \prod_{i=1}^k C_i \leq M, \quad (35)$$

and

$$\prod_{i=1}^k e^{m d_i} C_i \leq M, \quad k = 1, 2, \dots \quad (36)$$

Then the trivial solution of the RNIDE (10) is p -moment exponentially stable.

Proof: Let $x_0 \in \mathbb{R}^n$ be an arbitrary initial point and the stochastic process $x_\tau(t) = x(t; T_0, x_0, \{\tau_k\})$ be a solution of the initial value problem for the RNIDE (10).

Now consider the IVP for the scalar linear RNIDE (21) with $m_k = m, b_k = C_k$ for $k = 1, 2, \dots$, and $x_0 = V(T_0, x_0)$. According to Lemma 5.1 the solution $u(t; T_0, V(T_0, x_0), \{\tau_k\})$ of RNIDE (21) is given by (22).

Let t_k be an arbitrary value of the random variable $\tau_k, k = 1, 2, \dots$. Then $T_k = T_0 + \sum_{i=1}^k t_i + \sum_{i=1}^k d_i, k = 1, 2, \dots$ are values of the random variables ξ_k . Thus the corresponding function $x(t) = x(t; T_0, x_0, \{T_k\})$ is a sample path solution of the IVP for RNIDE (10), i.e., $x(t) = x(t; T_0, x_0, \{T_k\})$ is a solution of the IVP for the NIDE with fixed points of impulses (1).

For any given $\{T_k\}$ and x_0 we define

$$v(t) = V(t, x(t; T_0, x_0, \{T_k\})), \quad t \geq T_0, t \neq T_k$$

and

$$v(T_k) = V(T_k, x(T_k - 0; T_0, x_0, \{T_k\})), \quad k = 1, 2, \dots$$

Let $t \in (T_k, s_k], k = 0, 1, 2, \dots$. Using the continuity and monotonicity of the function $V(t, x)$ and condition 2(iii) we obtain

$$\begin{aligned} v(t) &= V(t, x(t)) = V(t, I_k(x(T_k - 0))) \\ &\leq w_k(t)V(t, x(T_k - 0)) \leq w_k(t)V(T_k, x(T_k - 0)) \\ &= w_k(t)v(T_k - 0) \\ &\leq C_k v(T_k - 0), \quad t \in (T_k, s_k], \quad k = 0, 1, 2, \dots \end{aligned} \quad (37)$$

Now, consider any interval $(s_k, T_{k+1}]$. Using $v(s_k) = V(s_k, x(s_k; T_0, x_0, \{T_k\}))$ we obtain

$$\begin{aligned} v'(t) &= D_+ v(t) = D_+ V(t, x(t; T_0, x_0, \{T_k\})) \\ &\leq -mV(t, x(t; T_0, x_0, \{T_k\})) = -mv(t), \quad t \in (s_k, T_{k+1}]. \end{aligned} \quad (38)$$

Therefore, from (37) and (38) it follows that the function $v(t)$ satisfies the linear impulsive differential inequalities with fixed points of non-instantaneous impulses:

$$\begin{aligned} v'(t) &\leq -m v(t) \quad \text{for } s_k < t < T_{k+1}, \quad k = 1, 2, \dots, \\ v(T_k+) &\leq C_k v(T_k) \quad \text{for } T_k < t \leq s_k, \quad k = 1, 2, \dots, \\ v(T_0) &= V(T_0, x_0). \end{aligned} \quad (39)$$

Consider the function

$$m(t) = v(t) - u(t; T_0, V(T_0, x_0), \{\tau_k\}) \quad \text{for } t \geq T_0.$$

The function $m(t)$ is piecewise continuous and according to Proposition 1 it is non-positive on the interval $[T_0, \infty)$, i.e.,

$$v(t) \leq u(t; T_0, x_0, \{T_k\}) \quad \text{for } t \geq T_0. \quad (40)$$

Note that inequality (40) is satisfied for any arbitrary given sequence of fixed points of impulses $\{T_k\}$. Therefore, the generated by $v(t)$ stochastic process $v_\tau(t)$ satisfies the inequality $v_\tau(t) \leq u(t; T_0, x_0, \{\tau_k\})$.

Inequality (36) proves the validity of (31) with $m_i = m$ and $b_i = C_i$. From condition 3 we obtain

$$\begin{aligned} \sum_{k=1}^{C_t-1} d_k \frac{\prod_{i=1}^k \left(e^{\lambda d_i} C_i \left(\frac{\lambda^2(t-T_0)}{m+\lambda} \right)^{\alpha_i} \right)}{(t-T_0 - \sum_{i=1}^{k-1} d_i) \Gamma(\sum_{i=1}^k \alpha_i)} &\leq 2 \sum_{k=1}^{C_t-1} d_k \frac{\prod_{i=1}^k \left(e^{\lambda d_i} C_i (\lambda(t-T_0))^{\alpha_i} \right)}{(t-T_0 - \sum_{i=1}^{k-1} d_i)} \\ &\leq 2 \sum_{k=1}^{\infty} d_k \frac{\prod_{i=1}^k \left(e^{\lambda d_i} C_i (\lambda(t-T_0))^{\alpha_i} \right)}{(t-T_0 - \sum_{i=1}^{k-1} d_i)} \leq M, \quad t > T_0. \end{aligned} \quad (41)$$

Therefore, inequality (30) holds with $m_i = m$ and $b_i = C_i$.

From Corollary 11 and inequality (33) with $m^* = m$ and condition 2(i) of Theorem 6.1 we obtain the inequalities

$$\begin{aligned} E(\|x_\tau(t)\|^p) &= \frac{1}{a} E(a\|x_\tau(t)\|^p) \leq \frac{1}{a} E(V(t, x_\tau(t))) \leq \frac{1}{a} E(v_\tau(t)) \\ &\leq \frac{1}{a} E(u(t; T_0, x_0, \{\tau_k\})) \leq \frac{2M}{a} V(T_0, x_0) e^{-m(t-T_0)} \leq \frac{2Mb}{a} \|x_0\|^p e^{-m(t-T_0)}, \quad t \geq T_0. \end{aligned} \quad (42)$$

Inequality (42) proves the p-moment exponential stability. □

Remark 7. Note that the inequalities (35) and (36) are satisfied, for example, for $C_i = e^{-\lambda d_i}$, $d_i = \frac{1}{i}$, $\alpha_i = e^{-i}$, because $\sum_{i=1}^k \alpha_i = \frac{1-e^{-k}}{e-1} < 1$ for $k \geq 1$.

Theorem 6.2. Let :

1. Conditions (H1), (H2), (H3), (H5) hold.
2. Condition 2 of Theorem 6.1 be satisfied.
3. There exists a positive constant $C < 1$, such that $w_k(t) \leq C$ for $t \geq T_0$ and

$$2B\lambda \sum_{k=1}^{\infty} \left(\lambda(t-T_0) \right)^{\sum_{i=1}^k \alpha_i - 1} \leq e^{mB} \quad \text{for any } t > T_0. \quad (43)$$

Then the trivial solution of the RNIDE (10) is p-moment exponentially stable.

Proof: Choose $M = e^{m \sum_{i=1}^{\infty} d_i} = e^{mB} > 1$. Then $Ce^{m \sum_{i=1}^k d_i} \leq Ce^{m \sum_{i=1}^{\infty} d_i} = Ce^{mB} \leq M$, i.e., inequality (31) holds with $m_i = m$, $b_i = C$.

Using $\Gamma(x) \geq 0.5$ for $x \geq 0$ and $\frac{\lambda}{m+\lambda} < 1$ we have

$$\begin{aligned} \sum_{k=1}^{C_l-1} d_k C^k \frac{\prod_{i=1}^k \left(e^{\lambda d_i} \left(\frac{\lambda^2(t-T_0)}{m+\lambda} \right)^{\alpha_i} \right)}{(t-T_0 - \sum_{i=1}^{k-1} d_i) \Gamma(\sum_{i=1}^k \alpha_i)} &\leq 2 \sum_{k=1}^{C_l-1} C^k d_k \frac{(\lambda(t-T_0))^{\sum_{i=1}^k \alpha_i}}{(t-T_0 - \sum_{i=1}^{k-1} d_i)} \\ &\leq 2\lambda \sum_{k=1}^{\infty} d_k (\lambda(t-T_0))^{\sum_{i=1}^k \alpha_i - 1} \\ &\leq 2B\lambda \sum_{k=1}^{\infty} (\lambda(t-T_0))^{\sum_{i=1}^k \alpha_i - 1} \leq e^{mB}, \quad t > T_0. \end{aligned}$$

Therefore, inequality (30) holds with $m_i = m$ and $b_i = C$ and we can apply Corollary 11. The rest of the proof is the same as the proof of Theorem 6.1 so we omit it. $\square$

Remark 8. Note that the inequality (43) is satisfied for example, for $\alpha_i = e^{-i}$, because $\sum_{i=1}^k \alpha_i = \frac{1-e^{-k}}{e-1} < 1$ for $k \geq 1$.

Example 2. (*p-exponential stability of RNIDE*). Let τ_i , $i = 1, 2, \dots$ be independent Gamma distributed random variables with the same rate parameter λ and shape parameter α_k respectively, i.e., $\tau_k \in \Gamma(\alpha_k, \lambda)$ with $\lambda \in (0, 0.1]$, $\alpha_k = e^{-k}$. Let $d_k = \frac{\pi}{4}$. Let the points $s_k = (4k+1)\frac{\pi}{2}$, $t_k = (4k-1)\frac{\pi}{2}$, $k = 1, 2, \dots$, $s_0 = 0$. Consider the following initial value problem for the scalar non-instantaneous impulsive differential equation

$$\begin{aligned} x'(t) &= xf(t) \quad \text{for } t \geq 0, \quad \xi_k + \frac{\pi}{2} < t < \xi_{k+1}, \quad k = 0, 1, 2, \dots, \\ x(t) &= I_k(t)x(t_k - 0) \quad \text{for } \xi_k < t < \xi_k + \frac{\pi}{2}, \quad k = 1, 2, \dots, \\ x(0) &= x_0, \end{aligned} \tag{44}$$

where $x \in \mathbb{R}$, $I_k(t) : \mathbb{R}_+^n \rightarrow \mathbb{R}^n$, $|I_k(t)| \leq C_k$, $C_k \leq e^{-\frac{\pi}{40}}$, $f(t) = 0.5 \frac{\cos(t) - \frac{t+1}{t+2}}{2 - \sin(t)}$, $k = 0, 1, 2, \dots$.

The zero solution of the corresponding ODE $x'(t) = xf(t)$ is exponentially stable (see Figure 1).

Also, to illustrate the behavior of the solution of (44) with $x_0 = 0.3$, $I_k(t) = \cos(t)e^{-\frac{\pi}{40}}$, we graph some of the path solutions for several randomly selected different values of τ_i , $i = 1, 2, \dots$, i.e. different values of the waiting time until the next impulse. Firstly, one value of each random variable τ_i , $i = 1, 2, 3, 4$ is chosen, i.e., $\tau_1 = 0.5$, $\tau_2 = 1$, $\tau_3 = 5$, $\tau_4 = 0.4$ and the corresponding path solution is graphed on Figure 3. Also, three values of each of the random variables τ_i , $i = 1, 2, 3, 4$ are chosen and the corresponding path solutions are graphed on Figure 4, i.e., for values $\tau_1 = 0.5$, $\tau_2 = 1$, $\tau_3 = 5$, $\tau_4 = 0.4$ the graph of the path solution is the black dashed curve; for $\tau_1 = 1$, $\tau_2 = 0.5$, $\tau_3 = 3$, $\tau_4 = 2$ the graph of the path solution is the black curve, and for $\tau_1 = 5$, $\tau_2 = 1$, $\tau_3 = 6$, $\tau_4 = 0.5$ the graph of the path solution is the black thick curve. From these graphs it is seen the path solutions approaches 0.

Let $V(t, x) = x^2$. Then $D_+V(t, x) = 2x^2f(t)$. Since the sign of the function $f(t)$ changes (see Figure 2) the quadratic Lyapunov function is not applicable to equation (44).

Let $V(t, x) = (2 - \sin(t))x^2$. The condition 2(i) of Theorem 6.1 is satisfied with $a = 1$, $b = 3$. Apply the Dini derivative and we get

$$D_+V(t, x) = 2x^2(2 - \sin(t))f(t) - x^2 \cos(t) = -\frac{0.5(t+1)}{(t+2)(2 - \sin(t))} V(t, x) \leq -0.1V(t, x),$$

i.e., Condition 2(ii) of Theorem 6.1 is satisfied with $m = 0.1$.

Let $k = 1, 2, \dots$, $x \in \mathbb{R}$ and we get $V(t, I_k(x)) \leq (2 - \sin(t))(c_k x)^2 = (2 - \sin(t))c_k^2 x^2 = c_k^2 V(t, x)$, i.e., condition 2(ii) is satisfied.

Figure 1: Solutions of ODE $x'(t) = 0.5x \frac{\cos(t) - \frac{t+1}{t+2}}{2-\sin(t)}$ with $x_0 = 0.1, 0.5, 1$.

Figure 2: Graph of $f(t) = 0.5 \frac{\cos(t) - \frac{t+1}{t+2}}{2-\sin(t)}$.

The inequalities (35) and (36) are satisfied with $d_i = \frac{\pi}{4}$, $m = 0.1$ and $T_0 = 0$. Indeed,

$$\begin{aligned} 2\lambda \sum_{k=1}^{\infty} d_k e^{\lambda \sum_{i=1}^k d_i} (\lambda(t - T_0))^{\sum_{i=1}^k \alpha_i - 1} \prod_{i=1}^k C_i &\leq \lambda \frac{\pi}{2} \sum_{k=1}^{\infty} e^{(\lambda - 0.1) \frac{k\pi}{4}} (\lambda t)^{\sum_{i=1}^k e^{-i} - 1} \\ &= \lambda \frac{\pi}{2} \sum_{k=1}^{\infty} e^{(\lambda - 0.1) \frac{k\pi}{4}} (\lambda t)^{\frac{1 - e^{-k}}{e - 1} - 1}. \end{aligned} \quad (45)$$

Since $\lambda - 0.1 \leq 0$, and $\frac{1 - e^{-k}}{e - 1} < 1$ for any $t > 0$ there exists $M = M(t) \geq 1$, such that

$$\frac{\lambda\pi}{2} \sum_{k=1}^{\infty} e^{(\lambda - 0.1) \frac{k\pi}{4}} (\lambda t)^{\frac{1 - e^{-k}}{e - 1} - 1} \leq M, \quad t > 0$$

and

$$\prod_{i=1}^k e^{md_i} C_i \leq \prod_{i=1}^k e^{\frac{\pi}{40}} C_i \leq 1 \leq M, \quad k = 1, 2, \dots \quad (46)$$

From Theorem 6.1 the zero solution of (44) is exponentially stable in mean square, i.e., the random non-instantaneous impulses could preserve the stability properties of the zero solution. $\square$

Figure 3: Graph of solution of (44) for one value of each random variable τ_k .

Figure 4: Graphs of solutions of (44) for various values of random variables τ_k .

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