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Stability of an additive-quadratic-quartic functional equation

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Abstract: In this paper, we investigate the stability of an additive-quadratic-quartic functional equation

$$f(x+2y) + f(x-2y) - 2f(x+y) - 2f(-x-y) - 2f(x-y) - 2f(y-x) \\ + 4f(-x) + 2f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) = 0$$

by the direct method in the sense of Găvruta.

Keywords: Hyers-Ulam stability, hyperstability, quadratic functional equation, fixed point theorem

MSC: 39B82, 39B52

1 Introduction

A. K. Hassan *et al.* [1], M. Mohamadi *et al.* [2] and C. Park *et al.* [3] investigated the stability of the AQQ (additive-quadratic-quartic) functional equation

$$f(x+2y) + f(x-2y) - 2f(x+y) - 2f(-x-y) - 2f(x-y) - 2f(y-x) \\ + 4f(-x) + 2f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) = 0 \quad (1.1)$$

in various spaces. For the terminology “AQQ (additive-quadratic-quartic) functional equation”, refer to the papers [1–3]. The second author [4] also studied different type of the additive-quadratic-quartic functional equation

$$f(x+ky) + f(x-ky) - k^2f(x+y) - k^2f(x-y) \\ + 2(k^2-1)f(x) + (k^2+k)f(y) + (k^2-k)f(-y) - 2f(ky) = 0,$$

where k is a fixed real constant with $k \neq 0, \pm 1$.

In this paper, let V and W be real vector spaces and Y be a real Banach space. For a given mapping $f: V \rightarrow W$, we use the following abbreviations

$$f_o(x) := \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2}, \\ Df(x, y) = f(x+2y) + f(x-2y) - 2f(x+y) - 2f(-x-y) - 2f(x-y) \\ - 2f(y-x) + 4f(-x) + 2f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y)$$

for all $x, y \in V$. In this paper, we will prove the stability of the functional equation (1.1) in the sense of Găvruta [5] (See also [6, 7]). In other words, from the given mapping f that approximately satisfies the functional

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equation (1.1), we will show that the mapping F , which is the solution of the functional equation (1.1), can be constructed using the formula

$$F(x) = \lim_{n \rightarrow \infty} \left(\frac{f_o(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 16^n} \right)$$

or

$$F(x) = \lim_{n \rightarrow \infty} \left(2^n f_o \left(\frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n}{3} f_e \left(\frac{x}{2^n} \right) - \frac{16^{n+1} - 4^{n+2}}{3} f_e \left(\frac{x}{2^{n+1}} \right) \right)$$

and we will prove that the mapping F is the unique solution mapping of functional equation (1.1) near the mapping f .

2 Main results

Lemma 1. If a mapping $f : V \rightarrow W$ satisfies $Df(x, y) = 0$ for all $x, y \in V$, then the equalities

$$f(x) = \frac{f_o(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 16^n}, \quad (2.1)$$

$$f(x) = 2^n f_o \left(\frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n}{3} f_e \left(\frac{x}{2^n} \right) - \frac{16^{n+1} - 4^{n+2}}{3} f_e \left(\frac{x}{2^{n+1}} \right) \quad (2.2)$$

hold for all $x \in V$ and all $n \in \mathbb{N} \cup \{0\}$.

Proof. If a mapping $f : V \rightarrow W$ satisfies $Df(x, y) = 0$ for all $x, y \in V$, then the equality (2.1) can be derived from the equalities

$$\begin{aligned} & f(x) - \frac{f_o(2^n x)}{2^n} - \frac{16f_e(2^n x) + f_e(2^{n+1}x)}{12 \cdot 4^n} + \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 16^n} \\ &= \sum_{i=0}^{n-1} \frac{2f_o(2^i x) - f_o(2^{i+1}x)}{2^{i+1}} + \sum_{i=0}^{n-1} \frac{64f_e(2^i x) - 20f_e(2^{i+1}x) + f_e(2^{i+2}x)}{12 \cdot 4^{i+1}} \\ &\quad + \sum_{i=0}^{n-1} -\frac{64f_e(2^i x) - 20f_e(2^{i+1}x) + f_e(2^{i+2}x)}{12 \cdot 16^{i+1}} \\ &= \sum_{i=0}^{n-1} \frac{-Df_o(2^i x, 2^{i-1}x)}{2^{i+1}} + \sum_{i=0}^{n-1} \frac{Df_e(2^{i+1}x, 2^i x) + 4Df_e(2^i x, 2^i x)}{12 \cdot 4^{i+1}} + \sum_{i=0}^{n-1} \frac{-Df_e(2^{i+1}x, 2^i x) - 4Df_e(2^i x, 2^i x)}{12 \cdot 16^{i+1}} \end{aligned}$$

for all $x \in V$ and $n \in \mathbb{N} \cup \{0\}$. The equality (2.2) can be easily obtained in a similar way. $\square$

In the following theorem, we can prove the generalized Hyers-Ulam stability of the functional equation (1.1) in the sense of Găvruta.

Theorem 1. Let $f : V \rightarrow Y$ be a mapping for which there exists a function $\varphi : (V \setminus \{0\})^2 \rightarrow [0, \infty)$ such that the inequality

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (2.3)$$

holds for all $x, y \in V \setminus \{0\}$ and let $f(0) = 0$. If φ has the property

$$\sum_{n=0}^{\infty} \frac{\varphi(2^n x, 2^n y)}{2^n} < \infty \quad (2.4)$$

for all $x, y \in V$, then there exists a unique solution mapping $F : V \rightarrow Y$ of the functional equation (1.1) satisfying the inequality

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \left(\frac{\varphi_e(2^i x, 2^{i-1}x)}{2^{i+1}} + \frac{\varphi_e(2^{i+1}x, 2^i x) + 4\varphi_e(2^i x, 2^i x)}{12 \cdot 4^{i+1}} \right) \quad (2.5)$$

for all $x \in V \setminus \{0\}$, where φ_e is the function defined by $\varphi_e(x, y) := \frac{\varphi(x, y) + \varphi(-x, -y)}{2}$. In particular, F is represented by

$$F(x) = \lim_{n \rightarrow \infty} \left(\frac{f_o(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 16^n} \right) \quad (2.6)$$

for all $x \in V$.

Proof. First, we define a set $A := \{f : V \rightarrow Y \mid f(0) = 0\}$ and a mapping $J_n : A \rightarrow A$ by

$$J_n f(x) := \frac{f_o(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 16^n}$$

for $x \in V$ and $n \in \mathbb{N} \cup \{0\}$. Notice that

$$\begin{aligned} \|J_n f(x) - J_{n+1} f(x)\| &= \left\| \frac{f_o(2^n x)}{2^n} - \frac{f_o(2^{n+1}x)}{2^{n+1}} + \frac{16f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 4^n} - \frac{16f_e(2^{n+1}x) - f_e(2^{n+2}x)}{12 \cdot 4^{n+1}} \right. \\ &\quad \left. - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 16^n} + \frac{4f_e(2^{n+1}x) - f_e(2^{n+2}x)}{12 \cdot 16^{n+1}} \right\| \\ &= \left\| \frac{-Df_o(2^n x, 2^{n-1}x)}{2^{n+1}} + \frac{Df_e(2^{n+1}x, 2^n x) + 4Df_e(2^n x, 2^n x)}{12 \cdot 4^{n+1}} \right. \\ &\quad \left. - \frac{Df_e(2^{n+1}x, 2^n x) - 4Df_e(2^n x, 2^n x)}{12 \cdot 16^{n+1}} \right\| \\ &\leq \frac{\varphi_e(2^n x, 2^{n-1}x)}{2^{n+1}} + \frac{\varphi_e(2^{n+1}x, 2^n x) + 4\varphi_e(2^n x, 2^n x)}{12 \cdot 4^{n+1}} \end{aligned} \quad (2.7)$$

for all $x \in V \setminus \{0\}$. It follows from (2.7) that

$$\begin{aligned} \|J_n f(x) - J_{n+m} f(x)\| &\leq \sum_{i=n}^{n+m-1} \|J_i f(x) - J_{i+1} f(x)\| \\ &\leq \sum_{i=n}^{n+m-1} \left(\frac{\varphi_e(2^i x, 2^{i-1}x)}{2^{i+1}} + \frac{\varphi_e(2^{i+1}x, 2^i x) + 4\varphi_e(2^i x, 2^i x)}{12 \cdot 4^{i+1}} \right) \end{aligned} \quad (2.8)$$

for all $x \in V \setminus \{0\}$. In view of (2.4) and (2.8), the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_n f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{n \rightarrow \infty} \left(\frac{f_o(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12 \cdot 16^n} \right)$$

for all $x \in V$. Moreover, letting $n = 0$ and passing the limit $m \rightarrow \infty$ in (2.8) we get the inequality (2.5). With the definition of F , we easily get the equality $DF(x, y) = 0$ from the relations

$$\begin{aligned} \|DF(x, y)\| &= \lim_{n \rightarrow \infty} \left\| \frac{Df_o(2^n x, 2^n y)}{2^n} + \frac{16Df_e(2^n x, 2^n y) - Df_e(2^{n+1}x, 2^{n+1}y)}{12 \cdot 4^n} \right. \\ &\quad \left. - \frac{4Df_e(2^n x, 2^n y) - Df_e(2^{n+1}x, 2^{n+1}y)}{12 \cdot 16^n} \right\| \\ &\leq \lim_{n \rightarrow \infty} \left(\left\| \frac{Df_o(2^n x, 2^n y)}{2^n} \right\| + \left\| \frac{16Df_e(2^n x, 2^n y)}{12 \cdot 4^n} \right\| + \left\| \frac{Df_e(2^{n+1}x, 2^{n+1}y)}{12 \cdot 4^n} \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} \left(\frac{\varphi_e(2^n x, 2^n y)}{2^n} + \frac{16\varphi_e(2^n x, 2^n y)}{12 \cdot 4^n} + \frac{\varphi_e(2^{n+1}x, 2^{n+1}y)}{12 \cdot 4^n} \right) \\ &= 0 \end{aligned}$$

for all $x, y \in V \setminus \{0\}$. From the equality $DF(x, y) = 0$ for all $x, y \in V \setminus \{0\}$ and $f(0) = 0$, it is easy to see that $DF(x, y) = 0$ for all $x, y \in V$. To prove the uniqueness of F , let $F' : V \rightarrow Y$ be another solution mapping of the functional equation (1.1) satisfying the inequality (2.5). Instead of the condition (2.5), it

is sufficient to show that there is a unique mapping F satisfying the simpler condition $\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} \frac{\varphi_e(2^i x, 2^{i-1} x) + \varphi_e(2^{i+1} x, 2^i x) + 4\varphi_e(2^i x, 2^i x)}{2^{i+1}}$. By (2.1), the equality $F'(x) = J_n F'(x)$ holds for all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \|J_n f(x) - F'(x)\| &= \|J_n f(x) - J_n F'(x)\| \\ &\leq \left\| \frac{f_o(2^n x)}{2^n} + \frac{16f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1} x)}{12 \cdot 16^n} \right. \\ &\quad \left. - \frac{F'_o(2^n x)}{2^n} - \frac{16F'_e(2^n x) - F'_e(2^{n+1} x)}{12 \cdot 4^n} + \frac{4F'_e(2^n x) - F'_e(2^{n+1} x)}{12 \cdot 16^n} \right\| \\ &\leq \left\| \frac{f_o(2^n x)}{2^n} - \frac{F'_o(2^n x)}{2^n} \right\| + \left(\frac{16}{12 \cdot 4^n} - \frac{4}{12 \cdot 16^n} \right) \|f_e(2^n x) - F'_e(2^n x)\| \\ &\quad + \left(\frac{1}{12 \cdot 4^n} - \frac{1}{12 \cdot 16^n} \right) \|f_e(2^{n+1} x) - F'_e(2^{n+1} x)\| \\ &\leq \sum_{i=0}^{\infty} \frac{\varphi_e(2^{i+n} x, 2^{i+n-1} x) + \varphi_e(2^{i+n+1} x, 2^{i+n} x) + 4\varphi_e(2^{i+n} x, 2^{i+n} x)}{2^{i+n+1}} \\ &\quad + 2 \sum_{i=0}^{\infty} \frac{\varphi_e(2^{i+n} x, 2^{i+n-1} x) + \varphi_e(2^{i+n+1} x, 2^{i+n} x) + 4\varphi_e(2^{i+n} x, 2^{i+n} x)}{2^{i+n+1}} \\ &\quad + \sum_{i=0}^{\infty} \frac{\varphi_e(2^{i+n+1} x, 2^{i+n} x) + \varphi_e(2^{i+n+2} x, 2^{i+n+1} x) + 4\varphi_e(2^{i+n+1} x, 2^{i+n+1} x)}{2^{i+n+2}} \\ &\leq 4 \sum_{i=n}^{\infty} \frac{\varphi_e(2^i x, 2^{i-1} x) + \varphi_e(2^{i+1} x, 2^i x) + 4\varphi_e(2^i x, 2^i x)}{2^i} \end{aligned}$$

for all $x \in V \setminus \{0\}$ and all $n \in \mathbb{N}$. Taking the limit in the above inequality as $n \rightarrow \infty$, we conclude that $F'(x) = \lim_{n \rightarrow \infty} J_n f(x)$ for all $x \in V \setminus \{0\}$. Because $F(0) = 0 = F'(0)$, this means that the equality $F(x) = F'(x)$ holds for all $x \in V$. $\square$

In the following corollary, we obtain the hyperstability of the functional equation (1.1).

Corollary 1. Let $p < 0$ be a real number and X be a real normed space. If $f : X \rightarrow Y$ is a mapping such that

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \quad (2.9)$$

for all $x, y \in X \setminus \{0\}$ and $f(0) = 0$, then $f : X \rightarrow Y$ satisfies the equality $Df(x, y) = 0$ for all $x, y \in X$.

Proof. According to Theorem 1, there is a unique solution mapping F of the functional equation $DF(x, y) = 0$ such that

$$\|f(x) - F(x)\| \leq \left(\frac{1 + 2^p}{2^p |2 - 2^p|} + \frac{2^p + 9}{12 |4 - 2^p|} \right) \theta \|x\|^p$$

for all $x \in X \setminus \{0\}$. From the equality

$$\begin{aligned} Df((2n+1)x, nx) &= Df((2n+1)x, nx) - DF((2n+1)x, nx) \\ &= (f - F)((4n+1)x) + (f - F)(x) - 2(f - F)((3n+1)x) \\ &\quad - 2(f - F)((-3n+1)x) - 2(f - F)((n+1)x) - 2(f - F)((-n+1)x) \\ &\quad + 4(f - F)((-2n+1)x) + 2(f - F)((2n+1)x) - (f - F)(2nx) \\ &\quad - (f - F)(-2nx) + 4(f - F)(nx) + 4(f - F)(-nx) \end{aligned}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$, we have the inequality

$$\|f(x) - F(x)\| \leq \|Df((2n+1)x, nx)\| + \|(f - F)((4n+1)x)\| + 2\|(f - F)((3n+1)x)\|$$

$$\begin{aligned}
& + 2\|(f - F)(-(3n + 1))\| + 2\|(f - F)((n + 1)x)\| + 2\|(f - F)(-(n + 1)x)\| \\
& + 4\|(f - F)(-(2n + 1)x)\| + 2\|(f - F)((2n + 1)x)\| + \|(f - F)(2nx)\| \\
& + \|(f - F)(-2nx)\| + 4\|(f - F)(nx)\| + 4\|(f - F)(-nx)\| \\
& \leq \left[\left(\frac{1 + 2^p}{2^p|2 - 2^p|} + \frac{2^p + 9}{12|4 - 2^p|} \right) ((4n + 1)^p + 4(3n + 1)^p + 6(2n + 1)^p \right. \\
& \quad \left. + 4(n + 1)^p + 2(2n)^p + 8 \cdot n^p) + n^p + (2n + 1)^p \right] \theta \|x\|^p
\end{aligned}$$

for all $x \in X \setminus \{0\}$ and $n \in \mathbb{N}$. Since $(4n + 1)^p$, $(3n + 1)^p$, $(2n + 1)^p$, $(n + 1)^p$ and $(2n)^p$ tend to 0 as $n \rightarrow \infty$ and $f(0) = F(0)$, we get $f(x) = F(x)$ for all $x \in X$. Therefore, the equality $Df(x, y) = DF(x, y) = 0$ holds for all $x, y \in X$. $\square$

Theorem 2. Let $f : V \rightarrow Y$ be a mapping for which there exists a function $\varphi : V^2 \rightarrow [0, \infty)$ such that the inequality

$$\|Df(x, y)\| \leq \varphi(x, y) \quad (2.10)$$

holds for all $x, y \in V$ and let $f(0) = 0$. If φ has the property

$$\sum_{n=0}^{\infty} 16^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) < \infty \quad (2.11)$$

for all $x, y \in V$, then there exists a unique solution mapping $F : V \rightarrow Y$ of the functional equation (1.1) satisfying the inequality

$$\|f(x) - F(x)\| \leq \sum_{n=0}^{\infty} \left(\frac{16^{n+1}}{3} \varphi_e\left(\frac{x}{2^{n+2}}, \frac{x}{2^{n+2}}\right) + \left(\frac{4 \cdot 16^n}{3} + 2^n\right) \varphi_e\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+2}}\right) \right) \quad (2.12)$$

for all $x \in V \setminus \{0\}$. In particular, F is represented by

$$F(x) = \lim_{n \rightarrow \infty} \left(2^n f_o\left(\frac{x}{2^n}\right) + \frac{4 \cdot 16^n - 4^n}{3} f_e\left(\frac{x}{2^n}\right) - \frac{16^{n+1} - 4^{n+2}}{3} f_e\left(\frac{x}{2^{n+1}}\right) \right) \quad (2.13)$$

for all $x \in V$.

Proof. First, we define $A := \{f : V \rightarrow Y \mid f(0) = 0\}$ and a mapping $J_n : A \rightarrow A$ by

$$J_n f(x) := 2^n f_o\left(\frac{x}{2^n}\right) + \frac{4 \cdot 16^n - 4^n}{3} f_e\left(\frac{x}{2^n}\right) - \frac{16^{n+1} - 4^{n+2}}{3} f_e\left(\frac{x}{2^{n+1}}\right)$$

for $x \in V$ and $n \in \mathbb{N} \cup \{0\}$. Notice that

$$\begin{aligned}
\|J_n f(x) - J_{n+1} f(x)\| & \leq \left\| \frac{(4 \cdot 16^n - 4^n)}{3} \left(f_e\left(\frac{x}{2^n}\right) - 20 f_e\left(\frac{x}{2^{n+1}}\right) + 64 f_e\left(\frac{x}{2^{n+2}}\right) \right) \right. \\
& \quad \left. + 2^n \left(f_o\left(\frac{x}{2^n}\right) - 2 f_o\left(\frac{x}{2^{n+1}}\right) \right) \right\| \\
& \leq \frac{4 \cdot 16^n}{3} \left\| Df_e\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+2}}\right) + 4 Df_e\left(\frac{x}{2^{n+2}}, \frac{x}{2^{n+2}}\right) \right\| + 2^n \left\| Df_o\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+2}}\right) \right\| \\
& \leq 2^n \varphi_e\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+2}}\right) + \frac{4 \cdot 16^n}{3} \left(\varphi_e\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+2}}\right) + 4 \varphi_e\left(\frac{x}{2^{n+2}}, \frac{x}{2^{n+2}}\right) \right)
\end{aligned} \quad (2.14)$$

for all $x \in V \setminus \{0\}$. It follows from (2.14) that

$$\|J_n f(x) - J_{n+m} f(x)\| \leq \sum_{i=n}^{n+m-1} \left(\frac{16^{i+1}}{3} \varphi_e\left(\frac{x}{2^{i+2}}, \frac{x}{2^{i+2}}\right) + \left(\frac{4 \cdot 16^i}{3} + 2^i\right) \varphi_e\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+2}}\right) \right) \quad (2.15)$$

for all $x \in V \setminus \{0\}$.

In view of (2.11) and (2.15), the sequence $\{J_n f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_n f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \rightarrow Y$ by

$$F(x) := \lim_{n \rightarrow \infty} \left(2^n f_o \left(\frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n}{3} f_e \left(\frac{x}{2^n} \right) - \frac{16^{n+1} - 4^{n+2}}{3} f_e \left(\frac{x}{2^{n+1}} \right) \right)$$

for all $x \in V$. Moreover, letting $n = 0$ and passing the limit $n \rightarrow \infty$ in (2.15) we get the inequality (2.12). From the definition of F , we easily get

$$\begin{aligned} \|DF(x, y)\| &= \lim_{n \rightarrow \infty} \left\| 2^n Df_o \left(\frac{x}{2^n}, \frac{y}{2^n} \right) + \frac{4 \cdot 16^n - 4^n}{3} Df_e \left(\frac{x}{2^n}, \frac{y}{2^n} \right) - \frac{16^{n+1} - 4^{n+2}}{3} Df_e \left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}} \right) \right\| \\ &\leq \lim_{n \rightarrow \infty} \left(\left\| 2^n Df_o \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\| + \left\| \frac{4 \cdot 16^n}{3} Df_e \left(\frac{x}{2^n}, \frac{y}{2^n} \right) \right\| + \left\| \frac{16^{n+1}}{3} Df_e \left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}} \right) \right\| \right) \\ &\leq \lim_{n \rightarrow \infty} \left(2^n \varphi_e \left(\frac{x}{2^n}, \frac{y}{2^n} \right) + \frac{4 \cdot 16^n}{3} \varphi_e \left(\frac{x}{2^n}, \frac{y}{2^n} \right) + \frac{16^{n+1}}{3} \varphi_e \left(\frac{x}{2^{n+1}}, \frac{y}{2^{n+1}} \right) \right) \\ &= 0 \end{aligned}$$

for all $x, y \in V \setminus \{0\}$, which means that $DF(x, y) = 0$ for all $x, y \in V$ from the same reason in Theorem 1.

To prove the uniqueness of F , let $F' : V \rightarrow Y$ be another solution of the functional equation (1.1) satisfying (2.12). Instead of the condition (2.12), it is sufficient to show that there is a unique mapping satisfying the simpler condition

$$\|f(x) - F(x)\| \leq \sum_{i=0}^{\infty} 16^{i+1} \left(\varphi_e \left(\frac{x}{2^{i+2}}, \frac{x}{2^{i+2}} \right) + \varphi_e \left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+2}} \right) \right)$$

for all $x \in V$. By (2.2), the equality $F'(x) = J_n F'(x)$ holds for all $x \in V$ and all $n \in \mathbb{N}$. Therefore, we have

$$\begin{aligned} \|J_n f(x) - F'(x)\| &= \|J_n f(x) - J_n F'(x)\| \\ &\leq \left\| 2^n f_o \left(\frac{x}{2^n} \right) + \frac{4 \cdot 16^n - 4^n}{3} f_e \left(\frac{x}{2^n} \right) - \frac{16^{n+1} - 4^{n+2}}{3} f_e \left(\frac{x}{2^{n+1}} \right) \right. \\ &\quad \left. - 2^n F'_o \left(\frac{x}{2^n} \right) - \frac{4 \cdot 16^n - 4^n}{3} F'_e \left(\frac{x}{2^n} \right) + \frac{16^{n+1} - 4^{n+2}}{3} F'_e \left(\frac{x}{2^{n+1}} \right) \right\| \\ &\leq 2^n \left\| f_o \left(\frac{x}{2^n} \right) - F'_o \left(\frac{x}{2^n} \right) \right\| + 16^{n+1} \left\| f_o \left(\frac{x}{2^{n+1}} \right) - F'_o \left(\frac{x}{2^{n+1}} \right) \right\| + 16^{n+1} \left\| f_e \left(\frac{x}{2^n} \right) - F'_e \left(\frac{x}{2^n} \right) \right\| \\ &\leq \sum_{i=0}^{\infty} 16^{i+n+1} \left(\varphi_e \left(\frac{x}{2^{i+n+2}}, \frac{x}{2^{i+n+2}} \right) + \varphi_e \left(\frac{x}{2^{i+n+2}}, \frac{x}{2^{i+n+1}} \right) \right) \\ &\quad + \sum_{i=0}^{\infty} 16^{i+n+2} \left(\varphi_e \left(\frac{x}{2^{i+n+3}}, \frac{x}{2^{i+n+3}} \right) + \varphi_e \left(\frac{x}{2^{i+n+3}}, \frac{x}{2^{i+n+2}} \right) \right) \\ &\quad + 16 \sum_{i=0}^{\infty} 16^{i+n+1} \left(\varphi_e \left(\frac{x}{2^{i+n+2}}, \frac{x}{2^{i+n+2}} \right) + \varphi_e \left(\frac{x}{2^{i+n+2}}, \frac{x}{2^{i+n+1}} \right) \right) \\ &\leq 17 \sum_{i=n+1}^{\infty} 16^i \left(\varphi_e \left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}} \right) + \varphi_e \left(\frac{x}{2^{i+1}}, \frac{x}{2^i} \right) \right) \\ &\quad + \sum_{i=n+2}^{\infty} 16^i \left(\varphi_e \left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}} \right) + \varphi_e \left(\frac{x}{2^{i+1}}, \frac{x}{2^i} \right) \right) \end{aligned}$$

for all $x \in V$ and all $n \in \mathbb{N}$. Taking the limit in the above inequality as $n \rightarrow \infty$, we can conclude that $F'(x) = \lim_{n \rightarrow \infty} J_n f(x)$ for all $x \in V$. This means that $F(x) = F'(x)$ for all $x \in V$. $\square$

3 Conclusions

We have proved the stability of an additive-quadratic-quartic functional equation

$$\begin{aligned} f(x+2y) + f(x-2y) - 2f(x+y) - 2f(-x-y) - 2f(x-y) - 2f(y-x) \\ + 4f(-x) + 2f(x) - f(2y) - f(-2y) + 4f(y) + 4f(-y) = 0 \end{aligned}$$

by the direct method in the sense of Găvruta.

References

- [1] H. Azadi-Kenary, J.R. Lee, and C. Park, *Non-Archimedean stability of an AQQ functional equation*, J. Comput. Anal. Appl. **14** (2012), no. 2, 211–227.
- [2] M. Mohamadi, Y.J. Cho, C. Park, P. Vetro, and R. Saadati, *Random stability of an additive-quadratic-quartic functional equation*, J. Inequal. Appl. **2010** (2010), 754210, DOI: 10.1155/2010/754210.
- [3] C. Park, *Fuzzy stability of an additive-quadratic-quartic functional equation*, J. Inequal. Appl. **2010** (2010), 253040, DOI: 10.1155/2010/253040.
- [4] Y.-H. Lee, *A fixed point approach to the stability of an additive-quadratic-quartic functional equation*, Honam Mathematical J. (2019), accepted.
- [5] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
- [6] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.
- [7] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.