Research Article

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Further results on Ulam stability for a system of first-order nonsingular delay differential equations

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Abstract: This paper is concerned with a system governed by nonsingular delay differential equations. We study the β -Ulam-type stability of the mentioned system. The investigations are carried out over compact and unbounded intervals. Before proceeding to the main results, we convert the system into an equivalent integral equation and then establish an existence theorem for the addressed system. To justify the application of the reported results, an example along with graphical representation is illustrated at the end of the paper.

Keywords: delayed exponential matrix, delay differential system, representation of solutions, β -Hyers-Ulam-Rassias stability

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1 Introduction

Delay systems are used to characterize the evolution processes in automatic engines, physiological systems and control theory. Shuklin and Khusainov [1] introduced a notion of delayed matrix exponential and used it to derive a representation of solutions to linear delay problems under the restriction of permutable matrices. Khusainov and Diblik [2] and Wang et al. [3] used the ideas of [1] to introduce a discrete matrix delayed exponential function and to consider a representation of solutions to linear delay systems.

Among the qualitative properties of differential systems, stability is an essential property. There are different types of stabilities, but recently researchers focused on the Ulam-Hyers-type stability. The idea of the aforesaid stability was initially introduced by Ulam [4], in 1940, when he addressed a mathematical colloquium. During his talk, he raised a problem regarding the stability of group homomorphisms. In the following year, Hyers [5] responded positively to this problem under the assumption that groups are Banach spaces. Since then, this stability was named as Ulam-Hyers stability. Rassias [6], in 1978, made an extension to the result of Hyer's theorem, where the bound of norm of Cauchy difference was presented in a more general form. This stability phenomenon is termed as Hyers-Ulam-Rassias stability. For more information about the topic, we refer the reader to [7–14].

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In 2019, You et al. [3] studied the exponential stability and relative controllability of nonsingular delay differential equations of the form:

$$\begin{cases} A\mathbb{G}'(t) = M\mathbb{G}(t) + N\mathbb{G}(t - \Theta) + \mathcal{F}(t, \mathbb{G}(t - \Theta)), & t \geq 0, \Theta \geq 0, \\ \mathbb{G}(t) = \phi(t), & -\Theta \leq t \leq 0, \end{cases} \quad (1.1)$$

where A , M and N are constant permutable matrices of dimension $n \times n$, A is a nonsingular and $\phi \in \mathbb{C}^1([-\Theta, 0], \mathcal{R}^n)$, where $\mathbb{C}^1([-\Theta, 0], \mathcal{R}^n)$ is the space of all continuously differentiable functions from $[-\Theta, 0]$ to $\mathcal{R}^n$. In addition, the nonlinearity $\mathcal{F} \in \mathbb{C}([0, +\infty] \times \mathcal{R}^n; \mathcal{R}^n)$, the space of all continuous functions from $[0, +\infty] \times \mathcal{R}^n$ to $\mathcal{R}^n$.

Motivated from [3], and using the techniques of [15], we analyze the β -Hyers-Ulam-Rassias stability [16] of solutions for the nonsingular delay differential system (1.1). We carry out our investigations in two folds: stability results over a compact interval and stability results over an unbounded interval. Before proceeding to the main results, we convert system (1.1) into an equivalent integral equation and establish an existence theorem for its solutions. To justify the application of the reported results, an example along with a graphical illustration is presented at the end of the paper.

2 Essential background

Here we present some basic concepts and definitions that are essential in proving the main results. Let $\mathcal{R}$ represent the set of all real numbers, $\mathcal{R}^+$ represent the set of all nonnegative real numbers and $\mathcal{R}^n$ the space of all n -tuples of $\mathcal{R}$. The interval $\mathbb{I} = [0, \theta] \subseteq \mathcal{R}$ and $\mathbb{S} = \mathcal{R}^n$, $\mathbb{C}(\mathbb{I}, \mathbb{S})$, the Banach space of all continuous functions from $\mathbb{I}$ to $\mathbb{S}$ with the norm

$$\|\mathbb{G}\|_{\mathbb{C}} = \{\sup_{t \in \mathbb{I}} \|\mathbb{G}(t)\|, \text{ for all } \mathbb{G} \in \mathbb{C}(\mathbb{I}, \mathbb{S})\}.$$

Furthermore, we define $\mathbb{C}^1(\mathbb{I}, \mathbb{S}) = \{\mathbb{G} \in \mathbb{C}(\mathbb{I}, \mathbb{S}) : \mathbb{G}' \in \mathbb{C}(\mathbb{I}, \mathbb{S})\}$.

Definition 2.1. [15] Consider the vector space $\mathcal{V}$ over some field K . A function $\|\cdot\|_{\beta} : \mathcal{V} \rightarrow [0, \infty)$ is said to be β -norm, with $0 < \beta \leq 1$, if:

- (i) $\|\mathbb{G}\|_{\beta} = 0$ if and only if $\mathbb{G} = 0$,
- (ii) $\|\eta\mathbb{G}\|_{\beta} = |\eta|^{\beta} \|\mathbb{G}\|_{\beta}$ for each $\eta \in K$ and $\mathbb{G} \in \mathcal{V}$,
- (iii) $\|\mathbb{G} + \mathbb{G}_1\|_{\beta} \leq \|\mathbb{G}\|_{\beta} + \|\mathbb{G}_1\|_{\beta}$ for all $\mathbb{G}, \mathbb{G}_1 \in \mathcal{V}$.

The space $(\mathcal{V}, \|\cdot\|_{\beta})$ is called β -norm space.

Remark 2.2. [3] The nonsingular delay differential system

$$\begin{cases} A\mathbb{G}'(t) = M\mathbb{G}(t) + N\mathbb{G}(t - \Theta) + \mathcal{F}(t, \mathbb{G}(t - \Theta)), & t \geq 0, \Theta \geq 0, \\ \mathbb{G}(t) = \phi(t), & -\Theta \leq t \leq 0 \end{cases}$$

has the solution

$$\mathbb{G}(t) = Z(t + \Theta)\phi(-\Theta) + A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)]ds + A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \mathbb{G}(s - \Theta))ds,$$

where $Z(t) = e^{MA^{-1}t}e_{\Theta}^{N_1A^{-1}(t-\Theta)}$, $N_1 = e^{-MA^{-1}\Theta}B$ and $e_{\Theta}^{N_1A^{-1}(t-\Theta)}$ is the delayed matrix exponential, also $MA = AM$, $MN = NM$ and $NA = AN$ are used.

From above, one can clearly conclude that $\dot{\phi}$ exists ($\phi \in \mathbb{C}^1([-\Theta, 0], \mathcal{R}^n)$).

Definition 2.3. [15] The trivial solution of (1.1) is said to be exponentially stable if there exist positive constants $\sigma_1, \sigma_2, \delta$ depending on A, M, N and $\|\phi\|_1$, where $\|\phi\|_1 := \max_{[-\Theta, 0]} \|\phi\| + \max_{[-\Theta, 0]} \|\phi\|$, such that

$$\|\mathbb{G}(t)\| \leq \sigma_1 e^{-\sigma_2 t}, \quad t \geq 0. \quad (2.1)$$

Choose $\varepsilon > 0$ and φ from $\mathbb{C}(\mathbb{I}, \mathbb{S})$. Consider the inequality

$$\begin{cases} \|A\mathbb{G}'(t) - M\mathbb{G}(t) - N\mathbb{G}(t - \Theta) - \mathcal{F}(t, \mathbb{G}(t - \Theta))\| \leq \varepsilon\varphi(t), & t \geq 0, \Theta \geq 0, \\ \|\mathbb{G}(t) - \phi(t)\| \leq \varepsilon, & -\Theta \leq t \leq 0. \end{cases} \quad (2.2)$$

Definition 2.4. System (1.1) is said to be β -Hyers-Ulam-Rassias stable with respect to φ^β if there exists positive number $\mathcal{K}_{f, M, \varphi, \beta}$ depending upon f, M, φ, β such that for any $\varepsilon > 0$ and for any solution $\mathbb{G} \in \mathbb{C}^1(\mathbb{I}, \mathbb{S})$ of (2.2), there exists a solution Υ of (1.1) in $\mathbb{C}(\mathbb{I}, \mathbb{S})$ such that

$$\|\Upsilon(t) - \mathbb{G}(t)\|^\beta \leq \mathcal{K}_{f, M, \varphi, \beta} \varepsilon^\beta (\varphi^\beta(t)), \quad t \in \mathbb{I}.$$

Remark 2.5. It is an immediate consequence of (2.2) that $\Upsilon \in \mathbb{C}^1(\mathbb{I}, \mathbb{S})$ satisfies (2.2) if and only if there exist $f \in \mathbb{C}(\mathbb{I}, \mathbb{S})$ satisfying

$$\begin{cases} \|f(t)\| \leq \varepsilon\varphi(t), & t \in \mathbb{I}, \\ A\Upsilon'(t) = M\Upsilon(t) + N\Upsilon(t - \Theta) + \mathcal{F}(t, \Upsilon(t - \Theta)) + f(t), & t \in \mathbb{I}, \\ \Upsilon(t) = \phi(t). \end{cases}$$

Assume that

$$\mathcal{M} = \sup_{0 \leq s \leq t \leq \theta} \|Z(t - s)\|, \quad (2.3)$$

where θ is the length of the interval $[0, \theta]$.

From Remark 2.5, the solution of

$$\begin{cases} A\mathbb{G}'(t) = M\mathbb{G}(t) + N\mathbb{G}(t - \Theta) + \mathcal{F}(t, \mathbb{G}(t - \Theta)) + f(t), & t \geq 0, \Theta \geq 0, \\ \mathbb{G}(t) = \phi(t), & -\Theta \leq t \leq 0 \end{cases}$$

is

$$\mathbb{G}(t) = Z(t + \Theta)\phi(-\Theta) + A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)]ds + A^{-1} \int_0^t Z(t - s)(\mathcal{F}(s, \mathbb{G}(s - \Theta)) + f(s))ds,$$

where $Z(t) = e^{MA^{-1}t}e_{\Theta}^{N_1A^{-1}(t-\Theta)}$, $N_1 = e^{-MA^{-1}\Theta}B$ and $MA = AM$, $MN = NM$, $NA = AN$. For inequality (2.2), we obtain

$$\begin{aligned} & \left\| \mathbb{G}(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)]ds - A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \mathbb{G}(s - \Theta))ds \right\| \\ &= \left\| A^{-1} \int_0^t Z(t - s)f(s)ds \right\| \leq \|A^{-1}\| \int_0^t \|Z(t - s)\| \|f(s)\|ds \leq \|A^{-1}\| \int_0^t \mathcal{M} \|f(s)\|ds \leq S \int_0^t \varepsilon \mathcal{M} \varphi(s)ds, \end{aligned}$$

where $S \geq 0$ such that $\|A^{-1}\| \leq S$.

Lemma 2.6. [17] Let J denote an interval of the real line of the form $[a, \infty)$, or $[a, b]$ or $[a, b)$ with $a < b$. Let α, ϖ and U be the real valued functions defined on J . Assume that ϖ and U are continuous and that the negative part of α is integrable on every closed and bounded subinterval of J .

(a) If ϖ is nonnegative and U satisfies the integral inequality

$$U(t) \leq \alpha(t) + \int_a^t \varpi(s)U(s)ds, \quad \forall t \in J,$$

then

$$U(t) \leq \alpha(t) + \int_a^t \alpha(s) \varpi(s) \exp \left(\int_s^t \varpi(r) dr \right) ds, \quad t \in J.$$

(b) If, in addition, the function α is nondecreasing, then

$$U(t) \leq \alpha(t) \exp \left(\int_a^t \varpi(s) ds \right), \quad t \in J.$$

3 Existence result

To discuss existence result of the given system, we need some assumptions:

A_1 : The linear system $A\mathbb{G}'(t) = M\mathbb{G}(t) + N\mathbb{G}(t - \Theta)$ is well posed.

A_2 : The continuous function $\mathcal{F} : \mathbb{I} \times \mathbb{S} \rightarrow \mathbb{S}$ satisfies the Caratheodory condition

$$\|\mathcal{F}(t, u) - \mathcal{F}(t, u')\| \leq \mathcal{L}_{\mathcal{F}} \|u - u'\|, \quad \mathcal{L}_{\mathcal{F}} > 0$$

for every $u, u' \in \mathbb{S}$.

A_3 : The inequality $SM\mathcal{L}_{\mathcal{F}}\theta < 1$ holds.

Theorem 3.1. *If Assumptions $A_1 - A_3$ along with (2.3) hold, then system (1.1) has only one solution $\mathbb{G} \in \mathbb{C}(\mathbb{I}, \mathbb{S})$.*

Proof. Define $\Lambda : \mathbb{C}(\mathbb{I}, \mathbb{S}) \rightarrow \mathbb{C}(\mathbb{I}, \mathbb{S})$ by

$$(\Lambda\mathbb{G})(t) = Z(t + \Theta)\phi(-\Theta) + A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)] ds + A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \mathbb{G}(s - \Theta)) ds.$$

For any $\mathbb{G}, G' \in \mathbb{C}(\mathbb{I}, \mathbb{S})$, we have

$$\begin{aligned} \|(\Lambda\mathbb{G})(t) - (\Lambda G')(t)\| &\leq \|A^{-1}\| \int_0^t \|Z(t - s)\| \|\mathcal{F}(s, \mathbb{G}(s - \Theta)) - \mathcal{F}(s, G'(s - \Theta))\| ds \\ &\leq S \int_0^t M\mathcal{L}_{\mathcal{F}} \|\mathbb{G}(s) - G'(s)\| ds \\ &\leq SM\mathcal{L}_{\mathcal{F}}\theta \|\mathbb{G} - G'\|_{\mathbb{C}}, < \|\mathbb{G} - G'\|_{\mathbb{C}}, \end{aligned}$$

where θ is the length of the interval $[0, \theta]$. Hence, Λ is contractive and it has only one fixed point (from contraction mapping theorem) which is the solution of system (1.1). $\square$

4 β -Hyers-Ulam-Rassias stability on a compact interval

To discuss the β -Hyers-Ulam-Rassias stability of system (1.1) on a compact interval, consider the following additional assumptions:

A_2^* : $\mathcal{F} : \mathbb{I} \times \mathbb{S} \rightarrow \mathbb{S}$ satisfies the Caratheodory conditions and

$$\|\mathcal{F}(t, \mu_1) - \mathcal{F}(t, \mu_2)\| \leq \mathcal{L}_{\mathcal{F}}(t) \|\mu_1 - \mu_2\|,$$

for every $\mathcal{L}_{\mathcal{F}} \in \mathbb{C}(\mathbb{I}, \mathbb{S})$, $t \in \mathbb{I}$ and $\mu_1, \mu_2 \in \mathbb{S}$.

A_4 : $\mathcal{L}_{\mathcal{F}}(t)$ is nonnegative.

A_5 : Assume the negative part of $S\epsilon M\eta_{\varphi}\varphi(t)$ is integrable on every closed and bounded subinterval of J and it is nondecreasing.

A_6 : There exists nondecreasing $\varphi \in \mathbb{C}(\mathbb{I}, \mathbb{S})$ with $\varphi(t) \geq 0$ and a constant η_{φ} such that

$$\int_0^t \varphi(s) ds \leq \eta_{\varphi} \varphi(t), \quad \text{for each } t \in \mathbb{I}.$$

Theorem 4.1. If A_1, A_2^* and A_3 – A_6 along with (2.3) hold, then system (1.1) is β -Hyers-Ulam-Rassias stable with respect to φ^{β} .

Proof. The unique solution of the Cauchy problem

$$\begin{cases} A\mathbb{G}'(t) = M\mathbb{G}(t) + N\mathbb{G}(t - \Theta) + \mathcal{F}(t, \mathbb{G}(t - \Theta)), & t \in [0, \tau], \quad s, t \geq 0, \Theta \geq 0, \\ \mathbb{G}(t) = \phi(t), & -\Theta \leq t \leq 0 \end{cases}$$

can be written as

$$\mathbb{G}(t) = \left\{ Z(t + \Theta)\phi(-\Theta) + A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)] ds + A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \mathbb{G}(s - \Theta)) ds, \right.$$

where $Z(t) = e^{MA^{-1}t}e^{N_1A^{-1}(t-\Theta)}$, $N_1 = e^{-MA^{-1}\Theta}B$ and $MA = AM$, $MN = NM$, $NA = AN$.

Let Υ satisfy (2.2). Then for each $t \in \mathcal{R}^+$, we obtain

$$\begin{aligned} & \left\| \Upsilon(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)] ds - A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \Upsilon(s - \Theta)) ds \right\| \\ & \leq S \int_0^t \epsilon M\varphi(s) ds \leq S\epsilon M\eta_{\varphi}\varphi(t). \end{aligned}$$

Now consider

$$\begin{aligned} \|\Upsilon(t) - \mathbb{G}(t)\|^{\beta} &= \left\| \Upsilon(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)] ds \right. \\ & \quad \left. - A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \mathbb{G}(s - \Theta)) ds \right\|^{\beta} \\ &= \left\| \Upsilon(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)] ds \right. \\ & \quad \left. - A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \mathbb{G}(s - \Theta)) ds + A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \Upsilon(s - \Theta)) ds \right. \\ & \quad \left. - A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \Upsilon(s - \Theta)) ds \right\|^{\beta} \\ &\leq \left(\left\| \Upsilon(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)] ds \right. \right. \end{aligned}$$

$$\begin{aligned}
& - A^{-1} \int_0^t Z(t-s) \mathcal{F}(s, \Upsilon(s-\Theta)) ds \left\| \right\|^\beta \\
& + \left(\left\| A^{-1} \int_0^t Z(t-s) (\mathcal{F}(s, \Upsilon(s-\Theta)) - \mathcal{F}(s, \mathbb{G}(s-\Theta))) ds \right\| \right)^\beta dt \\
& \leq (S\mathcal{E}M\eta_\varphi \varphi(t))^\beta + \left(\left\| A^{-1} \int_0^t Z(t-s) \|\mathcal{F}(s, \Upsilon(s-\Theta)) - \mathcal{F}(s, \mathbb{G}(s-\Theta))\| ds \right\| \right)^\beta \\
& \leq (S\mathcal{E}M\eta_\varphi \varphi(t))^\beta + \left(S \int_0^t \mathcal{M}\mathcal{L}_{\mathcal{F}}(s) \|\Upsilon(s-\Theta) - \mathbb{G}(s-\Theta)\| ds \right)^\beta,
\end{aligned}$$

where

$$\int_0^t \|Z(t-s)\| \|\mathcal{F}(s, \Upsilon(s-\Theta)) - \mathcal{F}(s, \mathbb{G}(s-\Theta))\| ds \leq S \int_0^t \mathcal{M}\mathcal{L}_{\mathcal{F}}(s) \|\Upsilon(s) - \mathbb{G}(s)\| ds.$$

Thus,

$$\|\Upsilon(t) - \mathbb{G}(t)\| \leq 3^{\frac{1}{\beta}-1} \left(S\mathcal{E}M\eta_\varphi \varphi(t) + S \int_0^t \mathcal{M}\mathcal{L}_{\mathcal{F}}(s) \|\Upsilon(s) - \mathbb{G}(s)\| ds \right).$$

Since we know that

$$(\varepsilon + \delta + \eta)^\gamma \leq 3^{\gamma-1}(\varepsilon^\gamma + \delta^\gamma + \eta^\gamma), \text{ where } \varepsilon, \delta, \eta \geq 0 \text{ and } \gamma > 1.$$

Taking $\mathcal{L}_{\mathcal{J}} = \max\{\mathcal{L}_{\mathcal{J}_1}, \mathcal{L}_{\mathcal{J}_2}, \dots, \mathcal{L}_{\mathcal{J}_m}\}$ and using Lemma 2.6, we get

$$\|\Upsilon(t) - \mathbb{G}(t)\| \leq 3^{\frac{1}{\beta}-1} (S\mathcal{E}M\eta_\varphi \varphi(t)) \exp \left(3^{\frac{1}{\beta}-1} S \int_0^t \mathcal{M}\mathcal{L}_{\mathcal{F}}(s) ds \right).$$

Hence, we have

$$\begin{aligned}
\|\Upsilon(t) - \mathbb{G}(t)\|^\beta & \leq 3^{1-\beta} (S\mathcal{E}M\eta_\varphi \varphi(t))^\beta \exp \left(3^{\frac{1}{\beta}-1} S \int_0^t \mathcal{M}\mathcal{L}_{\mathcal{F}}(s) ds \right)^\beta \\
& \leq 3^{1-\beta} (S\mathcal{E}M\eta_\varphi \varphi(t))^\beta \exp \left(3^{\frac{1}{\beta}-1} S \int_0^t \mathcal{M}\mathcal{L}_{\mathcal{F}}(s) ds \right) \\
& \leq \mathcal{K}_{\mathcal{F}, \mathcal{M}, \varphi} \varepsilon^\beta (\varphi^\beta(t)).
\end{aligned}$$

Since $(\varepsilon + \delta)^r \leq (\varepsilon^r + \delta^r)$, $\varepsilon, \delta \geq 0$, for any $r \in (0, 1]$, where

$$\mathcal{K}_{\mathcal{F}, \mathcal{M}, \varphi} \varepsilon^\beta = 3^{1-\beta} (S\mathcal{E}M\eta_\varphi \varphi(t))^\beta \exp \left(3^{\frac{1}{\beta}-1} S \int_0^\tau \mathcal{M}\mathcal{L}_{\mathcal{F}}(s) ds \right),$$

where τ is the maximum point in the finite interval.

Hence, system (1.1) is β -Hyers-Ulam-Rassias stable on the compact interval with respect to φ^β . $\square$

5 β -Hyers-Ulam-Rassias stability on unbounded interval

Here, we study the β -Hyers-Ulam-Rassias stability on an unbounded interval. Consider some more assumptions:

A_0 : The operator family $\{Z(t-s) : t \geq s \geq 0\}$ is exponentially stable, i.e.,

$$\|Z(t-s)\| \leq M e^{\kappa(t-s)}, \quad t > s \geq 0, \quad M \geq 1 \text{ and } \kappa < 0.$$

A_7 : $\mathcal{F} \in \mathbb{C}(\mathcal{R}^+ \times \mathbb{S}, \mathbb{S})$ and there exists $\mathcal{L}_{\mathcal{F}} \in \mathbb{C}(\mathcal{R}^+, \mathbb{S})$ satisfying the Caratheodory condition

$$\|\mathcal{F}(t, v) - \mathcal{F}(t, v')\| \leq \mathcal{L}_{\mathcal{F}}(t) \|v - v'\|,$$

for every $t \in \mathcal{R}^+$ and $v, v' \in \mathbb{S}$. Also, we assume that

$$\int_0^t \mathcal{L}_{\mathcal{F}}(s) ds \leq \eta_{\varphi} \varphi(t),$$

for each $t \in \mathcal{R}^+$, $\eta_{\varphi} \geq 0$ and $\varphi(t)$ is continuous.

By considering inequality (2.2) and the aforementioned assumptions, we are in a position to state and prove our second result.

Theorem 5.1. *Let A_0, A_1, A_4, A_5 and A_7 be satisfied. Then system (1.1) is β -Hyers-Ulam-Rassias stable with respect to φ^{β} on an unbounded interval.*

Proof. The unique solution of the semilinear nonautonomous differential system

$$\begin{cases} A\mathbb{U}'(t) = M\mathbb{U}(t) + N\mathbb{U}(t - \Theta) + \mathcal{F}(t, \mathbb{U}(t - \Theta)), & t \geq 0, \Theta \geq 0, \\ \mathbb{U}(t) = \phi(t), & -\Theta \leq t \leq 0 \end{cases}$$

is defined by

$$\mathbb{U}(t) = \left\{ Z(t + \Theta)\phi(-\Theta) + A^{-1} \int_{-\Theta}^0 Z(t-s)[A\dot{\phi}(s) - M\phi(s)]ds + A^{-1} \int_0^t Z(t-s)\mathcal{F}(s, \mathbb{U}(s - \Theta))ds. \right. \quad (5.1)$$

Let $\mathbb{Y}$ satisfy (2.2). Then for every $t \in \mathcal{R}^+$, we obtain that

$$\begin{aligned} & \left\| \mathbb{Y}(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t-s)[A\dot{\phi}(s) - M\phi(s)]ds - A^{-1} \int_0^t Z(t-s)\mathcal{F}(s, \mathbb{Y}(s - \Theta))ds \right\| \\ & \leq \|A^{-1}\| \int_0^t \|Z(t-s)\| \|\mathcal{F}(s, \mathbb{Y}(s - \Theta)) - \mathcal{F}(s, \mathbb{U}(s - \Theta))\| ds, \\ & \left\| \mathbb{Y}(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t-s)[A\dot{\phi}(s) - M\phi(s)]ds - A^{-1} \int_0^t Z(t-s)\mathcal{F}(s, \mathbb{Y}(s - \Theta))ds \right\| \\ & \leq S \int_0^t M e^{\kappa(t-s)} \varepsilon \varphi(s) ds. \end{aligned}$$

Thus for each $t \in \mathcal{R}^+$, we get that,

$$\begin{aligned}
& \|\Upsilon(t) - \mathbb{U}(t)\|^\beta \\
&= \left\| \Upsilon(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t-s)[A\dot{\phi}(s) - M\phi(s)]ds - A^{-1} \int_0^t Z(t-s)\mathcal{F}(s, \Upsilon(s - \Theta))ds \right\|^\beta \\
&= \left\| \Upsilon(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t-s)[A\dot{\phi}(s) - M\phi(s)]ds - A^{-1} \int_0^t Z(t-s)\mathcal{F}(s, \mathbb{U}(s - \Theta))ds \right. \\
&\quad \left. + A^{-1} \int_0^t Z(t-s)\mathcal{F}(s, \Upsilon(s - \Theta))ds - A^{-1} \int_0^t Z(t-s)\mathcal{F}(s, \Upsilon(s - \Theta))ds \right\|^\beta \\
&\leq \left(\left\| \Upsilon(t) - Z(t + \Theta)\phi(-\Theta) - A^{-1} \int_{-\Theta}^0 Z(t-s)[A\dot{\phi}(s) - M\phi(s)]ds - A^{-1} \int_0^t Z(t-s)\mathcal{F}(s, \mathbb{U}(s - \Theta))ds \right\| \right. \\
&\quad \left. + \left\| A^{-1} \int_0^t Z(t-s)(\mathcal{F}(s, \Upsilon(s - \Theta)) - \mathcal{F}(s, \mathbb{U}(s - \Theta)))ds \right\| \right)^\beta \\
&\leq \left(\mathcal{S} \int_0^t \mathcal{M}e^{\kappa(t-s)}\varepsilon\varphi(s)ds \right)^\beta + \left(\mathcal{S} \int_0^t \mathcal{M}e^{\kappa(t-s)}\mathcal{L}_{\mathcal{F}}(s)\|\Upsilon(s) - \mathbb{U}(s)\|ds \right)^\beta \\
&\leq \left(\mathcal{S} \int_0^t \mathcal{M}e^{\kappa(t-s)}\varepsilon\varphi(s)ds \right)^\beta + \left(\mathcal{S} \int_0^t \mathcal{M}e^{\kappa(t-s)}\mathcal{L}_{\mathcal{F}}(s)\|\Upsilon(s) - \mathbb{U}(s)\|ds \right)^\beta.
\end{aligned}$$

If we set $\bar{\Upsilon}(t) := e^{-\kappa t}\Upsilon(t)$ and $\bar{\mathbb{U}}(t) := e^{-\kappa t}\mathbb{U}(t)$, then we have

$$\|\bar{\Upsilon}(t) - \bar{\mathbb{U}}(t)\|^\beta \leq \left(\mathcal{S} \int_0^t \mathcal{M}e^{\kappa(t-s)}\varepsilon\varphi(s)ds \right)^\beta + \left(\mathcal{S} \int_0^t \mathcal{M}\mathcal{L}_{\mathcal{F}}(s)\|\bar{\Upsilon}(s) - \bar{\mathbb{U}}(s)\|ds \right)^\beta,$$

and with the help of

$$(\varepsilon + \delta)^\gamma \leq 3^{\gamma-1}(\varepsilon^\gamma + \delta^\gamma), \quad \text{where } \varepsilon, \delta \geq 0 \text{ and } \gamma > 1,$$

we get that

$$\|\bar{\Upsilon}(t) - \bar{\mathbb{U}}(t)\| \leq 3^{\frac{1}{\beta}-1} \left(\mathcal{S} \int_0^t \mathcal{M}e^{\kappa(t-s)}\varepsilon\varphi(s)ds \right) + 3^{\frac{1}{\beta}-1} \mathcal{S} \int_0^t \mathcal{M}\mathcal{L}_{\mathcal{F}}(s)\|\bar{\Upsilon}(s) - \bar{\mathbb{U}}(s)\|ds.$$

Using Lemma 2.6, we obtain

$$\|\bar{\Upsilon}(t) - \bar{\mathbb{U}}(t)\| \leq 3^{\frac{1}{\beta}-1} \left(\mathcal{S} \int_0^t \mathcal{M}e^{\kappa(t-s)}\varepsilon\varphi(s)ds \right) \exp \left(3^{\frac{1}{\beta}-1} \mathcal{S} \mathcal{M} \int_0^t \mathcal{L}_{\mathcal{F}}(s)ds \right),$$

and resubmitting some values we end up with

$$\begin{aligned}
\|\Upsilon(t) - \mathbb{U}(t)\| &\leq 3^{\frac{1}{\beta}-1} e^{\kappa t} \left(\mathcal{S} \int_0^t \mathcal{M}e^{\kappa(t-s)}\varepsilon\varphi(s)ds \right) \exp \left(3^{\frac{1}{\beta}-1} \mathcal{S} \mathcal{M} \int_0^t \mathcal{L}_{\mathcal{F}}(s)ds \right) \\
&\leq 3^{\frac{1}{\beta}-1} e^{\kappa t} \left(\mathcal{S} \int_0^t \mathcal{M}e^{\kappa(t-s)}\varepsilon\varphi(s)ds \right) \exp \left(3^{\frac{1}{\beta}-1} \mathcal{S} \mathcal{M} \eta_\varphi(t) \right),
\end{aligned}$$

which implies

$$\|\Upsilon(t) - \mathbb{U}(t)\|^\beta \leq \mathcal{K}_{\mathcal{F}, \mathcal{M}, \varphi, \beta} \varepsilon^\beta \eta_\varphi \varphi^\beta(t),$$

where

$$\mathcal{K}_{\mathcal{F}, \mathcal{M}, \varphi, \beta} := 3^{\frac{1}{\beta}-1} e^{\kappa t} \left(S \int_0^t \mathcal{M} e^{\kappa(t-s)} \varepsilon \varphi(s) ds \right) \exp \left(3^{\frac{1}{\beta}-1} S \mathcal{M} \eta_\varphi \varphi(t) \right).$$

Hence, system (1.1) is β -Hyers-Ulam-Rassias stable on an unbounded interval with respect to φ^β . $\square$

6 An example

Consider the following nonsingular delay differential system

$$\begin{cases} A\mathbb{G}'(t) = M\mathbb{G}(t) + N\mathbb{G}(t - 0.2) + \mathcal{F}(t, \mathbb{G}(t - 0.2)), & \mathbb{G}(0) = 1, t \in [0, 2], \\ \mathbb{G}(t) = \phi(t), & -0.2 \leq t \leq 0 \end{cases} \quad (6.1)$$

and the associated inequality

$$\begin{cases} |A\mathbb{G}'(t) - M\mathbb{G}(t) - N\mathbb{G}(t - 0.2) - \mathcal{F}(t, \mathbb{G}(t - 0.2))| \leq 1, & t \in [0, 2], \\ \mathbb{G}(t) = \phi(t), & -0.2 \leq t \leq 0, \end{cases} \quad (6.2)$$

where $\Theta = 0.2$. If we set

$$M = \begin{bmatrix} -3.3 & 0.6 \\ 0 & -3.15 \end{bmatrix}, \quad N = \begin{bmatrix} 1.5 & 1.2 \\ 0 & 1.8 \end{bmatrix}, \quad A = \begin{bmatrix} 1.5 & 0 \\ 0 & 1.5 \end{bmatrix},$$

$$\mathcal{F}(t, \mathbb{G}(t - 0.2)) = \begin{bmatrix} 0.2 \mathbb{G}_1(t - 0.2) \sin t \\ 0.15 \mathbb{G}_2(t - 0.2) \sin t \end{bmatrix}$$

and $\phi(t) = [\cos(t + \pi/2) \cos(t + \pi/2)]^t$ (clearly, $\phi(t) = [0 \ 0]^t$, when $t = 0$), then we get

$$MN = \begin{bmatrix} -4.95 & -2.88 \\ 0 & -5.67 \end{bmatrix} = NM, \quad MA = \begin{bmatrix} -4.95 & 0.9 \\ 0 & -4.725 \end{bmatrix} = AM,$$

$$NA = \begin{bmatrix} -4.95 & 0.9 \\ 0 & -4.725 \end{bmatrix} = AN, \quad A^{-1} = \begin{bmatrix} 0.6667 & 0 \\ 0 & 0.6667 \end{bmatrix},$$

$$N_1 A^{-1} = \begin{bmatrix} 1.5527 & 1.0946 \\ 0 & 1.8264 \end{bmatrix}, \quad MA^{-1} = \begin{bmatrix} 1.5527 & 1.0946 \\ 0 & 1.8264 \end{bmatrix}.$$

Furthermore, if $\mathbb{G}$ satisfies (6.2), then there exists $f(t) = [\cos(\pi/2 + t) \cos(\pi/2 + t)]^t$ such that $|f(t)| \leq 1$. Hence, we have

$$\begin{cases} A\mathbb{G}'(t) = M\mathbb{G}(t) + N\mathbb{G}(t - 0.2) + \mathcal{F}(t, \mathbb{G}(t - 0.2)) + f(t), & \mathbb{G}(0) = 1, t \in [0, 2], \\ \mathbb{G}(t) = \phi(t), & -0.2 \leq t \leq 0, \end{cases}$$

and the solution of (6.1) is

$$\mathbb{G}(t) = Z(t + \Theta)\phi(-\Theta) + A^{-1} \int_{-\Theta}^0 Z(t - s)[A\dot{\phi}(s) - M\phi(s)] ds + A^{-1} \int_0^t Z(t - s)\mathcal{F}(s, \mathbb{G}(s - \Theta)) ds,$$

where $Z(t) = e^{MA^{-1}t} e^{N_1 A^{-1}(t-\Theta)}$, $N_1 = e^{-MA^{-1}\Theta} B$ and $MA = AM$, $MN = NM$, $NA = AN$ are used.

By virtue of Remark 2.3 of [3], the aforementioned solution has the form

$$\mathbb{G}(t) = Z(t + \Theta)\phi(0) + B \int_{-\Theta}^0 Z(t - \Theta - s)\phi(s)ds.$$

Using MATLAB, we conclude that

$$G(t) = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix},$$

where

$$a_{11} = 2 \left(\sin \frac{1}{10} \right)^2 \left(1 / \left(\exp \left(\frac{1748192912675783}{5629499534213120} \right) \exp \left(\frac{3643934411889949t}{5629499534213120} \right) + 1 \right) \right),$$

$$a_{21} = 2 \left(\sin \frac{1}{10} \right)^2 \left(1 / \left(\exp \left(\frac{4112583296853779}{11258999068426240} \right) \exp \left(\frac{3080981559426209t}{11258999068426240} \right) + 1 \right) \right).$$

As $f(t) = \begin{bmatrix} \cos\left(t + \frac{\pi}{2}\right) & \cos\left(t + \frac{\pi}{2}\right) \end{bmatrix}^t$, then we have

$$\mathbb{G}^*(t) = Z(t + \Theta)\phi(0) + B \int_{-\Theta}^0 Z(t - \Theta - s)\phi(s)ds + \begin{bmatrix} \cos\left(t + \frac{\pi}{2}\right) & \cos\left(t + \frac{\pi}{2}\right) \end{bmatrix}^t,$$

and again using MATLAB, we get

$$\mathbb{G}^*(t) = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix},$$

where

$$\begin{cases} b_{11} = \sin \frac{1}{5} + 2 \sin \left(\frac{1}{10} \right)^2 + (2 \sin (1/10)^2) / \left(\exp \left(\frac{1748192912675783}{5629499534213120} \right) \exp \left(\frac{3643934411889949t}{5629499534213120} \right) + 1 \right), \\ b_{21} = \sin \frac{1}{5} + 2 \sin \left(\frac{1}{10} \right)^2 + (2 \sin \left(\frac{1}{10} \right)^2) / \left(\exp \left(\frac{4112583296853779}{11258999068426240} \right) \exp \left(\frac{3080981559426209t}{11258999068426240} \right) + 1 \right). \end{cases}$$

From above, we end up with $[\varepsilon \ \varepsilon]^t = [0.1987 \ 0.1987]^t$. Therefore, equation (6.1) has a unique solution in $\mathbb{C}([0, 2], \mathcal{R})$ which is Hyers-Ulam stable on $[0, 2]$.

7 Conclusion

In the last few years and along with the explosion in studying differential equations, the notion of stability has gained extensive interest by many mathematicians. Following the trend, in this paper we discuss the β -Hyers-Ulam-Rassias stability of a nonsingular differential system over compact and unbounded intervals. Different types of conditions were established for the sake of proving the main results. An example with specific parameters and matrices and graphical representation demonstrate consistency to our theoretical findings.

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