Research Article

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Hasimoto surfaces for two classes of curve evolution in Minkowski 3-space

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Abstract: In this work, we study Hasimoto surfaces for the second and third classes of curve evolution corresponding to a Frenet frame in Minkowski 3-space. Later, we derive two formulas for the differentials of the second and third Hasimoto-like transformations associated with the repulsive-type nonlinear Schrödinger equation.

Keywords: Hasimoto surface, nonlinear Schrödinger equation, time evolution

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1 Introduction

Many mathematicians have been interested for a long time in studying connections between integrable equations (soliton equations) and geometric motions of a curve in various spaces. In particular, Hasimoto [1] discovered a connection between thin vortex filament without stretching in an incompressible inviscid fluid and the nonlinear Schrödinger equation. Lakshmanan [2] presented a connection between the nonlinear Schrödinger equation and integrable Landau-Lifshitz equation for a time evolution of a spin vector X as follows:

$$X_u = X \times X_{ss}. \quad (1.1)$$

Recently, some geometers studied connections between moving frame of space curves and soliton equations in ambient spaces (see [4–13]).

On the other hand, Lamb [3] introduced the Hasimoto transformation as a complex function and he studied a certain class of moving space curves with soliton equations. Muruges and Balakrishnan [5] presented two Hasimoto transformations as two other complex functions. Also, they showed that there are two other classes of curve evolution that may be so identified. Hence, three distinct classes of curve evolution are associated with the nonlinear Schrödinger equation in Euclidean 3-space. The differential formulas of Hasimoto transformations in Euclidean 3-space have been presented by Langer and Perline [4]. For the extension to the Lorentz version, Gürbüz suggested three classes of curve evolution associated with the nonlinear Schrödinger equation in Minkowski 3-space (see [6,8,9,12]). Also, Gürbüz [6] extended the results of Langer and Perline in [4] in Minkowski 3-space, and she derived differential formulas of Hasimoto transformations for the first class of curve evolutions associated with the repulsive-type nonlinear Schrödinger equation in Minkowski 3-space.

In this paper, we give differential formulas of two Hasimoto-like transformations of the second and third classes associated with the repulsive-type nonlinear Schrödinger equation in Minkowski 3-space.

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2 Hasimoto surface for second class

2.1 Time evolution of second class

Let β be a spacelike curve parametrized by arc-length s in Minkowski 3-space $\mathbb{R}_1^3$. Then, the Frenet frame $\{t, n, b\}$ satisfies the following formulas:

$$\begin{aligned}t_s &= \kappa n, \\n_s &= -\kappa t - \tau b, \\b_s &= -\tau n,\end{aligned}$$

where the binormal vector is timelike. Here κ and τ are the curvature and the torsion of the spacelike curve.

Consider a second frame $\{P_1, P_2, P_2^*\}$ for the second class of the spacelike curve evolution associated with the repulsive-type nonlinear Schrödinger equation (NLS⁻) as follows [9]:

$$\begin{aligned}P_1 &= b, \\P_2 &= (n + it)e^{-i\int \kappa}, \\P_2^* &= (n - it)e^{i\int \kappa}.\end{aligned}\tag{2.1}$$

Now we take the second Hasimoto-like transformation Ψ and the second Darboux vector ξ_2 as follows, respectively:

$$\begin{aligned}\Psi &= \tau e^{i\int \kappa}, \\\xi_2 &= At + Bn + \hat{\kappa}_0 b.\end{aligned}$$

Considering the derivatives of $\{P_1, P_2, P_2^*\}$ with respect to s and u , and using the second Hasimoto-like transformation, we obtain, respectively:

$$\begin{aligned}P_{1s} &= -\frac{\Psi}{2}P_2 - \frac{\Psi^*}{2}P_2^*, \\P_{2s} &= -\Psi^*P_1, \\P_{2s}^* &= -\Psi P_1,\end{aligned}$$

and

$$\begin{aligned}P_{1u} &= -\frac{i\Psi_s}{2}P_2 + \frac{i\Psi_s^*}{2}P_2^*, \\P_{2u} &= i\Psi_s^*P_1 + iR_2P_2, \\P_{2u}^* &= -i\Psi_sP_1 - iR_2P_2^*,\end{aligned}$$

where u is the time and $R_2 = -\frac{\Psi\Psi^*}{2}$. The compatibility conditions $P_{1su} = P_{1us}$ and $P_{2su} = P_{2us}$ give the repulsive-type nonlinear Schrödinger equation as:

$$\Psi_u = i\Psi_{ss} + i\Psi R_2.$$

On the other hand, the time evolution $\{t_u, n_u, b_u\}$ of the Frenet frame for the second class of the curve evolution is expressed as:

$$\begin{aligned}t_u &= \xi_2 \times t = \hat{\kappa}_0 n + \bar{\omega} b, \\n_u &= \xi_2 \times n = -\hat{\kappa}_0 t + \lambda b, \\b_u &= \xi_2 \times b = \bar{\omega} t + \lambda n,\end{aligned}\tag{2.2}$$

where $A = -\lambda$ and $B = \bar{\omega}$. Using $t_{us} = t_{su}$ and $b_{us} = b_{su}$, the time evolution of the curvature and the torsion of the curve β are obtained as follows:

$$\begin{aligned}\kappa_u &= \hat{\kappa}_{0s} - \bar{\omega}\tau, \\ \tau_u &= \lambda_s - \bar{\omega}\kappa.\end{aligned}\quad (2.3)$$

We also have

$$\bar{\omega}_s = \lambda\kappa + \hat{\kappa}_0\tau.$$

Consider

$$X = \beta_s = P_1$$

and the vector field $\beta_u = f_2t + g_2n + h_2b$. Here f_2, g_2 , and h_2 are smooth functions. Then, $\beta_{us} = \beta_{su}$ implies

$$\begin{aligned}f_{2s} - g_2\kappa &= \bar{\omega}, \\ h_{2s} - g_2\tau &= 0, \\ f_2\kappa + g_{2s} &= h_2\tau + \lambda.\end{aligned}\quad (2.4)$$

In other words, the vortex filament flow for the second class associated with NLS^- is defined as

$$\beta_u = \beta_s \times \beta_{ss} = \tau t, \quad (2.5)$$

which gives

$$f_2 = \tau, \quad g_2 = h_2 = 0. \quad (2.6)$$

Using (2.4) and (2.6), we have

$$\bar{\omega} = \kappa\tau, \quad \hat{\kappa}_0 = \frac{\tau_{ss}}{\tau} - \kappa^2, \quad \lambda = \tau_s. \quad (2.7)$$

Thus, from (2.2), (2.3), and (2.7), we obtain the following results.

Theorem 2.1. *Let β be a spacelike curve with a second frame in Minkowski 3-space. Then the vortex filament flow (2.5) implies that the time evolution of the Frenet frame $\{t, n, b\}$ along the spacelike curve β is as follows:*

$$\begin{aligned}t_u &= \left(\frac{\tau_{ss}}{\tau} - \kappa^2\right)n + \tau_sb, \\ n_u &= -\left(\frac{\tau_{ss}}{\tau} - \kappa^2\right)t + \kappa\tau b, \\ b_u &= \tau_st + \kappa\tau n.\end{aligned}\quad (2.8)$$

Theorem 2.2. *Let β be a spacelike curve with a second frame in Minkowski 3-space. Then the vortex filament flow (2.5) implies that the time evolution of the curvature κ and the torsion τ for the spacelike curve are as follows:*

$$\begin{aligned}\kappa_u &= \left(\frac{\tau_{ss}}{\tau} - \kappa^2\right)_s - \kappa\tau^2, \\ \tau_u &= \tau_{ss} - \kappa^2\tau.\end{aligned}\quad (2.9)$$

The soliton surface associated with the equation for the second class is called a second Hasimoto-like surface which is timelike. The first and the second fundamental forms of a second timelike Hasimoto-like surface are obtained as follows, respectively:

$$\begin{aligned}I &= -ds^2 + \tau^2 du^2, \\ II &= \tau ds^2 - 2\kappa\tau ds du - \tau\left(\frac{\tau_{ss}}{\tau} - \kappa^2\right) du^2,\end{aligned}$$

which imply that the Gauss curvature K and the mean curvature H of the second timelike Hasimoto-like surface are given by:

$$K = \frac{\tau_{ss}}{\tau},$$

$$H = -\frac{\tau_{ss} - \kappa^2\tau + \tau^3}{2\tau^2}.$$

2.2 Second Hasimoto-like transformation

Consider a vector field

$$\beta_u = f_2 t + g_2 n + h_2 b = Z_2$$

along a spacelike curve with a timelike binormal vector for the second class of curve evolution associated with NLS^- . Since Z_2 satisfies the arc length preserving condition, $\langle Z_{2s}, b \rangle = 0$ implies $h_{2s} = g_2 \tau$.

On the other hand, a second normalization operator N_2 for the second class according to a Frenet frame is introduced as:

$$N_2 Z_2 = f_2 t + g_2 n + \left(\int_0^s g_2 \tau ds \right) b. \quad (2.10)$$

Also, the second linear recursion operator R_2 for the second class is given by:

$$R_2 Z_2 = N_2(P_1 \times Z_{2s}). \quad (2.11)$$

The differential formula of the second Hasimoto-like transformation $H_2(\beta) = \Psi$ can be presented as follows:

$$dH_2(Z_2) = \langle P_2^*, P_1 \times (R_2 Z_2)_s \rangle. \quad (2.12)$$

As an application for the second Hasimoto-like transformation Ψ , we consider the vortex filament flow

$$Z_2 = \tau t. \quad (2.13)$$

Then from (2.11) and (2.13), the second linear recursion operator is derived as:

$$R_2 Z_2 = \tau_s t + \kappa \tau + \frac{\tau^2}{2} b. \quad (2.14)$$

From (2.12) and (2.14), we have the following theorem.

Theorem 2.3. *Let β be a spacelike curve with a second frame in Minkowski 3-space. Then the vortex filament flow (2.13) implies a differential formula of a second Hasimoto-like transformation as*

$$dH_2(Z_2) = e^{-i \int \kappa} \left(i \left(\frac{\tau^3}{2} - \tau_{ss} + \kappa^2 \tau \right) - (\kappa \tau_s + (\kappa \tau)_s) \right). \quad (2.15)$$

Remark 2.4. For the vortex filament flow $Z_2 = \tau t$, we can check the nonlinear Schrödinger equation of repulsive-type NLS^- as

$$\Psi_u = dH_2(Z_2) = i\Psi_{ss} - \frac{i}{2}|\Psi|^2\Psi. \quad (2.16)$$

3 Hasimoto surface for third class

3.1 Time evolution of third class

A third frame $\{S_1, S_2, S_2^*\}$ for the third class of a spacelike curve evolution with a timelike binormal vector associated with the NLS⁻ equation is given by [9]:

$$\begin{aligned} S_1 &= n, \\ S_2 &= b + it, \\ S_2^* &= b - it. \end{aligned} \quad (3.1)$$

Consider the third Hasimoto-like transformation Φ with NLS⁻ as

$$\Phi = \tau + i\kappa$$

and the third Darboux vector ξ_3 associated with the NLS⁻ equation for the third class as $\xi_3 = At + \gamma^*n + Cb$. Taking derivatives of $\{S_1, S_2, S_2^*\}$ with respect to s and u , we obtain

$$\begin{aligned} S_{1s} &= -\frac{\Phi}{2}S_2 - \frac{\Phi^*}{2}S_2^*, \\ S_{2s} &= -\Phi^*S_1, \\ S_{2s}^* &= -\Phi S_1, \end{aligned}$$

and

$$\begin{aligned} S_{1u} &= -\frac{i\Phi_s}{2}S_2 + \frac{i\Phi_s^*}{2}S_2^*, \\ S_{2u} &= i\Phi_s^*S_1 + iR_2S_2, \\ S_{2u}^* &= -i\Phi_sS_1 - iR_2S_2^*, \end{aligned}$$

where $R_2 = \frac{\Phi\Phi^*}{2}$. It follows that the compatibility conditions $S_{1su} = S_{1us}$ and $S_{2su} = S_{2us}$ give the repulsive-type Schrödinger equation as

$$\Phi_u = i\Phi_{ss} + i\Phi R_2.$$

On the other hand, the time evolution $\{t_u, n_u, b_u\}$ of the Frenet frame for the third class of the curve evolution is expressed by:

$$\begin{aligned} t_u &= \xi_3 \times t = \bar{\omega}n - \gamma^*b, \\ n_u &= \xi_3 \times n = \bar{\omega}t + \mu b, \\ b_u &= \xi_3 \times b = \gamma^*t + \mu n, \end{aligned} \quad (3.2)$$

where $A = \mu$ and $B = -\bar{\omega}$. From the compatibility conditions $t_{us} = t_{su}$ and $b_{us} = b_{su}$, it follows that the time evolution of the curvature and the torsion with the spacelike curve β for the third class are given by:

$$\begin{aligned} \kappa_u &= -\bar{\omega}_s + \gamma^*\tau, \\ \tau_u &= \mu_s - \gamma^*\kappa. \end{aligned} \quad (3.3)$$

We also have

$$\gamma^* = \bar{\omega}\tau + \kappa\mu. \quad (3.4)$$

Now, consider

$$X = \beta_s = S_1$$

and the vector field $\beta_u = f_3t + g_3n + h_3b$, where f_3, g_3 , and h_3 are smooth functions. Then, $\beta_{us} = \beta_{su}$ implies

$$\begin{aligned} f_{3s} - g_3\kappa &= \bar{\omega}, \\ g_{3s} &= f_3\kappa - h_3\tau, \\ \mu &= h_{3s} + g_3\tau. \end{aligned} \quad (3.5)$$

In other words, the vortex filament flow for the third class associated with the NLS⁻ equation is defined as:

$$\beta_u = \beta_s \times \beta_{ss} = \tau t + \kappa b, \quad (3.6)$$

which gives

$$f_3 = \tau, \quad g_3 = 0, \quad h_3 = \kappa. \quad (3.7)$$

From (3.5) and (3.7), one finds

$$\bar{\omega} = \tau_s, \quad \mu = \kappa_s. \quad (3.8)$$

Using (3.4) and (3.8), we obtain

$$\gamma^* = \frac{\kappa^2 + \tau^2}{2}. \quad (3.9)$$

Thus, we obtain the following theorem.

Theorem 3.1. *Let β be a spacelike curve with a third frame in Minkowski 3-space. Then the vortex filament flow (3.6) implies that the time evolutions of the Frenet frame and the curvature, the torsion for the third class of the curve β are as follows, respectively:*

$$\begin{cases} t_u = \tau_s n - \left(\frac{\kappa^2 + \tau^2}{2} \right) b, \\ n_u = \tau_s t + \kappa_s b, \\ b_u = \left(\frac{\kappa^2 + \tau^2}{2} \right) t + \kappa_s n, \end{cases} \quad (3.10)$$

$$\begin{cases} \kappa_u = -\tau_{ss} + \tau \left(\frac{\kappa^2 + \tau^2}{2} \right), \\ \tau_u = \kappa_{ss} - \kappa \left(\frac{\kappa^2 + \tau^2}{2} \right). \end{cases} \quad (3.11)$$

The soliton surface associated with the NLS⁻¹ equation for the third class is called the third Hasimoto-like surface which is timelike. The first fundamental form of the third timelike Hasimoto-like surface connected with the NLS⁻¹ equation is given by:

$$I = -ds^2 + (\kappa^2 + \tau^2)du^2.$$

Also, the second fundamental form of the surface is derived as

$$II = -x_2 ds^2 + (\tau x_1 + \kappa x_3 - y_2) ds du + (\tau y_1 + \kappa y_3) du^2,$$

where

$$x_1 = \frac{\kappa_s \tau^2 - \kappa \tau \tau_s}{(\kappa^2 + \tau^2)^{3/2}}, \quad x_2 = -(\kappa^2 + \tau^2)^{1/2}, \quad x_3 = \frac{\kappa \tau \kappa_s - \tau_s \kappa^2}{(\kappa^2 + \tau^2)^{3/2}},$$

and

$$\begin{aligned} y_1 &= \frac{\kappa_u \tau^2 - \kappa \tau \tau_u - \frac{\tau(\kappa^2 + \tau^2)^2}{2}}{(\kappa^2 + \tau^2)^{3/2}}, \\ y_2 &= \frac{\kappa \tau_s - \tau \kappa_s}{(\kappa^2 + \tau^2)^{1/2}}, \\ y_3 &= \frac{\kappa \tau \kappa_u - \tau_u \kappa^2 - \frac{\kappa(\kappa^2 + \tau^2)^2}{2}}{(\kappa^2 + \tau^2)^{3/2}}. \end{aligned}$$

Therefore, the Gauss curvature K and the mean curvature H of the third timelike Hasimoto surface are given by, respectively:

$$K = \frac{\tau_{ss} + \kappa_{ss}}{\kappa^2 + \tau^2} - \frac{\kappa_s \tau - \tau_s \kappa}{(\kappa^2 + \tau^2)^2},$$

$$H = -\frac{\kappa \kappa_{ss} + \tau \tau_{ss}}{2(\kappa^2 + \tau^2)^{3/2}}.$$

3.2 Third Hasimoto-like transformation

Consider a vector field

$$\beta_u = f_3 t + g_3 n + h_3 b = Z_3$$

along a spacelike curve β with a timelike binormal vector for the third class. Since Z_3 satisfies the arc length preserving condition, we get $g_{3s} = h_3 \tau - f_3 \kappa$.

Now, we introduce a third normalization operator N_3 and the third linear recursion operator R_3 for the third class according to a Frenet frame as, respectively:

$$N_3 Z_3 = f_3 t + \left(\int_0^s (h_3 \tau - f_3 \kappa) ds \right) n + h_3 b, \quad (3.12)$$

$$R_3 Z_3 = N_3 (S_1 \times Z_{3s}). \quad (3.13)$$

Also, the differential formula of the third Hasimoto-like transformation $H_3(\beta) = \Phi$ can be presented as follows:

$$dH_3(Z_3) = \langle S_2^*, S_1 \times (R_3 Z_3)_s \rangle. \quad (3.14)$$

If the differential formula (3.14) is applied to the vortex filament flow

$$Z_3 = \tau t + \kappa b, \quad (3.15)$$

then the third linear recursion operator is derived as

$$R_3 Z_3 = \kappa_s t + \left(\int_0^s \kappa_s \kappa + \tau_s \tau d\bar{s} \right) n - \tau_s b. \quad (3.16)$$

From (3.14) and (3.15), we have the following theorem.

Theorem 3.2. *Let β be a spacelike curve with a third frame in Minkowski 3-space. Then the vortex filament flow (3.15) gives a differential formula of the third Hasimoto-like transformation*

$$dH_3(Z_3) = \left(-\kappa_{ss} + \kappa \left(\frac{\kappa^2 + \tau^2}{2} \right) \right) + i \left(\tau_{ss} - \tau \left(\frac{\kappa^2 + \tau^2}{2} \right) \right). \quad (3.17)$$

Remark 3.3. For the vortex filament flow $Z_3 = \tau t + \kappa b$, we can check the nonlinear Schrödinger equation of repulsive-type NLS⁺ as

$$\Phi_u = i\Phi_{ss} - \frac{i}{2}|\Phi|^2\Phi. \quad (3.18)$$

4 Conclusion

In this paper, Hasimoto surfaces for the second and third classes of a spacelike curve evolution were studied. Also, two formulas for the differentials of two Hasimoto-like transformations corresponding to a Frenet frame in Minkowski 3-space have been derived.

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