Research Article

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Hyers-Ulam stability of a nonautonomous semilinear equation with fractional diffusion

<https://doi.org/10.1515/dema-2020-0021>

received July 1, 2020; accepted August 27, 2020

Abstract: In this paper, we study the Hyers-Ulam stability of a nonautonomous semilinear reaction-diffusion equation. More precisely, we consider a nonautonomous parabolic equation with a diffusion given by the fractional Laplacian. We see that such a stability is a consequence of a Gronwall-type inequality.

Keywords: parabolic partial differential equations, Hyers-Ulam stability, fractional Laplacian, Gronwall type inequalities

MSC 2020: 35B20, 35B35, 45H05, 47H10

1 Introduction and statement of the result

In 1940, S. M. Ulam raised the stability problem for automorphism of metric groups. D. H. Hyers in 1941 solved the stability problem in the case of additive functions (also known as Cauchy functional equation). Since then there has been an intense study of stability problems with emphasis on different disciplines, now they are usually called Hyers-Ulam stability problems. The study of Hyers-Ulam stability, in the context of differential equations, was initiated by C. Alsina and C. R. Ger [1]. Even more, the Hyers-Ulam stability problem for partial differential equations is now a well-known topic, see for example [2–5] and references therein.

On the other hand, there are some preliminary studies on the Hyers-Ulam stability of differential equations involving Riemann-Liouville-type fractional operators, see [6,7]. The study of partial differential equations with fractional Laplacian diffusion is relatively recent. Indeed, L. Caffarelli, and some of his coauthors, promoted the systematic study of the properties of the solutions of these types of equations. Today, it is an important area of mathematics that has received great attention, see for example [8–10] and the references mentioned in such works.

Recently, the fundamental role that Gronwall-type inequalities play in the study of the Hyers-Ulam stability problem has been appreciated, an example of such a fact can be seen in [11,12]. Our contribution is also in this regard. That is to say, in this paper we study the Hyers-Ulam stability of nonautonomous semilinear reaction-diffusion equations with fractional Laplacian diffusion using a convenient version of the Gronwall inequality. Ten equivalent definitions of fractional Laplacian are presented in [13]; however, in the books [14,15] we usually find the most relevant properties of such an integral operator. This background motivates the present work, apart from the fact, to the best of our knowledge, that there are no previous results studying the Hyers-Ulam stability problem for equations of the type considered here.

We state precisely the main result.

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Theorem 1. Let $f : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function, for which

$$|f(t, x) - f(t, y)| \leq h(t)|x - y|, \quad \text{for all } t > 0, x, y \in \mathbb{R}^d,$$

where $h : (0, \infty) \rightarrow \mathbb{R}$ and $h \in L^1((0, \infty))$. We also assume $\psi : (0, \infty) \rightarrow \mathbb{R}$, $\psi \in L^1((0, \infty))$ and $v \in C^{1,2}((0, \infty) \times \mathbb{R}^d)$, $v(0, \cdot) \in C_b(\mathbb{R}^d)$ is such that

$$\left| \frac{\partial}{\partial t} v(x, t) - \psi(t) \Delta_\alpha v(x, t) - f(v(x, t), t) \right| \leq \varphi(t) \Phi(x), \quad \text{for all } (x, t) \in (0, \infty) \times \mathbb{R}^d, \quad (1)$$

where $\varphi : (0, \infty) \rightarrow \mathbb{R}$ and $\Phi : \mathbb{R}^d \rightarrow [0, \infty)$. If $\varphi \in L^1((0, \infty))$ and $\Phi \in L^1(\mathbb{R}^d)$, then there exists a unique solution $u \in C^{1,2}((0, \infty) \times \mathbb{R}^d)$, $u(0, \cdot) \in C_b(\mathbb{R}^d)$ of

$$\frac{\partial}{\partial t} u(t, x) = \psi(t) \Delta_\alpha u(t, x) + f(t, u(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad (2)$$

where $\Delta_\alpha = -(-\Delta/2)^{\alpha/2}$, $0 < \alpha \leq 2$, such that

$$|v(x, t) - u(x, t)| \leq K, \quad \text{for all } (x, t) \in (0, \infty) \times \mathbb{R}^d, \quad (3)$$

for some constant $K > 0$ that depends on $\|h\|_1$, $\|\varphi\|_1$, $\|\psi\|_1$, $v(0, \cdot)$ and $u(0, \cdot)$.

Currently, when the above conditions (1) and (3) are met, we say that the corresponding differential equation (2) has the *generalized Hyers-Ulam stability* or that it has the Hyers-Ulam-Rassias stability. In the literature, there are other definitions for stability of this type, see for example [16]. In order to facilitate the manuscript reading, in the next section we remember some of the notations introduced in the statement of Theorem 1.

We divide the proof of Theorem 1 into two parts. In Section 2, we prove the existence of classical solutions to equation (2) and in Section 3 we study the Hyers-Ulam stability problem for solutions of equation (2). Finally, in Section 4 we prove the validity of the required Gronwall-type inequality for studying the stability problem.

2 Existence

First let us see when equation (2) has a unique solution, as we said before, this will allow us to introduce some notations. By $C_b(\mathbb{R}^d)$ we denote the space of all real-valued continuous and bounded functions defined on $\mathbb{R}^d$. For each $t > 0$, we denote by $p(t, \cdot)$ the real-valued function determined by (its Fourier transform)

$$\int_{\mathbb{R}^d} p(t, y) e^{iy \cdot x} dz = e^{-t|x|^\alpha}, \quad \text{for all } x \in \mathbb{R}^d. \quad (4)$$

The family of operators $\{S_t, t > 0\}$, where

$$S_t g(x) = \int_{\mathbb{R}^d} p(t, x - z) g(z) dz, \quad x \in \mathbb{R}^d,$$

is a strongly continuous semigroup, on the Banach space $C_b(\mathbb{R}^d)$. The fractional Laplacian, Δ_α , can be defined as the infinitesimal generator of $\{S_t, t > 0\}$, see [15].

Let $T > 0$ be arbitrary and fixed. Define the space

$$E_T = \{u : [0, T] \rightarrow C_b(\mathbb{R}^d), \|u\| < \infty\},$$

where $\|u\| = \sup\{\|u(t, \cdot)\|_\infty : 0 \leq t \leq T\}$ and $\|\cdot\|_\infty$ is the usual uniform (or supreme) norm. It is clear that E_T is a Banach space. Let us also set the function

$$\Psi(s, t) = \int_s^t \psi(r) dr, \quad 0 \leq s < t. \quad (5)$$

Proof of existence. For each $u \in E_T$, let us set the function $G(u) : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ as

$$G(u)(t, x) = p(\Psi(0, t), \cdot) * u(0, \cdot)(x) + \int_0^t p(\Psi(s, t), \cdot) * f(s, u(s, \cdot))(x) ds,$$

where $*$ is the convolution operator. It easily follows that (see Lemma 3)

$$|||G(u)||| \leq \|u(0, \cdot)\|_u + T \sup\{f(t, z) : |z| \leq |||u|||, r \leq T\},$$

we thus have $G : E_T \rightarrow E_T$. Otherwise, for each $u, \tilde{u} \in E_T$ we get

$$\begin{aligned} |G(u)(t, x) - G(\tilde{u})(t, x)| &\leq \int_0^t \int_{\mathbb{R}^d} p(\Psi(s, t), x - y) |f(s, u(s, y)) - f(s, \tilde{u}(s, y))| dy ds \\ &\leq \int_0^t \int_{\mathbb{R}^d} p(\Psi(s, t), x - y) h(s) |u(s, y) - \tilde{u}(s, y)| dy ds \\ &\leq \left(\int_0^t h(s) ds \right) |||u - \tilde{u}|||. \end{aligned}$$

Using mathematical induction we can verify that

$$|G^n(u)(x, t) - G^n(\tilde{u})(x, t)| \leq I_n(t) |||u - \tilde{u}|||,$$

where $G^n := G \circ \dots \circ G$ and

$$I_n(t) := \int_0^t \int_0^{s_1} \dots \int_0^{s_{n-1}} h(s_1) h(s_2) \dots h(s_n) ds_n \dots ds_2 ds_1 = \frac{1}{n!} \left(\int_0^t h(s) ds \right)^n. \quad (6)$$

Taken n large enough we have $I_n(T) < 1$, accordingly the mapping R has a unique fixed point $u \in E_T$, see for example Lemma 5.10.4 in [17]. Such a fixed point, u , is the desired solution of (2), usually called mild solution. Using basic properties of the convolution operator (see [18]) and the Lebesgue dominated convergence theorem, we can see that $u \in C^{1,2}((0, \infty) \times \mathbb{R}^d)$, the space of continuously differentiable in time and two times continuously differentiable in space, and such a function satisfies the differential equation (2), for details see, for example, [19].

Since the time $T > 0$ was arbitrary and $\int_0^\infty h(s) ds < \infty$, it follows easily that u is a global solution (we mean u is defined on all intervals $[0, T]$). $\square$

3 Hyers-Ulam stability

The proof of the Hyers-Ulam stability of equation (2) is based on the following fundamental inequality, which will be proved in the next section.

Lemma 2. (Gronwall-type inequality). *Let $w : (0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a non-negative measurable function. If there exists a constant $L > 0$ such that*

$$w(t, x) \leq L + \int_0^t h(s) \left(\int_{\mathbb{R}^d} p(\Psi(s, t), x - y) w(s, y) dy \right) ds, \quad \text{for all } (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad (7)$$

then

$$w(t, x) \leq L \exp \left(\int_0^t h(s) ds \right), \quad \text{for all } (t, x) \in (0, \infty) \times \mathbb{R}^d, \quad (8)$$

where the function $h : (0, \infty) \rightarrow \mathbb{R}$ is in $L^1((0, \infty))$, $p(t, \cdot)$ and Ψ are given in (4) and (5), respectively.

With this result in hand we are in position to deal with the stability result.

Proof of stability. Let $t > 0$ be fixed. As usual, in the semigroup theory, let us introduce the function

$$G(s, x) = p(\Psi(s, t), \cdot) * v(s, \cdot)(x) = \int_{\mathbb{R}^d} p(\Psi(s, t), x - y) v(s, y) dy, \quad (s, x) \in (0, t) \times \mathbb{R}^d.$$

Deriving G with respect to the variable s and using that $\frac{\partial}{\partial t}(p(t, \cdot) * \varphi(x)) = p(t, \cdot) * \Delta_\alpha \varphi(x)$ (this means that Δ_α is the infinitesimal generator of $\{S_t, t > 0\}$, see page 329 in [19]) we arrived to

$$\frac{\partial}{\partial s} G(s, x) = -p(\Psi(s, t), \cdot) * (k(s) \Delta_\alpha v(s, \cdot))(x) + p(\Psi(s, t), \cdot) * \frac{\partial}{\partial s} v(s, \cdot)(x).$$

Accordingly, from (1) we obtain

$$\frac{\partial}{\partial s} G(s, x) \leq p(\Psi(s, t), \cdot) * f(s, v(s, \cdot))(x) + \varphi(s) p(\Psi(s, t), \cdot) * \psi(x).$$

Integrating, the previous inequality, from 0 to t we have (see Theorem 8.14 in [18])

$$v(t, x) \leq p(\Psi(0, t), \cdot) * v(0, \cdot)(x) + \int_0^t p(\Psi(s, t), \cdot) * f(s, v(s, \cdot))(x) ds + \int_0^t \varphi(s) p(\Psi(s, t), \cdot) * \psi(x) ds.$$

Using similar arguments we can verify that

$$\begin{aligned} v(t, x) &\geq p(\Psi(0, t), \cdot) * v(0, \cdot)(x) + \int_0^t p(\Psi(s, t), \cdot) * f(s, v(s, \cdot))(x) ds - \int_0^t \varphi(s) p(\Psi(s, t), \cdot) * \psi(x) ds, \\ u(t, x) &= p(\Psi(0, t), \cdot) * u(0, \cdot)(x) + \int_0^t p(\Psi(s, t), \cdot) * f(s, u(s, \cdot))(x) ds. \end{aligned}$$

From these inequalities we can deduce that

$$\begin{aligned} |v(t, x) - u(t, x)| &\leq p(\Psi(0, t), \cdot) * |v(0, \cdot) - u(0, \cdot)|(x) + \int_0^t \varphi(s) p(\Psi(s, t), \cdot) * \psi(x) ds \\ &\quad + \int_0^t p(\Psi(s, t), \cdot) * |f(s, v(s, \cdot)) - f(s, u(s, \cdot))|(x) ds \\ &\leq p(\Psi(0, t), \cdot) * |v(0, \cdot) - u(0, \cdot)|(x) + \int_0^t \varphi(s) p(\Psi(s, t), \cdot) * \psi(x) ds \\ &\quad + \int_0^t h(s) p(\Psi(s, t), \cdot) * |v(s, \cdot) - u(s, \cdot)|(x) ds. \end{aligned}$$

From Lemma 3(i) and Young's inequality for convolutions (see Theorem 8.7 in [18]), we get

$$p(\Psi(0, t), \cdot) * |v(0, \cdot) - u(0, \cdot)|(x) \leq \|v(0, \cdot) - u(0, \cdot)\|_u,$$

$$\int_0^t \varphi(s) p(\Psi(s, t), \cdot) * \psi(x) ds \leq \|\varphi\|_1 \|\psi\|_1.$$

Let us introduce the auxiliary function $w(x, t) = |v(x, t) - u(x, t)|$. From the above inequalities we conclude

$$w(x, t) \leq L + \int_0^t h(s) \left(\int_{\mathbb{R}^d} p(\Psi(s, t), x - y) w(y, s) dy \right) ds,$$

where $L = \|v(0, \cdot) - u(0, \cdot)\|_u + \|\varphi\|_1 \|\psi\|_1$. Gronwall-type inequality (8) yields

$$|v(x, t) - u(x, t)| \leq L \exp \left(\int_0^\infty h(s) ds \right), \quad \text{for all } (x, t) \in (0, \infty) \times \mathbb{R}^d.$$

Taking the constant K as $L \exp(\|h\|_1)$ we get the desired result. $\square$

4 Proof of Gronwall-type inequality

Before starting the proof of the inequality, it is convenient to present some previous facts. The functions p , in the context of the probability theory, are called α -stable densities and they are symmetric, positive and continuous functions; moreover, they are smooth functions. In particular, (4) is called characteristic function of p .

Lemma 3. Let $p(t, \cdot)$, $t > 0$, be the α -stable densities, given in (4):

- (i) For each $t > 0$, $\int_{\mathbb{R}^d} p(x, t) dx = 1$.
- (ii) For each $s, t > 0$ and $x, y \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} p(s, x - z) p(t, z - y) dy = p(s, x - y)$.

Proof. A proof, of these results, can be obtained from Proposition 11.3 of [15], see also [19]. $\square$

Statement (ii) is known as the semigroup property or as the Chapman-Kolmogorov equation.

Proof of Lemma 2. For $(t, x) \in (0, \infty) \times \mathbb{R}^d$ fixed let us prove, by mathematical induction, that

$$w(t, x) \leq L \sum_{j=0}^{n-1} I_j(t) + R_n(t, x), \quad (9)$$

where $I_0(t) \equiv 1$, $I_j(t)$ is given by (6), and

$$R_n(t, x) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} h(s_1) h(s_2) \cdots h(s_n) \left(\int_{\mathbb{R}^d} p(\Psi(s_n, t), x_n - x) w(s_n, x_n) dx_n \right) ds_n \cdots ds_2 ds_1.$$

When $n = 1$, inequality (9) is just the previous inequality (7). Let us see that inequality (9) is also true for $n + 1$. Iterating (7), using Lemma 3, Fubini's theorem and (5) we obtain

$$\begin{aligned}
& w(t, x) - L \sum_{j=0}^{n-1} I_j(t) \\
& \leq \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} h(s_1) h(s_2) \cdots h(s_n) \left(\int_{\mathbb{R}^d} p(\Psi(s_n, t), x_n - x) w(s_n, x_n) dx_n \right) ds_n \cdots ds_2 ds_1 \\
& = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} h(s_1) h(s_2) \cdots h(s_n) \left(\int_{\mathbb{R}^d} p(\Psi(s_n, t), x_n - x) \right. \\
& \quad \times \left. \left(L + \int_0^{s_n} h(s_{n+1}) \left(\int_{\mathbb{R}^d} p(\Psi(s_{n+1}, s_n), x_{n+1} - x_n) w(s_{n+1}, x_{n+1}) dx_{n+1} \right) ds_{n+1} \right) dx_n \right) ds_n \cdots ds_2 ds_1 \\
& = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} h(s_1) h(s_2) \cdots h(s_n) \left(L \int_{\mathbb{R}^d} p(\Psi(s_n, t), x_n - x) dx_n + \int_{\mathbb{R}^d} p(\Psi(s_n, t), x_n - x) \right. \\
& \quad \times \left. \int_0^{s_n} h(s_{n+1}) \left(\int_{\mathbb{R}^d} p(\Psi(s_{n+1}, s_n), x_{n+1} - x_n) w(s_{n+1}, x_{n+1}) dx_{n+1} \right) ds_{n+1} dx_n \right) ds_n \cdots ds_2 ds_1 \\
& = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} h(s_1) h(s_2) \cdots h(s_n) \left(L + \int_0^{s_n} h(s_{n+1}) \right. \\
& \quad \times \left. \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} p(\Psi(s_n, t), x_n - x) p(\Psi(s_{n+1}, s_n), x_{n+1} - x_n) dx_n \right) w(s_{n+1}, x_{n+1}) dx_{n+1} ds_{n+1} \right) ds_n \cdots ds_2 ds_1 \\
& = L \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} h(s_1) h(s_2) \cdots h(s_n) ds_n \cdots ds_2 ds_1 + \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \int_0^{s_n} h(s_1) h(s_2) \cdots h(s_n) h(s_{n+1}) \\
& \quad \times \left(\int_{\mathbb{R}^d} p(\Psi(s_{n+1}, t), x_{n+1} - x) w(s_{n+1}, x_{n+1}) dx_{n+1} \right) ds_{n+1} ds_n \cdots ds_2 ds_1 = LI_n(t) + R_{n+1}(t, x).
\end{aligned}$$

Now let us verify that $\lim_{n \rightarrow \infty} R_n = 0$. To this end, we integrated (7) with respect to x , and using Fubini's theorem we arrive to

$$\int_{\mathbb{R}^d} w(t, x) dx \leq L + \int_0^t h(s) \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p(\Psi(s, t), y - x) w(s, y) dy dx \right) ds = L + \int_0^t h(s) \left(\int_{\mathbb{R}^d} w(s, y) dy \right) ds.$$

The classical Gronwall inequality yields

$$\int_{\mathbb{R}^d} w(x, t) dx \leq L \exp \left(\int_0^t h(s) ds \right). \quad (10)$$

On the other hand, Young's inequality for convolutions and (10) turns out, for $s_n < t$,

$$\int_{\mathbb{R}^d} p(\Psi(s_n, t), x_n - x) w(s_n, x_n) dx_n \leq L \exp \left(\int_0^t h(s) ds \right),$$

accordingly

$$R_n(t, x) \leq L \exp \left(\int_0^t h(s) ds \right) \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} h(s_1) h(s_2) \cdots h(s_n) ds_n \cdots ds_2 ds_1.$$

In this way, using definition (6), we obtain

$$w(t, x) \leq L \sum_{j=0}^{n-1} \frac{1}{j!} \left(\int_0^t h(s) ds \right)^j + \frac{L}{n!} \left(\int_0^t h(s) ds \right)^n \exp \left(\int_0^t h(s) ds \right).$$

Letting $n \rightarrow \infty$, in the above inequality, we get the desired result, (8). $\square$

5 Conclusions

In this paper, we have studied the Hyers-Ulam stability of a nonautonomous semilinear equation with diffusion determined by the fractional Laplacian. More accurately, we have used the Banach contraction principle to prove the existence of the solutions of the partial differential equation (2) and in the study of the Hyers-Ulam stability we used a Gronwall-type inequality. In this way, we recognized the fundamental role played by such an inequality in the study of stability, first observed in [11].

Acknowledgement: The author was partially supported by the grant PIM20-1 of Universidad Autónoma de Aguascalientes.

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