Research Article

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Direct and strong converse inequalities for approximation with Fejér means

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Abstract: We present upper and lower estimates of the error of approximation of periodic functions by Fejér means in the Lebesgue spaces $L_{2\pi}^p$. The estimates are given in terms of a K -functional for $1 \leq p \leq \infty$ and in terms of the first modulus of continuity in the case $1 < p < \infty$. We pay attention to the involved constants.

Keywords: Fejér operators, rate of convergence, direct and converse results

MSC 2010: 42A10, 42A20, 41A25, 41A27

1 Introduction

Let $C_{2\pi}$ denote the Banach space of all 2π -periodic, continuous functions f defined on the real line $\mathbb{R}$ with the sup norm

$$\|f\|_{\infty} = \max_{x \in [-\pi, \pi]} |f(x)|.$$

For $1 \leq p < \infty$, the Banach space $\mathbb{L}^p$ consists of all 2π -periodic, p th power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^p dx \right)^{1/p}.$$

In order to simplify, we write $X^p = \mathbb{L}^p$ for $1 \leq p < \infty$ and $X^{\infty} = C_{2\pi}$.

Recall that for $f \in \mathbb{L}^1$ and $k \in \mathbb{N}_0$, the Fourier coefficients are defined by

$$a_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(kt) dt \quad \text{and} \quad b_k(f) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(kt) dt,$$

and the (formal) Fourier series is given by

$$f(x) \sim \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos(kx) + b_k(f) \sin(kx)) := \sum_{n=0}^{\infty} A_n(f). \quad (1)$$

For $n \in \mathbb{N}$ and $f \in X^1$, the Fejér sum of order n is defined by

$$\sigma_n(f, x) = \sum_{k=0}^n \left(1 - \frac{k}{n+1} \right) A_k(f, x).$$

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Recall that for $f \in X^p$ ($1 \leq p \leq \infty$) and $t > 0$, the usual modulus of continuity of f is defined by

$$\omega_1(f, t)_p = \sup_{|h| \leq t} \|f(\circ + h) - f(\circ)\|_p.$$

In [1], Ching and Chui proved that, for each $f \in X^2$ and $n \in \mathbb{N}$,

$$\frac{1}{\pi} \omega_1\left(f, \frac{\pi}{n+1}\right)_2 \leq \|f - \sigma_n(f)\|_2 \leq \frac{1}{\sqrt{2}} \omega_1\left(f, \frac{\pi}{n+1}\right)_2.$$

In this note, we show that a similar relation holds for all X^p spaces with $1 < p < \infty$. In the cases $p = 1$ and $p = \infty$, we provide another kind of characterization. At the end of the article, we compare our results with similar ones given in [2].

2 Main results

We need a result of Alexits [3].

Proposition 1. (Alexits, [3, Lemma 1]) *Let X be a Banach space with norm $\|\cdot\|$ and let $\{x_n\}$ be a sequence in X . If there exists a constant K such that, for each $n \in \mathbb{N}$,*

$$\left\| \frac{1}{n+1} \sum_{k=1}^n \sum_{j=1}^k x_j \right\| = \left\| \sum_{k=1}^n \left(1 - \frac{k}{n+1}\right) x_k \right\| \leq K,$$

then there exists $x \in X$ such that

$$\left\| x - \frac{1}{n+1} \sum_{k=1}^n \sum_{j=1}^k \frac{1}{j} x_j \right\| \leq \frac{3K}{n}.$$

For $f \in X^1$, the conjugate function is defined by

$$\tilde{f}(x) = -\frac{1}{2\pi} \int_0^\pi \frac{f(x+t) - f(x-t)}{\tan(t/2)} dt = -\lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi} \int_\varepsilon^\pi \frac{f(x+t) - f(x-t)}{\tan(t/2)} dt,$$

whenever the limit exists. It is known that if $f \in X^p$ with $1 < p < \infty$, then $\tilde{f} \in X^p$, and that is not the case for $p = 1$ and $p = \infty$. It can be proved that if $f \in X^p$ ($1 < p < \infty$) is absolutely continuous and $f' \in X^p$, then $\tilde{f}' = (\tilde{f})'$.

The associated series $\tilde{S}(f)$ is defined by [4, p. 49]

$$\tilde{S}(f, x) = \sum_{k=1}^{\infty} (-b_k \cos(kx) + a_k \sin(kx)). \quad (2)$$

Recall that the conjugate Fejér mean of order n of a function $f \in \mathbb{L}^1$ with Fourier series (1) is defined by [4, p. 85]

$$\tilde{\sigma}_n(f, x) = \sum_{k=1}^n \left(1 - \frac{k}{n+1}\right) (-b_k \cos(kx) + a_k \sin(kx)).$$

Proposition 2. *If $1 < p < \infty$, $f \in X^p$ is absolutely continuous and $f' \in X^p$, then $\tilde{f}$ is absolutely continuous, $(\tilde{f})' \in X^p$ and $(\tilde{f})' = \tilde{f}'$.*

Proof. Since (see [4, p. 40])

$$S(f', x) = S'(f, x) \sim \sum_{k=1}^{\infty} k(b_k \cos(jx) - a_k \sin(kx)), \quad (3)$$

we have

$$\tilde{S}(f', x) \sim \sum_{k=1}^{\infty} k(a_k \cos(jx) + b_k \sin(kx)).$$

Therefore,

$$\tilde{\sigma}_n(f', x) = \sum_{k=1}^n \left(1 - \frac{k}{n+1}\right) k(a_k \cos(kx) + b_k \sin(kx)).$$

From the Riesz theorem, we know that $\tilde{f}$ exists a.e. and $\tilde{f} \in X^p$. Since

$$\tilde{f}(x) \sim \sum_{k=1}^{\infty} (-b_k \cos(kx) + a_k \sin(kx)),$$

one has

$$\sigma_n(\tilde{f}, x) = \sum_{k=1}^n \left(1 - \frac{k}{n+1}\right) (-b_k \cos(kx) + a_k \sin(kx)),$$

and

$$\sigma'_n(\tilde{f}, x) = \sum_{k=1}^n \left(1 - \frac{k}{n+1}\right) k(a_k \cos(kx) + b_k \sin(kx)).$$

We have proved that $\sigma'_n(\tilde{f}, x) = \tilde{\sigma}_n(f', x)$. Hence,

$$\|\sigma'_n(\tilde{f})\|_p = \|\tilde{\sigma}_n(f')\|_p \leq C_p \|\sigma_n(f')\|_p \leq C_p \|f'\|_p$$

(at the end of the article we explain the nature of the constant C_p). Since $\|\sigma_n(\tilde{f}) - \tilde{f}\|_p \rightarrow 0$, and for any $h > 0$ (we set $\Delta_h f(x) = f(x+h) - f(x)$)

$$\|\Delta_h \tilde{f}\|_p = \lim_{n \rightarrow \infty} \|\Delta_h \sigma_n(\tilde{f})\|_p \leq \sup_{n \in \mathbb{N}} \|\sigma'_n(\tilde{f})\|_p h \leq C_p \|f'\|_p h,$$

we conclude that $\tilde{f} \in \text{Lip}(1, p)$. This means that there is a constant C such that

$$\sup_{0 < h \leq t} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\tilde{f}(x+h) - \tilde{f}(x)|^p dx \right)^{1/p} \leq Ct.$$

But it is known that (see [4, p. 180]), for $1 < p < \infty$, $\tilde{f} \in \text{Lip}(1, p)$ if and only if there exists $g \in X^p$ such that

$$\tilde{f}(x) = \int_{-\pi}^x g(s) ds.$$

Therefore, $(\tilde{f})'(x)$ exists a.e. and $(\tilde{f})'(x) = g(x) \in X^p$. In this case, equation (3) can be written as

$$S((\tilde{f})', x) = S'(\tilde{f}, x) = \sum_{k=1}^{\infty} k(a_k \cos(kx) + b_k \sin(kx)).$$

We know that, for any $h \in X^p$ ($1 < p < \infty$) $\tilde{\sigma}_n(h, x) = \sigma_n(\tilde{h}, x)$ [4, p. 92] and $\|\sigma_n(\tilde{h}) - \tilde{h}\|_p \rightarrow 0$ as $n \rightarrow \infty$. By taking $h = f'$, this yields

$$\tilde{S}(f', x) = S(\tilde{f}', x) = \sum_{k=1}^{\infty} k(a_k \cos(kx) + b_k \sin(kx)).$$

We have proved that $(\tilde{f})'(x) = \tilde{f}'(x)$ a.e. $\square$

Theorem 1. Assume $1 \leq p \leq \infty$. If $f, \tilde{f}, (\tilde{f})' \in X^p$ (we assume that f and $\tilde{f}$ are absolutely continuous) and $n \in \mathbb{N}$, then

$$\|(I - \sigma_n)(f)\|_p \leq \frac{3}{n} \|(\tilde{f})'\|_p.$$

Proof. It is known that if f is absolutely continuous, then

$$f'(x) \sim \sum_{k=1}^{\infty} k(b_k(f) \cos(kx) - a_k(f) \sin(kx)),$$

see [4, p. 41]. If $\tilde{f}$ exists and $\tilde{f} \in \mathbb{L}^1$, then $\tilde{S}(f, x) = S(\tilde{f}, x)$ a.e. [4, p. 51].

Set $g = f - C$, where $C = a_0(f)/2$.

Note that $(\tilde{f})'(x) = (\tilde{g})'(x) \sim -\sum_{k=1}^{\infty} k(a_k(f) \cos(kx) + b_k(f) \sin(kx))$.

We will apply Proposition 2 with $x_k = k(a_k(f) \cos(kx) + b_k(f) \sin(kx))$. Take into account that

$$\left\| \frac{1}{n+1} \sum_{k=1}^n \sum_{j=1}^k x_k \right\|_p = \left\| \frac{1}{n+1} \sum_{k=1}^n (n+1-k)x_k \right\|_p = \|\sigma_n(\tilde{f}')\|_p \leq \|\tilde{f}'\|_p.$$

On the other hand, with the notations given above one has

$$\frac{1}{n+1} \sum_{k=1}^n \sum_{j=1}^k \frac{1}{j} x_j = \frac{1}{n+1} \sum_{j=1}^n \sum_{k=1}^j (a_k(f) \cos(kx) + b_k(f) \sin(kx)) = \frac{1}{n+1} \sum_{j=1}^n \sum_{k=1}^j A_k(f) = \sigma_n(g).$$

Now it follows from Proposition 2 (with $K = \|\tilde{f}'\|_p$) that there exists $h \in X^p$ satisfying

$$\|h - \frac{1}{n+1} \sum_{k=1}^n \sum_{j=1}^k \frac{1}{j} x_j\|_p = \|h - \sigma_n(g)\|_p \leq \frac{3\|\tilde{f}'\|_p}{n}.$$

Since $\sigma_n(g) \rightarrow g$, $g = h$ a.e. But $\sigma_n(g) - g = \sigma_n(f) - f$. $\square$

Remark 1. Note that, if $f \in \mathbb{T}_n$ is not a constant polynomial, say $f(x) = \sum_{k=0}^n (a_k \cos(kx) + b_k \sin(kx))$, then

$$(\tilde{f})'(x) = \sum_{k=0}^n k(a_k \cos(kx) + b_k \sin(kx)),$$

while

$$(I - \sigma_n)(f, x) = \frac{1}{n+1} \sum_{k=1}^n k(a_k \cos(kx) + b_k \sin(kx)) = \frac{1}{n+1} (\tilde{f})'(x).$$

Therefore,

$$\frac{n\|(I - \sigma_n)(f)\|_p}{\|(\tilde{f})'\|_p} = \frac{n}{n+1}$$

and

$$\sup_{n \in \mathbb{N}} \sup_{f \in X^p, f \neq \text{const}} \frac{n\|(I - \sigma_n)(f)\|_p}{\|(\tilde{f})'\|_p} \geq 1.$$

This provides a lower bound for the constant in Theorem 1.

Theorem 2. Assume $1 \leq p \leq \infty$. If $n \in \mathbb{N}$ and $f \in X^p$, then

$$\frac{1}{2} \|f - \sigma_n(f)\|_p \leq \tilde{K}\left(f, \frac{3}{n+1}\right)_p \leq 4 \|f - \sigma_n(f)\|_p, \quad (4)$$

where

$$\tilde{K}(f, t)_p = \inf\{\|f - g\|_p + t\|(\tilde{g})'\|_p : g, \tilde{g} \in X^p, \tilde{g} \in AC, (\tilde{g})' \in X^p\}.$$

Moreover, if $1 < p < \infty$, there exists a constant C_p such that, for each $f \in X^p$,

$$\frac{1}{2C_p} \|f - \sigma_n(f)\|_p \leq \omega_1\left(f, \frac{3}{n+1}\right)_p \leq 8C_p \|f - \sigma_n(f)\|_p. \quad (5)$$

Proof. A direct computation shows that, for any trigonometric polynomial T of degree not greater than n ,

$$(\sigma_n - I)(T) = -\frac{1}{n+1}(\tilde{T})'.$$

Since $\tilde{\sigma}_n(f)$ is a trigonometric polynomial of degree not greater than n and $\|\sigma_n\| = 1$, we know that

$$\|(\tilde{\sigma}_n(f))'\|_p = (n+1)\|(I - \sigma_n)(\sigma_n f)\|_p = (n+1)\|\sigma_n(f - \sigma_n(f))\|_p \leq (n+1)\|f - \sigma_n(f)\|_p.$$

Hence,

$$\tilde{K}\left(f, \frac{3}{n+1}\right)_p \leq \|f - \sigma_n(f)\|_p + \frac{3}{n+1}\|(\tilde{\sigma}_n(f))'\|_p \leq 4\|f - \sigma_n(f)\|_p.$$

On the other hand, it follows from Theorem 1 that, for each $n \in \mathbb{N}$ and every $g \in X^p$ such that $(\tilde{g})' \in X^p$,

$$\|f - \sigma_n(f)\|_p \leq \|f - g - \sigma_n(f - g)\|_p + \|g - \sigma_n(g)\|_p \leq 2\|f - g\|_p + \frac{3}{n}\|(\tilde{g})'\|_p \leq 2\|f - g\|_p + \frac{6}{n+1}\|(\tilde{g})'\|_p$$

and this yields $\|f - \sigma_n(f)\|_p \leq 2\tilde{K}(f, 3/(n+1))_p$.

If $1 < p < \infty$, $h \in L^p$ and $S(h)$ is the Fourier series of h , then $h(x) = S(h, x)$ a.e. In particular, if $h = \tilde{g}'$, then $\tilde{h} = -g'$ a.e. and it follows from the Riesz theorem (see [5, p. 336] or [6]) that there exists a constant $C_p \geq 1$ such that

$$\|\tilde{g}'\|_p \leq C_p \|g'\|_p \quad \text{and} \quad \|g'\|_p = \|\tilde{h}\|_p \leq C_p \|h\|_p = C_p \|\tilde{g}'\|_p. \quad (6)$$

Therefore,

$$\frac{1}{C_p} \tilde{K}(f, t)_p \leq \inf\{\|f - g\|_p + t\|g'\|_p : g \in X^p, g \in AC, g' \in X^p\} \leq C_p \tilde{K}(f, t)_p.$$

But it is known that (see [7, pp. 175–177])

$$\frac{1}{2} \omega_1(f, t)_p \leq \inf\{\|f - g\|_p + t\|g'\|_p : g \in X^p, g \in AC, g' \in X^p\} \leq 2\omega_1(f, t)_p.$$

Hence,

$$\frac{1}{4C_p} \|f - \sigma_n(f)\|_p \leq \frac{1}{2C_p} \tilde{K}\left(f, \frac{3}{n+1}\right)_p \leq \omega_1\left(f, \frac{3}{n+1}\right)_p \leq 2C_p \tilde{K}\left(f, \frac{3}{n+1}\right)_p \leq 8C_p \|f - \sigma_n(f)\|_p,$$

and this yields the result. $\square$

Remark 2. The best constant C_p in inequality (6) was obtained by Pichorides in [6]:

$$P(p) = \begin{cases} \tan \frac{\pi}{2p} & \text{if } 1 < p \leq 2, \\ \cot \frac{\pi}{2p} & \text{if } 2 < p < \infty. \end{cases}$$

In particular, when $p = 2$ and $C_p = 1$, equation (5) can be written as

$$\frac{1}{4} \|f - \sigma_n(f)\|_2 \leq \omega_1\left(f, \frac{3}{n+1}\right)_2 \leq 8 \|f - \sigma_n(f)\|_2.$$

Thus, we obtain a result similar to the one of Ching and Chui.

Remark 3. Ditzian and Ivanov [2, Theorem 2.1] proved (4) in the case of continuous functions. They used some ideas of Lorentz in [8], instead of the Alexits result recalled in Proposition 2. The general case of X^p ($1 \leq p \leq \infty$) is included in [2, Theorem 2.2]. In both theorems, no information is given concerning the constants. Hence, equation (4) is an improvement of the Ditzian-Ivanov results. We also remark that there is not an analogous of (5) in [2].

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