Research Article

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Chandrasekhar quadratic and cubic integral equations via Volterra-Stieltjes quadratic integral equation

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Abstract: In this work, we study the existence of one and exactly one solution $x \in C[0, 1]$, for a delay quadratic integral equation of Volterra-Stieltjes type. As special cases we study a delay quadratic integral equation of fractional order and a Chandrasekhar cubic integral equation.

Keywords: Volterra-Stieltjes type, continuous solution, delay functional integral equation, continuous dependence

MSC 2020: 74H10, 45G10, 47H30

1 Introduction

Quadratic integral equations arise in many applications in the real world. For example, problems in the theory of radioactive transfer, in the theory of neutron transport and the kinetic theory of gasses lead to quadratic integral equations (see [1–5]).

The existence of solutions of the integral equations of Volterra-Stieltjes type has been studied by J. Banaś in [6–12].

Consider the delay quadratic integral equation of Volterra-Stieltjes type

$$x(t) = a(t) + \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) + \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s), \quad t \in I = [0, 1]. \quad (1)$$

Here we study the existence of solutions of the quadratic integral equation (1) in the class of continuous functions. The continuous dependence of the unique solution of the quadratic integral equation (1) on the functions g_1 , g_2 and φ will be studied. As an application, we use the concept and properties of the fractional-order integral (see, for example, [13, 14] and [15–22]) to obtain the solutions of the fractional-order quadratic integral equation

$$x(t) = a(t) + I^\alpha f_1(t, s, x) \cdot \int_0^1 \frac{t}{t+s} f_2(t, s, x(s)) ds,$$

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the Chandrasekhar quadratic integral equation

$$x(t) = a(t) + x(t) \cdot \int_0^1 \frac{t}{t+s} a(s)x(s) ds$$

and the Chandrasekhar cubic integral equation

$$x(t) = a(t) + x^2(t) \cdot \int_0^1 \frac{t}{t+s} a(s)x(s) ds$$

will be considered.

2 Existence of at least one solution

Now we consider the delay quadratic integral equation of Volterra-Stieltjes type (1) under the following assumptions:

- (i) $\varphi : [0, 1] \rightarrow [0, 1]$ is continuous and increasing such that $\varphi(t) \leq t$.
- (ii) $a \in C[0, 1]$.
- (iii) $f_i : [0, 1] \times [0, 1] \times R \rightarrow R$ are continuous such that

$$|f_1(t, s, x)| \leq b_1(t, s) + k_1(t, s)|x|^p, \quad p \geq 1$$

and

$$|f_2(t, s, x)| \leq b_2(t, s) + k_2(t, s)|x|,$$

where $b_i, k_i : [0, 1] \times [0, 1] \rightarrow R_+$ are continuous and

$$b = \sup_t \{b_i(t, s) : t, s \in [0, 1], i = 1, 2\},$$

$$k = \sup_t \{k_i(t, s) : t, s \in [0, 1], i = 1, 2\}.$$

- (iv) (1) The function g_1 is continuous on Λ , where

$$\Lambda = (t, s) : 0 \leq s \leq t \leq 1.$$

- (2) The function $g_2 : [0, 1] \times R \rightarrow R$ is continuous with

$$\mu = \max\{\sup\{|g_2(t, 1)| : t \in [0, 1]\}, \sup\{|g_2(t, 0)| : t \in [0, 1]\}\}.$$

- (v) (1) For each $\varepsilon > 0$ there exists $\delta > 0$ for all $t_1, t_2 \in I$ such that $t_1 < t_2$ and $t_2 - t_1 < \delta$ the following inequality holds:

$$\bigvee_0^{t_1} [g_1(t_2, s) - g_1(t_1, s)] \leq \varepsilon.$$

- (2) For all $t_1, t_2 \in I$ such that $t_1 < t_2$ the function $s \rightarrow g_2(t_2, s) - g_2(t_1, s)$ is nondecreasing on $[0, 1]$.

- (vi) (1) $g_1(t, 0) = 0$ for any $t \in [0, 1]$.

- (2) $g_2(0, s) = 0$ for any $s \in [0, 1]$.

- (vii) The function $s \rightarrow g_1(t, s)$ is of bounded variation on $[0, t]$ for each fixed $t \in I$.

- (viii) There exists a positive root r of the algebraic equation

$$k^2 K \mu r^{p+1} + b k K \mu r^p + (b k \mu K - 1)r + (a + b^2 K \mu) = 0. \quad (2)$$

If $p = 1$, then the algebraic equation (2) has the form

$$k^2 K \mu r^2 + (2bkK\mu - 1)r + (a + b^2 K \mu) = 0.$$

If $p = 2$, the algebraic equation (2) will be of the form

$$k^2 K \mu r^3 + bkK\mu r^2 + (bk\mu K - 1)r + (a + b^2 K \mu) = 0.$$

Remark 1. The function $z \rightarrow \bigvee_{s=0}^s g_1(t, s)$ is continuous on $[0, t]$ for any fixed $t \in I$ [7].

Lemma 1. For an arbitrary fixed $0 < t_2 < I$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $t_1 \in I$, $t_1 < t_2$ and $t_2 - t_1 < \delta$ then [7]

$$\bigvee_{s=t_1}^{t_2} g_1(t_2, s) \leq \delta.$$

Lemma 2. The function $t \rightarrow \bigvee_{s=0}^t g_1(t, s)$ is continuous on I . Then there exists a finite positive constant K such that

$$K = \sup \left\{ \bigvee_{s=0}^t g_1(t, s) : t \in I \right\}.$$

Remark 2. (see [7]) Observe that the function $s \rightarrow g_2(t, s)$ is nondecreasing on the interval $[0, 1]$. In fact, for $s_1, s_2 \in [0, 1]$, with $s_1 < s_2$, from assumptions (v) and (vi), we obtain

$$g_2(t, s_2) - g_2(t, s_1) = [g_2(t, s_2) - g_2(0, s_2)] - [g_2(t, s_1) - g_2(0, s_1)] \geq 0.$$

Lemma 3. (see [7]) Assume that the function g_2 satisfies assumption (vi). Then for arbitrary $s_1, s_2 \in I$, such that $s_1 < s_2$, the function $t \rightarrow g_2(t, s_2) - g_2(t, s_1)$ is nondecreasing on the interval I .

In fact, take $t_1, t_2 \in [0, 1]$ such that $t_1 < t_2$. Then, by assumption (vi), we get

$$[g_2(t_2, s_2) - g_2(t_2, s_1)] - [g_2(t_1, s_2) - g_2(t_1, s_1)] = [g_2(t_2, s_2) - g_2(t_1, s_2)] - [g_2(t_2, s_1) - g_2(t_1, s_1)] \geq 0.$$

For the existence of at least one solution of the quadratic integral equation (1), we have the following theorem.

Theorem 1. Let assumptions (i)–(viii) be satisfied, then the functional integral equation (1) has at least one continuous solution $x \in C[0, 1]$.

Proof. Define the operator

$$Fx(t) = a(t) + \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s).$$

Define the set Q by $Q = \{x \in C[0, 1] : |x| \leq r\}$, where r is a positive solution of the algebraic equation $(a + (b + kr)^2 \mu K = r)$.

It is clear that the set Q is a nonempty, bounded, closed and convex set.

Now, let $x \in C[0, 1]$, then

$$\begin{aligned} |Fx(t)| &= \left| a(t) + \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \right| \\ &\leq a + \int_0^{\varphi(t)} |f_1(t, s, x(s))| d_s g_1(t, s) \int_0^1 |f_2(t, s, x(s))| d_s g_2(t, s) \end{aligned}$$

$$\begin{aligned}
&\leq a + \int_0^{\varphi(t)} (b_1(t, s) + k_1(t, s)|x|^p) d_s \left(\bigvee_{z=0}^s g_1(t, z) \right) \int_0^1 (b_2(t, s) + k_2(t, s)|x|) d_s \left(\bigvee_{z=0}^s g_2(t, z) \right) \\
&\leq a + (b + k\|x\|^p) \int_0^{\varphi(t)} d_s g_1(t, s) \cdot (b + k\|x\|) \int_0^1 d_s g_2(t, s) \\
&\leq a + (b + kr^p) \sup_{t \in I} \bigvee_{s=0}^s g_1(t, s) \cdot (b + kr) (g_2(t, 1) - g_2(t, 0)) \\
&\leq a + (b + kr^p)(b + kr)K\mu = r.
\end{aligned}$$

Hence, $Fx \in Q$ which proves that the operator F maps Q into itself and the class of functions $\{Fx\}$ is uniformly bounded in Q .

Let $x \in Q$ and define

$$\theta(\delta) = \sup_{x \in Q_r} \{|f_i(t_2, s, x(s)) - f_i(t_1, s, x(s))| : t_1, t_2 \in [0, 1], t_1 < t_2, |t_2 - t_1| < \delta, s \in I, i = 1, 2\},$$

then from the uniform continuity of the function $f_i : [0, 1] \times [0, 1] \times Q \rightarrow R$ and assumption (iii), we deduce that $\theta(\delta) \rightarrow 0$, as $\delta \rightarrow 0$ independently on $x \in Q$.

Now, to prove the operator F maps $C[0, 1]$ into itself, let $t_1, t_2 \in [0, 1]$, such that $|t_2 - t_1| < \delta$, then we have

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| &= \left| a(t_2) + \int_0^{\varphi(t_2)} f_1(t_2, s, x(s)) d_s g_1(t_2, s) \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) \right. \\
&\quad \left. - a(t_1) - \int_0^{\varphi(t_1)} f_1(t_1, s, x(s)) d_s g_1(t_1, s) \int_0^1 f_2(t_1, s, x(s)) d_s g_2(t_1, s) \right| \\
&\leq |a(t_2) - a(t_1)| \\
&\quad + \left| \int_0^{\varphi(t_2)} f_1(t_2, s, x(s)) d_s g_1(t_2, s) \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) \right. \\
&\quad \left. - \int_0^{\varphi(t_1)} f_1(t_1, s, x(s)) d_s g_1(t_1, s) \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) \right. \\
&\quad \left. + \int_0^{\varphi(t_1)} f_1(t_1, s, x(s)) d_s g_1(t_1, s) \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) \right. \\
&\quad \left. - \int_0^{\varphi(t_1)} f_1(t_1, s, x(s)) d_s g_1(t_1, s) \int_0^1 f_2(t_1, s, x(s)) d_s g_2(t_1, s) \right| \\
&= |a(t_2) - a(t_1)| \\
&\quad + \left| \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) \left[\int_0^{\varphi(t_2)} f_1(t_2, s, x(s)) d_s g_1(t_2, s) - \int_0^{\varphi(t_1)} f_1(t_1, s, x(s)) d_s g_1(t_1, s) \right] \right. \\
&\quad \left. + \int_0^{\varphi(t_1)} f_1(t_1, s, x(s)) d_s g_1(t_1, s) \left[\int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) - \int_0^1 f_2(t_1, s, x(s)) d_s g_2(t_1, s) \right] \right|
\end{aligned}$$

$$\begin{aligned}
&\leq |a(t_2) - a(t_1)| + \left| \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) \left[\int_0^{\varphi(t_1)} f_1(t_2, s, x(s)) d_s g_1(t_2, s) \right. \right. \\
&\quad + \left. \int_{\varphi(t_1)}^{\varphi(t_2)} f_1(t_2, s, x(s)) d_s g_1(t_2, s) - \int_0^{\varphi(t_1)} f_1(t_1, s, x(s)) d_s g_1(t_1, s) \right] \\
&\quad + \int_0^1 f_2(t_1, s, x(s)) d_s g_2(t_1, s) \left[\int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) \right. \\
&\quad \left. \left. - \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_1, s) + \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_1, s) - \int_0^1 f_2(t_1, s, x(s)) d_s g_2(t_1, s) \right] \right| \\
&\leq |a(t_2) - a(t_1)| + \left| \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) \left[\int_0^{\varphi(t_1)} f_1(t_2, s, x(s)) d_s g_1(t_2, s) \right. \right. \\
&\quad + \int_{\varphi(t_1)}^{\varphi(t_2)} f_1(t_2, s, x(s)) d_s g_1(t_2, s) - \int_0^{\varphi(t_1)} f_1(t_1, s, x(s)) d_s g_1(t_1, s) \\
&\quad + \left. \int_0^{\varphi(t_1)} f_1(t_2, s, x(s)) d_s g_1(t_1, s) - \int_0^{\varphi(t_1)} f_1(t_2, s, x(s)) d_s g_1(t_1, s) \right] \\
&\quad + \int_0^1 f_2(t_1, s, x(s)) d_s g_2(t_1, s) \left[\int_0^1 f_2(t_2, s, x(s)) d_s [g_2(t_2, s) - g_2(t_1, s)] \right. \\
&\quad \left. + \int_0^1 [f_2(t_2, s, x(s)) - f_2(t_1, s, x(s))] d_s g_2(t_1, s) \right] \right| \\
&\leq |a(t_2) - a(t_1)| + \left| \int_0^1 f_2(t_2, s, x(s)) d_s g_2(t_2, s) \left[\int_{\varphi(t_1)}^{\varphi(t_2)} f_1(t_2, s, x(s)) d_s g_1(t_2, s) \right. \right. \\
&\quad + \left. \int_0^{\varphi(t_1)} f_1(t_2, s, x(s)) d_s [g_1(t_2, s) - g_1(t_1, s)] + \int_0^{\varphi(t_1)} f_1(t_2, s, x(s)) - f_1(t_2, s, x(s)) d_s g_1(t_1, s) \right] \\
&\quad + \int_0^1 f_2(t_1, s, x(s)) d_s g_2(t_1, s) \left[\int_0^1 f_2(t_2, s, x(s)) d_s [g_2(t_2, s) - g_2(t_1, s)] \right. \\
&\quad \left. + \int_0^1 [f_2(t_2, s, x(s)) - f_2(t_1, s, x(s))] d_s g_2(t_1, s) \right] \right| \\
&\leq |a(t_2) - a(t_1)| \\
&\quad + (b + kr) [g_2(t_2, 1) - g_2(t_2, 0)] \left[(b + kr^p) \bigvee_{\varphi(t_1)}^{\varphi(t_2)} g_1(t_2, s) + (b + kr^p) N(\varepsilon) + \theta(\varepsilon) \sup_{t \in I} \left(\bigvee_{s=0}^{\varphi(t_1)} g_1(t, s) \right) \right] \\
&\quad + (b + kr) [g_2(t_1, 1) - g_2(t_1, 0)] \left[(b + kr) \bigvee_{z=0}^s [g_2(t_2, z) - g_2(t_1, z)] + \theta(\varepsilon) (g_2(t_2, 1) - g_2(t_2, 0)) \right],
\end{aligned}$$

where

$$N(\varepsilon) = \sup \left\{ \bigvee_{s=0}^{t_1} (g_1(t_2, s) - g_1(t_1, s)) : t_1, t_2 \in I, t_1 < t_2, t_2 - t_1 \leq \varepsilon \right\}.$$

The above inequality means that $Fx : C[0, 1] \rightarrow C[0, 1]$.

Then FQ is compact.

Now we prove that the operator F is continuous.

Let $\{x_n\} \subset Q$, and $\{x_n\} \rightarrow x$, in $Q \subseteq R$ then

$$\begin{aligned} Fx_n(t) &= a(t) + \int_0^{\varphi(t)} f_1(t, s, x_n(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_n(s)) d_s g_2(t, s), \\ \lim_{n \rightarrow \infty} Fx_n(t) &= a(t) + \lim_{n \rightarrow \infty} \int_0^{\varphi(t)} f_1(t, s, x_n(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_n(s)) d_s g_2(t, s). \end{aligned}$$

Applying Lebesgue dominated convergence theorem (see [23]), then

$$\begin{aligned} &= a(t) + \int_0^{\varphi(t)} f_1(t, s, \lim_{n \rightarrow \infty} x_n(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, \lim_{n \rightarrow \infty} x_n(s)) d_s g_2(t, s) \\ &= a(t) + \int_0^{\varphi(t)} f_1(t, s, x_0(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_0(s)) d_s g_2(t, s) = Fx_0(t) \end{aligned}$$

which means that the operator F is continuous.

Since all conditions of Schauder fixed point theorem [23] (see also [2, 3, 6, 13]) are satisfied, the operator F has at least one fixed point $x \in Q$, and the integral equation (1) has at least one solution $x \in C[0, 1]$. This completes the proof. $\square$

3 Uniqueness of the solution

To study the uniqueness of the solution of the functional integral equation (1), we replace the assumption (iii) by:

(iii)* $f_i : [0, 1] \times [0, 1] \times R \rightarrow R$, $i = 1, 2$ are continuous and satisfy the Lipschitz condition

$$|f_1(t, s, x) - f_1(t, s, y)| \leq k_1 |x - y|^p \leq 2r^{p-1} k_1 |x - y|, \quad x, y \in Q,$$

$$|f_2(t, s, x) - f_2(t, s, y)| \leq k_2 |x - y|, \quad x, y \in Q$$

$$\text{and } k = \max\{2r^{p-1} k_1, k_2\}.$$

From assumption (iii)* we have consecutively

$$\begin{aligned} |f_1(t, s, x(s))| - |f_1(t, s, 0)| &\leq |f_1(t, s, x(s)) - f_1(t, s, 0)| \leq k|x|^p, \\ |f_1(t, s, x(s))| &\leq k|x|^p + |f_1(t, s, 0)|. \end{aligned}$$

Hence,

$$|f_1(t, s, x(s))| \leq k|x|^p + b, \quad b = \sup_t \{|f_1(t, s, 0)| : t, s \in [0, 1], i = 1, 2\}.$$

Similarly,

$$|f_2(t, s, x(s))| \leq k|x| + b.$$

For the uniqueness of the solution of the functional integral equation (1) we have the following theorem.

Theorem 2. Let assumptions (i)-(ii)-(iii)*-(iv)-(v)-(vi)-(vii)-(viii) be satisfied, if $kK[(b + kr) + (b + kr^p)]\mu < 1$, then the solution $x \in C[0, 1]$ of the functional integral equation (1) is unique.

Proof. Let x_1, x_2 be two solutions of the integral equation (1), then

$$\begin{aligned}
 |F x_1 - F x_2| &= |F x_1(t) - F x_2(t)| \\
 &= \left| a(t) + \int_0^{\varphi(t)} f_1(s, x_1(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_1(s)) d_s g_2(t, s) \right. \\
 &\quad \left. - a(t) - \int_0^{\varphi(t)} f_1(t, s, x_2(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_2(s)) d_s g_2(t, s) \right| \\
 &= \left| \int_0^{\varphi(t)} f_1(t, s, x_1(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_1(s)) d_s g_2(t, s) \right. \\
 &\quad \left. - \int_0^{\varphi(t)} f_1(t, s, x_2(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_2(s)) d_s g_2(t, s) \right| \\
 &= \left| \int_0^{\varphi(t)} f_1(t, s, x_1(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_1(s)) d_s g_2(t, s) \right. \\
 &\quad \left. - \int_0^{\varphi(t)} f_1(t, s, x_2(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_2(s)) d_s g_2(t, s) \right. \\
 &\quad \left. + \int_0^{\varphi(t)} f_1(t, s, x_1(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_2(s)) d_s g_2(t, s) \right. \\
 &\quad \left. - \int_0^{\varphi(t)} f_1(t, s, x_1(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x_2(s)) d_s g_2(t, s) \right| \\
 &\leq \left| \int_0^{\varphi(t)} f_1(t, s, x_1(s)) d_s g_1(t, s) \left[\int_0^1 (f_2(t, s, x_1(s)) - f_2(t, s, x_2(s))) d_s g_2(t, s) \right] \right| \\
 &\quad + \left| \int_0^{\varphi(t)} f_2(t, s, x_2(s)) d_s g_2(t, s) \left[\int_0^{\varphi(t)} (f_1(t, s, x_1(s)) - f_1(t, s, x_2(s))) d_s g_1(t, s) \right] \right| \\
 &\leq (b + kr^p) K \mu k |x_1 - x_2| + (b + kr) K \mu k |x_1 - x_2|,
 \end{aligned}$$

$$\|F x_1 - F x_2\| \leq k K \mu [(b + kr) + (b + kr^p)] \|x_1 - x_2\|.$$

This proves that the map $F : C[0, 1] \rightarrow C[0, 1]$ is a contraction. Then by Banach fixed point theorem the solution of the functional integral equation (1) is unique. $\square$

4 Continuous dependence of the solution

In this section, we are going to study the continuous dependence of the unique solution $x \in C[0, 1]$ of the functional integral equation (1) on the delay function and the functions g_1, g_2 .

4.1 Continuous dependence on the delay functions $\varphi(t)$

Definition 1. The solution of the functional integral equation (1), depends continuously on the delay function $\varphi(t)$ if $\forall \varepsilon > 0, \exists \delta > 0$, such that

$$|\varphi(t) - \varphi^*(t)| \leq \delta \Rightarrow \|x - x^*\| \leq \varepsilon.$$

Theorem 3. Let the assumptions of Theorem 2 be satisfied, then the solution of the functional integral equation (1) depends continuously on the delay function $\varphi(t)$.

Proof. Let $\delta > 0$ be given such that $|\varphi(t) - \varphi^*(t)| \leq \delta, \forall t \geq 0$, then

$$\begin{aligned} |x(t) - x^*(t)| &\leq \left| a(t) + \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \right. \\ &\quad \left. - a(t) - \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x^*(s)) d_s g_2(t, s) \right| \\ &\leq \left| \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \right. \\ &\quad \left. - \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x^*(s)) d_s g_2(t, s) \right. \\ &\quad \left. + \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \right. \\ &\quad \left. - \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \right| \\ &\leq \left| \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \left[\int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) - \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \right] \right. \\ &\quad \left. + \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \left[\int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) - \int_0^1 f_2(t, s, x^*(s)) d_s g_2(t, s) \right] \right| \\ &\leq \left| \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \left[\int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) - \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \right] \right. \\ &\quad \left. + \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \left[\int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) - \int_0^1 f_2(t, s, x^*(s)) d_s g_2(t, s) \right] \right| \\ &\leq \left| \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \left[\int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) - \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \right] \right. \\ &\quad \left. + \int_0^{\varphi^*(t)} f_1(t, s, x^*(s)) d_s g_1(t, s) \left[\int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) - \int_0^1 f_2(t, s, x^*(s)) d_s g_2(t, s) \right] \right| \\ &\leq \int_0^1 |f_2(t, s, x(s))| d_s g_2(t, s) \left[\int_0^{\varphi(t)} k|x(s) - x^*(s)| d_s g_1(t, s) + \int_0^{\varphi^*(t)} |f_1(t, s, x^*(s))| d_s g_1(t, s) \right. \\ &\quad \left. + \int_0^{\varphi^*(t)} |f_1(t, s, x^*(s))| d_s g_1(t, s) \left[\int_0^1 k|x(s) - x^*(s)| d_s g_2(t, s) \right] \right] \end{aligned}$$

$$\begin{aligned}
&\leq (b + kr)[g_2(t, 1) - g_2(t, 0)] \left\{ k|x - x^*| \sup_{t \in I} \left(\bigvee_{s=0}^{\varphi(t)} g_1(t, s) \right) + (b + kr^p)[g_1(t, \varphi(t)) - g_1(t, \varphi^*(t))] \right\} \\
&\quad + (b + kr^p) \sup_{t \in I} \left(\bigvee_{s=0}^{\varphi^*(t)} g_1(t, s) \right) \{k|x - x^*|[g_2(t, 1) - g_2(t, 0)]\} \\
&\leq (b + kr)\mu\{k\|x - x^*\|K + (b + kr^p)[g_1(t, \varphi(t)) - g_1(t, \varphi^*(t))]\} + (b + kr^p)Kk\|x - x^*\|\mu, \\
\|x - x^*\| &\leq \frac{(b + kr)\mu(b + kr^p)[g_1(t, \varphi(t)) - g_1(t, \varphi^*(t))]}{1 - kK\mu[(b + kr) + (b + kr^p)]}.
\end{aligned}$$

Using the continuity of g_1 we have

$$|\varphi(t) - \varphi^*(t)| \leq \delta \Rightarrow |g_1(t, \varphi(t)) - g_1(t, \varphi^*(t))| < \varepsilon_1,$$

then

$$\leq \frac{(b + kr)(b + kr^p)\mu\varepsilon_1}{1 - kK\mu[(b + kr) + (b + kr^p)]} = \varepsilon.$$

This completes the proof. $\square$

4.2 Continuous dependence on the functions g_i

Definition 2. The solution of the quadratic functional integral equation (1), depends continuously on the functions $g_i(t, s)$, $i = 1, 2$ if $\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$|g_i(t, s) - g_i^*(t, s)| \leq \delta \Rightarrow \|x - x^*\| \leq \varepsilon.$$

Theorem 4. Let the assumptions of Theorem 2 be satisfied, then the solution of the delay quadratic functional integral equation (1) depends continuously on the functions g_1, g_2 .

Proof. Let $\delta > 0$ be given such that $|g_i(t, s) - g_i^*(t, s)| \leq \delta$, $\forall t \geq 0$, then

$$\begin{aligned}
|x(t) - x^*(t)| &= \left| a(t) + \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \right. \\
&\quad \left. - a(t) + \int_0^{\varphi(t)} f_1(t, s, x^*(s)) d_s g_1^*(t, s) \int_0^1 f_2(t, s, x^*(s)) d_s g_2^*(t, s) \right| \\
&\leq \left| \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) \right. \\
&\quad - \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x^*(s)) d_s g_2^*(t, s) \\
&\quad + \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \int_0^1 f_2(t, s, x^*(s)) d_s g_2^*(t, s) \\
&\quad \left. - \int_0^{\varphi(t)} f_1(t, s, x^*(s)) d_s g_1^*(t, s) \int_0^1 f_2(t, s, x^*(s)) d_s g_2^*(t, s) \right|
\end{aligned}$$

$$\begin{aligned}
&\leq \left| \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \left[\int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) - \int_0^1 f_2(t, s, x^*(s)) d_s g_2^*(t, s) \right] \right. \\
&\quad \left. + \int_0^1 f_2(t, s, x^*(s)) d_s g_2^*(t, s) \left[\int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) - \int_0^{\varphi(t)} f_1(t, s, x^*(s)) d_s g_1^*(t, s) \right] \right| \\
&\leq \left| \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \left[\int_0^1 f_2(t, s, x(s)) d_s g_2(t, s) - \int_0^1 f_2(t, s, x(s)) d_s g_2^*(t, s) \right] \right. \\
&\quad \left. + \int_0^1 f_2(t, s, x(s)) d_s g_2^*(t, s) - \int_0^1 f_2(t, s, x^*(s)) d_s g_2^*(t, s) \right] \\
&\quad + \int_0^1 f_2(t, s, x^*(s)) d_s g_2^*(t, s) \left[\int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) - \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1^*(t, s) \right. \\
&\quad \left. + \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1^*(t, s) - \int_0^{\varphi(t)} f_1(t, s, x^*(s)) d_s g_1^*(t, s) \right] \right| \\
&\leq \left| \int_0^{\varphi(t)} f_1(t, s, x(s)) d_s g_1(t, s) \left[\int_0^1 f_2(t, s, x(s)) d_s [g_2(t, s) - g_2^*(t, s)] \right. \right. \\
&\quad \left. + \int_0^1 [f_2(t, s, x(s)) - f_2(t, s, x^*(s))] d_s g_2^*(t, s) \right] \\
&\quad + \int_0^1 f_2(t, s, x^*(s)) d_s g_2^*(t, s) \left[\int_0^{\varphi(t)} f_1(t, s, x(s)) d_s [g_1(t, s) - g_1^*(t, s)] \right. \\
&\quad \left. + \int_0^{\varphi(t)} [f_1(t, s, x(s)) - f_1(t, s, x^*(s))] d_s g_1^*(t, s) \right] \right| \\
&\leq (b + kr^p) \sup_{t \in I} \left(\bigvee_{s=0}^{\varphi(t)} g_1(t, s) \right) \{ (b + kr) \| [g_2(t, 1) - g_2^*(t, 1)] - [g_2(t, 0) - g_2^*(t, 0)] \| \\
&\quad + k \| x - x^* \| [g_2^*(t, 1) - g_2^*(t, 0)] \} \\
&\quad + (b + kr) [g_2^*(t, 1) - g_2^*(t, 0)] \left\{ (b + kr^p) \| [g_1(t, \varphi(t)) - g_1^*(t, \varphi(t))] - [g_1(t, 0) - g_1^*(t, 0)] \| \right. \\
&\quad \left. + k \| x - x^* \| \sup_{t \in I} \left(\bigvee_{s=0}^{\varphi(t)} g_1(t, s) \right) \right\} \\
&\leq 2(b + kr^p)(b + kr)K\delta + (b + kr^p)kK\mu \| x - x^* \| + 2(b + kr^p)(b + kr)\mu\delta + (b + kr)kK\mu \| x - x^* \|,
\end{aligned}$$

then

$$\| x - x^* \| [1 - kK\mu[(b + kr) + (b + kr^p)]] \leq 2(b + kr^p)(b + kr)[K + \mu]\delta$$

and

$$\| x - x^* \| \leq \frac{2(b + kr^p)(b + kr)[K + \mu]\delta}{1 - kK\mu[(b + kr) + (b + kr^p)]} = \varepsilon.$$

This completes the proof. $\square$

5 Applications

- (I) In this section, we use the concept and properties of fractional-order integral to show that the delay Volterra integral equation of fractional-order $\alpha \in (0, 1]$

$$x(t) = a(t) + \int_0^{\varphi(t)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f_1(t, s, x(s)) ds + \int_0^1 f_2(t, s, x(s)) d_s g_2(t, s), \quad t \in [0, 1] \quad (3)$$

can be considered as a special case of the Volterra-Stieltjes integral equation (1).

Let the function g_1 be given by

$$g_1(t, s) = \frac{t^\alpha - (t-s)^\alpha}{\Gamma(\alpha+1)},$$

then

$$\frac{\partial g}{\partial s} = \frac{1}{(t-s)^{1-\alpha}} > 0,$$

$$d_s g_1(t, s) = \frac{(t-s)^\alpha - 1}{\Gamma(\alpha)} ds$$

and

$$\begin{aligned} \bigvee_{s=0}^{t_1} [g_1(t_2, s) - g_1(t_1, s)] &= \sum_{i=1}^n |[g_1(t_2, s_i) - g_1(t_1, s_i)] - [g_1(t_2, s_{i-1}) - g_1(t_1, s_{i-1})]| \\ &= \sum_{i=1}^n \{|[g_1(t_2, s_{i-1}) - g_1(t_1, s_{i-1})] - [g_1(t_2, s_i) - g_1(t_1, s_i)]\}| \\ &= g_1(t_1, t_1) - g_1(t_2, t_1) \\ &= \frac{1}{\Gamma(\alpha+1)} [t_1^\alpha - t_2^\alpha - (t_2 - s)^\alpha + (t_1 - s)^\alpha] \leq \varepsilon. \end{aligned}$$

It is clear that the function g_1 satisfies assumptions (iv), (vi), (vii) and (v) (see [9,11]). This proves that the integral equation (3) is a special case of the integral equation (1).

- (II) Chandrasekhar integral equations

Let $\varphi(t) = t$, $f_1(t, s, x) = x^p(t)$, $p = 1, 2$ in (3) and

$$g_2(t, s) = \begin{cases} t \ln \frac{t+s}{t}, & \text{for } t \in (0, 1], s \in I, \\ 0, & \text{for } t = 0, s \in I. \end{cases}$$

Then the function g_2 satisfies assumptions (iv), (v) and (vi) (see [9,11]).

Then equation (3) will be (see [24])

$$x(t) = a(t) + I^\alpha x^p(t) + \int_0^1 \frac{t}{t+s} f_2(t, s, x(s)) ds. \quad (4)$$

Letting $\alpha \rightarrow 0$, then

$$x(t) = a(t) + x^p(t) + \int_0^1 \frac{t}{t+s} f_2(t, s, x(s)) ds. \quad (5)$$

Let $f_2(t, s, x) = a(s)x(s)$, at $p = 1$ we obtain the Chandrasekhar quadratic integral equation

$$x(t) = a(t) + x(t) + \int_0^1 \frac{t}{t+s} a(s)x(s) ds \quad (6)$$

and at $p = 2$ we obtain the Chandrasekhar cubic integral equation

$$x(t) = a(t) + x^2(t) \cdot \int_0^1 \frac{t}{t+s} a(s)x(s)ds, \quad (7)$$

which are special cases of equation (1).

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