Research Article

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Solving two-dimensional nonlinear fuzzy Volterra integral equations by homotopy analysis method

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Abstract: The purpose of the paper is to find an approximate solution of the two-dimensional nonlinear fuzzy Volterra integral equation, as homotopy analysis method (HAM) is applied. Studied equation is converted to a nonlinear system of Volterra integral equations in a crisp case. Using HAM we find approximate solution of this system and hence obtain an approximation for the fuzzy solution of the nonlinear fuzzy Volterra integral equation. The convergence of the proposed method is proved. An error estimate between the exact and the approximate solution is found. The validity and applicability of the HAM are illustrated by a numerical example.

Keywords: homotopy analysis method, two-dimensional nonlinear fuzzy Volterra integral equation, convergence, error estimation

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1 Introduction

Mathematical models of many physical phenomena and engineering problems are described by means of differential, integral and integro-differential equations. The topic of fuzzy integral equations is one of the important branches of fuzzy analysis theory and plays a major role in numerical analysis. Many mathematical models used in biology, chemistry, physics and engineering are based on integral equations.

One of the first applications of fuzzy integration was given by Wu and Ma [1] who investigated the fuzzy Fredholm integral equation of the second kind. In recent years, many mathematicians have studied a solution to fuzzy integral equations by numerical and analytical methods [2–8].

In 1992, Liao [9] employed the basic idea of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM) [10–12]. The method is based on the concept of creating function series. If the series converges, its sum is the solution of this system of equations. This method has been successfully applied to solve many types of nonlinear problems [13–16]. In [17,18], the HAM is applied to solve two-dimensional linear Volterra fuzzy integral equations and mixed Volterra-Fredholm fuzzy integral equations.

In this paper, we present an application of the HAM for solving the nonlinear fuzzy Volterra integral equations in two variables.

The paper is organized as follows: in Section 2, we give the basic notations of fuzzy numbers, fuzzy functions and fuzzy integrals. In Section 3, we present the basic idea of the HAM. In Section 4, the two-dimensional nonlinear fuzzy Volterra integral equation (2D-NFVIE) and its parametric form are discussed

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and then we apply HAM to it. In Section 5, we obtain sufficient conditions for convergence of the proposed method and error estimate. In Section 6, we give an example. In Section 7, the conclusion is drawn.

2 Basic concepts

First, we present some notions and results about fuzzy numbers and fuzzy-number-valued functions, and fuzzy integrals.

Definition 1. [19] A fuzzy number is a fuzzy set $u : \mathbf{R} \rightarrow [0, 1]$ such that

- (i) u is upper semi-continuous function on $\mathbf{R}$;
- (ii) $u(x) = 0$ outside of some interval $[c, d]$;
- (iii) there are the real numbers a and b with $c \leq a \leq b \leq d$, such that u is increasing on $[c, a]$, decreasing on $[b, d]$ and $u(x) = 1$ for each $x \in [a, b]$.

In [20], the convexity of the fuzzy numbers $u(rx + (1 - r)y) \geq \min\{u(x), u(y)\}$, for any $x, y \in \mathbf{R}$, $r \in [0, 1]$ is shown.

We denote with E^1 the set of all fuzzy numbers and $\mathbf{R}_+ = (0, \infty)$.

The set $[u]^r = \{x \in \mathbf{R} : u(x) \geq r\}$ is called r -level set of the fuzzy number u . Then lower and upper representation of fuzzy number is $[u]^r = [\underline{u}(r), \bar{u}(r)]$ for all $r \in [0, 1]$, where $\underline{u}, \bar{u} : [0, 1] \rightarrow \mathbf{R}$ are functions, such that $\underline{u}$ is increasing and $\bar{u}$ is decreasing.

Let $u, v \in E^1$, $k \in \mathbf{R}$. The addition and the scalar multiplication are defined by

$$[u \oplus v]^r = [u]^r + [v]^r = [\underline{u}(r) + \underline{v}(r), \bar{u}(r) + \bar{v}(r)]$$

and

$$[k \odot u]^r = k \cdot [u]^r = \begin{cases} [k\underline{u}(r), k\bar{u}(r)], & k \geq 0, \\ [k\bar{u}(r), k\underline{u}(r)], & k < 0. \end{cases}$$

With respect to $\oplus$ in E^1 the neutral element is denoted by $\tilde{0} = \chi_{\{0\}}$.

In [21], the algebraic properties of fuzzy numbers are given. We use the Hausdorff metric to define a distance between fuzzy numbers.

Definition 2. [21] For arbitrary fuzzy numbers $u, v \in E^1$ the quantity

$$D(u, v) = \sup_{r \in [0, 1]} \max\{|\underline{u}(r) - \underline{v}(r)|, |\bar{u}(r) - \bar{v}(r)|\}$$

is the distance between u, v .

For any fuzzy-number-valued function $f : A = [a, b] \times [c, d] \rightarrow E^1$ we define the left and right r -level functions, respectively, $\underline{f}(\cdot, \cdot, r), \bar{f}(\cdot, \cdot, r) : A \rightarrow \mathbf{R}$ for each $(s, t) \in A$ and $r \in [0, 1]$.

Definition 3. [22] A fuzzy-number-valued function $f : A \rightarrow E^1$ is said to be continuous at $(s_0, t_0) \in A$ if for each $\varepsilon > 0$ there is $\delta > 0$ such that $D(f(s, t), f(s_0, t_0)) < \varepsilon$ whenever $((s - s_0)^2 + (t - t_0)^2)^{\frac{1}{2}} < \delta$. If f be continuous for each $(s, t) \in A$, then we say that f is continuous on A .

Define the metric $D^*(f, g) = \max_{(s, t) \in A} D(f(s, t), g(s, t))$ on the set

$$C(A, E^1) = \{f : A \rightarrow E^1; \ f \text{ is continuous}\}.$$

We see that $(C(A, E^1), D^*)$ is a complete metric space.

In [23], the notion of Henstock integral for fuzzy-number-valued functions is defined as follows.

Let $f : A \rightarrow E^1$, for $\Delta_x^n : a = x_0 < x_1 < \dots < x_n = b$, $\Delta_y^n : c = y_0 < y_1 < \dots < y_n = d$, be two partitions of the intervals $[a, b]$ and $[c, d]$, respectively. Let one consider the intermediate points $\xi_i \in [x_{i-1}, x_i]$ and $\eta_j \in [y_{j-1}, y_j]$, $i = 1, \dots, n$; $j = 1, \dots, n$, and $\delta : [a, b] \rightarrow \mathbf{R}_+$ and $\sigma : [c, d] \rightarrow \mathbf{R}_+$. The divisions $P_x = ([x_{i-1}, x_i]; \xi_i)$, $i = 1, \dots, n$ and $P_y = ([y_{j-1}, y_j]; \eta_j)$, $j = 1, \dots, n$ are said to be δ -fine and σ -fine, respectively, if $[x_{i-1}, x_i] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$ and $[y_{j-1}, y_j] \subseteq (\eta_j - \sigma(\eta_j), \eta_j + \sigma(\eta_j))$.

Definition 4. [23] The function f is said to be two-dimensional Henstock integrable to $I \in E^1$ if for every $\varepsilon > 0$ there are functions $\delta : [a, b] \rightarrow \mathbf{R}_+$ and $\sigma : [c, d] \rightarrow \mathbf{R}_+$ such that for any δ -fine and σ -fine divisions we have

$$D \left(\sum_{j=1}^n \sum_{i=1}^n (x_i - x_{i-1})(y_j - y_{j-1}) \odot f(\xi_i, \eta_j), I \right) < \varepsilon,$$

where $\sum$ denotes the fuzzy summation.

Then, I is called the two-dimensional Henstock integral of f and is denoted by

$$I(f) = (FH) \int_c^d (FH) \int_a^b f(s, t) ds dt.$$

If the above δ and σ are constant functions, then one recaptures the concept of Riemann integral. In this case, $I \in E^1$ will be called two-dimensional integral of f on A and will be denoted by $(FR) \int_c^d (FR) \int_a^b f(s, t) ds dt$.

Lemma 1. [23] Let $f : A \rightarrow E^1$, then f is (FH) -integrable if and only if $\underline{f}$ and $\bar{f}$ are Henstock integrable for any $r \in [0, 1]$. Moreover,

$$\left[(FH) \int_c^d (FH) \int_a^b f(s, t) ds dt \right]^r = \left[(H) \int_c^d (H) \int_a^b \underline{f}(s, t, r) ds dt, (H) \int_c^d (H) \int_a^b \bar{f}(s, t, r) ds dt \right].$$

Also, if f is continuous, then $\underline{f}(\dots, r)$ and $\bar{f}(\dots, r)$ are continuous for any $r \in [0, 1]$, and consequently, they are Henstock integrable.

3 Basic idea of HAM

According to [6], we will give a brief overview of the main used method. HAM transforms the considered equation into the corresponding deformation equation. Using this method we solve the operator equation

$$N(u(z)) = 0, \quad z \in \Omega, \quad (1)$$

where N is the nonlinear operator, u is the unknown function and Ω is the domain of z . Define the homotopy operator $\mathcal{H}$ in the following way:

$$\mathcal{H}(\Phi(z; p)) \equiv (1 - p)L(\Phi(z; p) - u_0(z)) - pHN(\Phi(z; p)), \quad (2)$$

where $p \in [0, 1]$ is an embedding parameter, $h \neq 0$ is the convergence control parameter [24, 25] and u_0 is the initial approximation of the solution of equation (1). The linear operator L is the auxiliary with property $L(0) = 0$. This operator can be arbitrarily selected. The practice is to choose L so that the equations, obtained in the next stages of the procedure, would be as simple to solve as possible.

From the equation $\mathcal{H}(\Phi(z; p)) = 0$, we get the so-called zero-order deformation equation:

$$(1 - p)L(\Phi(z; p) - u_0(z)) = pHN(\Phi(z; p)). \quad (3)$$

For $p = 0$ we have $\Phi(z;0) = u_0(z)$ and for $p = 1$ we get the searched solution of equation (1). By expanding function $\Phi(z;p) : \Omega \times [0, 1] \rightarrow \mathbf{R}$ into the Maclaurin series with respect to parameter p we have

$$\Phi(z;p) = u_0(z) + \sum_{m=1}^{\infty} u_m(z)p^m, \quad (4)$$

where $u_0(z) = \Phi(z;0)$ and

$$u_m(z) = \frac{1}{m!} \frac{\partial^m \Phi(z;p)}{\partial p^m} \bigg|_{p=0}, \quad m = 1, 2, 3, \dots \quad (5)$$

If series (4) converges for $p = 1$, then we get the sought solution

$$u(z) = \sum_{m=0}^{\infty} u_m(z). \quad (6)$$

In order to determine the function u_m we differentiate m -times, with respect to parameter p , the left and right-hand side of formula (3), then the obtained result is divided by $m!$ and substituted with $p = 0$, which gives the so-called m th-order deformation equation ($m \geq 1$)

$$L(u_m(z) - \chi_m u_{m-1}(z)) = h R_m(\tilde{u}_{m-1}(z)), \quad (7)$$

where $\tilde{u}_{m-1}(z) = \{u_0(z), u_1(z), \dots, u_{m-1}(z)\}$.

Applying L^{-1} on both sides of equation (7), we get

$$u_m(z) - \chi_m u_{m-1}(z) = h L^{-1}(R_m(\tilde{u}_{m-1}(z))),$$

where

$$\chi_m = \begin{cases} 0, & m = 1, \\ 1, & m \geq 2, \end{cases} \quad (8)$$

and

$$R_m(\tilde{u}_{m-1}(z)) = \frac{1}{(m-1)!} \left(\frac{\partial^{m-1} N[\Phi(z;p)]}{\partial p^{m-1}} \right) \bigg|_{p=0}. \quad (9)$$

If we are not able to determine the sum of series in (6), then as the approximate solution of considered equation we accept the partial sum of this series

$$S_n(z) = \sum_{m=0}^n u_m(z). \quad (10)$$

By appropriate selection of the convergence control parameter h , we can influence the convergence region of series and the rate of this convergence [15,26]. One of the methods for selecting the value of convergence control parameter is the so-called h -curve. To obtain this curve we need to investigate the behavior of a certain quantity of the exact solution as a function of parameter h [27]. Another method is the so-called “optimization method” proposed in the study by Liao [15]. In this method, we define the squared residual of the governing equation:

$$E_n(h) = \int_{\Omega} (N(s_n(z)))^2 dz. \quad (11)$$

The optimum value of the convergence control parameter is obtained by finding the minimum of this squared residual, whereas the effective region of the convergence control parameter h is defined as

$$R_h = \{h : \lim_{n \rightarrow \infty} E_n(h) = 0\}. \quad (12)$$

To speed up the calculations, Liao [15] suggested to replace the integral in formula 11 by its approximate value obtained by applying the quadrature rules. By choosing a different value of the control parameter h

than the optimal one, but still belonging to the effective region, we also obtain the convergent series, only the rate of convergence is lower. A version of the method with the above described selection of optimal value of the convergence control parameter is called the basic optimal HAM [15].

4 Using of HAM

We consider 2D-NFVIE of the following form:

$$u(s, t) = g(s, t) \oplus (FR) \int_c^t (FR) \int_a^s k(s, t, x, y) \odot G(u(x, y)) dx dy, \quad (13)$$

where $g, u : A = [a, b] \times [c, d] \rightarrow E^1$ are continuous fuzzy-number valued functions, $k : A \times A \rightarrow R_+$ is continuous function and $G : E^1 \rightarrow E^1$ is continuous function on E^1 .

According to [2], we introduce the parametric form of the integral equation (13). Let $u(s, t, r) = (\underline{u}(s, t, r), \bar{u}(s, t, r))$, $g(s, t, r) = (\underline{g}(s, t, r), \bar{g}(s, t, r))$, $0 \leq r \leq 1$ and $(s, t) \in A$ be the parametric form of the functions $u(s, t)$ and $g(s, t)$.

So, the parametric form of equation (13) is as follows:

$$\begin{aligned} \bar{u}(s, t, r) &= \bar{g}(s, t, r) + \int_c^t \int_a^s \overline{k(s, t, x, y) G(u(x, y, r))} dx dy, \\ \underline{u}(s, t, r) &= \underline{g}(s, t, r) + \int_c^t \int_a^s \underline{k(s, t, x, y) G(u(x, y, r))} dx dy. \end{aligned}$$

Let for $(x, y) \in A$, we have

$$\begin{aligned} H(\underline{u}(x, y, r), \bar{u}(x, y, r)) &= \min\{G(\beta) : \underline{u}(x, y, r) \leq \beta \leq \bar{u}(x, y, r)\}, \\ F(\underline{u}(x, y, r), \bar{u}(x, y, r)) &= \max\{G(\beta) : \underline{u}(x, y, r) \leq \beta \leq \bar{u}(x, y, r)\}. \end{aligned}$$

Then,

$$\begin{aligned} \overline{k(s, t, x, y) G(u(x, y, r))} &= \begin{cases} k(s, t, x, y) F(\underline{u}(x, y, r), \bar{u}(x, y, r)), & \text{if } k(s, t, x, y) \geq 0, \\ k(s, t, x, y) H(\underline{u}(x, y, r), \bar{u}(x, y, r)), & \text{if } k(s, t, x, y) < 0, \end{cases} \\ \underline{k(s, t, x, y) G(u(x, y, r))} &= \begin{cases} k(s, t, x, y) H(\underline{u}(x, y, r), \bar{u}(x, y, r)), & \text{if } k(s, t, x, y) \geq 0, \\ k(s, t, x, y) F(\underline{u}(x, y, r), \bar{u}(x, y, r)), & \text{if } k(s, t, x, y) < 0, \end{cases} \end{aligned}$$

for $a \leq x \leq s \leq b$, $c \leq y \leq t \leq d$ and $0 \leq r \leq 1$.

Let the function $G(\beta)$ be increasing for $\beta \in [\underline{u}(x, y, r), \bar{u}(x, y, r)]$ and $k(s, t, x, y) \geq 0$ for all $a \leq x \leq s \leq b$, $c \leq y \leq t \leq d$ and $0 \leq r \leq 1$. Then the parametric form of equation (13) is as follows:

$$\underline{u}(s, t, r) = \underline{g}(s, t, r) + \int_c^t \int_a^s k(s, t, x, y) G(\underline{u}(x, y, r)) dx dy, \quad (14)$$

$$\bar{u}(s, t, r) = \bar{g}(s, t, r) + \int_c^t \int_a^s k(s, t, x, y) G(\bar{u}(x, y, r)) dx dy. \quad (15)$$

We consider the operators L and N can be defined in the following way:

$$\begin{aligned} L(\underline{u}(s, t, r)) &= \underline{u}(s, t, r), \quad L(\bar{u}(s, t, r)) = \bar{u}(s, t, r), \\ N(\underline{u}(s, t, r)) &= \underline{u}(s, t, r) - \underline{g}(s, t, r) - \int_c^t \int_a^s k(s, t, x, y) G(\underline{u}(x, y, r)) dx dy, \end{aligned}$$

$$N(\bar{u}(s, t, r)) = \bar{u}(s, t, r) - \bar{g}(s, t, r) - \int_c^t \int_a^s k(s, t, x, y) G(\bar{u}(x, y, r)) dx dy.$$

Then we get the following formula for the function $\underline{u}_m(s, t, r)$

$$\underline{u}_m(s, t, r) = \chi_m \underline{u}_{m-1}(s, t, r) + h R_m(\tilde{\underline{u}}_{m-1}(s, t, r)), \quad (16)$$

where χ_m is defined by (8).

For the operator R_m , $m \geq 1$ from (9) we obtain

$$\begin{aligned} R_m(\tilde{\underline{u}}_{m-1}(s, t, r)) &= \frac{1}{(m-1)!} \left(\frac{\partial^{m-1}}{\partial p^{m-1}} N \left(\sum_{i=0}^{\infty} \underline{u}_i(s, t, r) p^i \right) \right) \Bigg|_{p=0} \\ &= \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \left(\sum_{i=0}^{\infty} \underline{u}_i(s, t, r) p^i - \bar{g}(s, t, r) \right) \Bigg|_{p=0} \\ &\quad - \frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} \left(\int_c^t \int_a^s k(s, t, x, y) G \left(\sum_{i=0}^{\infty} \underline{u}_i(x, y, r) p^i \right) dx dy \right) \Bigg|_{p=0} \\ &= \underline{u}_{m-1}(s, t, r) - \frac{1 - \chi_m}{(m-1)!} \bar{g}(s, t, r) \\ &\quad - \frac{1}{(m-1)!} \int_c^t \int_a^s k(s, t, x, y) \left(\frac{\partial^{m-1}}{\partial p^{m-1}} G \left(\sum_{i=0}^{\infty} \underline{u}_i(x, y, r) p^i \right) \right) dx dy \Bigg|_{p=0}. \end{aligned}$$

By using (16) and the definitions of appropriate operators for $m \geq 1$ we obtain

$$\underline{u}_1(s, t, r) = h(\underline{u}_0(s, t, r) - \bar{g}(s, t, r) - \int_c^t \int_a^s k(s, t, x, y) G(\underline{u}_0(x, y, r)) dx dy), \quad (17)$$

where $\underline{u}_0 \in C(A \times [0, 1], \mathbf{R})$ and for $m \geq 2$

$$\underline{u}_m(s, t, r) = (1 + h) \underline{u}_{m-1}(s, t, r) - \frac{h}{(m-1)!} \int_c^t \int_a^s k(s, t, x, y) \left(\frac{\partial^{m-1}}{\partial p^{m-1}} G \left(\sum_{i=0}^{\infty} \underline{u}_i(x, y, r) p^i \right) \right) dx dy. \quad (18)$$

5 Existence and convergence

In this section, we prove convergence of the presented method and obtain the error estimation between the exact and the approximate solution.

Lemma 2. [28] *Let the functions $g \in C(A, E^1)$ and $h \in C(A, R_+)$. Then the function $h.g : A \rightarrow E^1$ given by $(h.g)(s, t) = h(s, t) \odot g(s, t)$ is continuous on A .*

Lemma 3. [29] *Let the functions $k \in C(A \times A, R_+)$, $u \in C(A, E^1)$ and $G \in C(E^1, E^1)$. Then the function $F_u : A \rightarrow E^1$ defined by*

$$F_u(s, t) = (FR) \int_c^t (FR) \int_a^s k(s, t, x, y) \odot G(u(x, y)) dx dy$$

is continuous on A .

We introduce the following conditions:

- (i) $g \in C(A, E^1)$, $k \in C(A \times A, \mathbf{R}_+)$ and $G \in C(E^1, E^1)$;
- (ii) there exists $L_G \geq 0$ such that $D(G(u), G(v)) \leq L_G D(u, v)$ for all $u, v \in E^1$;
- (iii) $\alpha = M_k L_G \Delta < 1$, where $|k(s, t, x, y)| \leq M_k$ for all $(s, t), (x, y) \in A$, according to the continuity of k and $\Delta = (b - a)(d - c)$.

Theorem 1. *Let conditions (i)–(iii) be fulfilled. Then the integral equation (13) has a unique solution.*

Proof. Let $\mathfrak{F}(A, E^1) = \{f : A \rightarrow E^1\}$ and $X = C(A, E^1)$.

We define the operator $T : X \rightarrow \mathfrak{F}(A, E^1)$ by

$$T(u)(s, t) = g(s, t) \oplus (FR) \int_c^t (FR) \int_a^s k(s, t, x, y) \odot G(u(x, y)) dx dy, \quad (s, t) \in A, \quad u \in X.$$

First, we prove that $T(X) \subset X$. To this purpose, we show that the operator T is uniformly continuous. Let arbitrary $u \in X$, $(t_0, s_0) \in A$ and $\varepsilon > 0$. Since u is continuous it follows that for $\frac{\varepsilon}{L_G} > 0$ there exists $\delta(\varepsilon) > 0$ such that $D(u(s, t), u(s_0, t_0)) \leq \frac{\varepsilon}{L_G}$ for any $(s, t) \in A$ with $|s - s_0| + |t - t_0| < \delta(\varepsilon)$.

From condition (ii), we have

$$D(G(u(s, t)), G(u(s_0, t_0))) \leq L_G D(u(s, t), u(s_0, t_0)) \leq \varepsilon$$

and the function $G_u : A \rightarrow E^1$, defined by $G_u(s, t) = G(u(s, t))$, is continuous in (s_0, t_0) . We conclude that G_u is continuous on A for any $u \in X$. From Lemma 2, it follows that the function $k(s, t, \cdot, \cdot) \odot G_u(\cdot, \cdot) : A \rightarrow E^1$ is continuous on A for any $u \in X$, $(s, t) \in A$. Applying Lemma 3, it follows that the function $F_u : A \rightarrow E^1$, defined by

$$F_u(s, t) = (FR) \int_c^t (FR) \int_a^s k(s, t, x, y) \odot G_u(x, y) dx dy$$

is continuous on A for any $u \in X$. Since $g \in X$, we conclude that the operator $T(u)$ is continuous on A for any $u \in X$.

Now, we prove that $T : X \rightarrow X$ is a contraction. Let arbitrary $u, v \in X$. From conditions (ii) and (iii) we have

$$\begin{aligned} & D(A(u)(s, t), A(v)(s, t)) \\ & \leq D \left((FR) \int_c^t (FR) \int_a^s k(s, t, x, y) \odot G(u(x, y)) dx dy, (FR) \int_c^t (FR) \int_a^s k(s, t, x, y) \odot G(v(x, y)) dx dy \right) \\ & \leq \int_c^t \int_a^s |k(s, t, x, y)| D(G(u(x, y)), G(v(x, y))) dx dy \\ & \leq M_k L \int_c^d \int_a^b D(u(x, y), v(x, y)) dx dy \leq M_k L_G \Delta D^*(u, v) \quad \text{for all } (s, t) \in A. \end{aligned}$$

Under the condition $\alpha < 1$ the operator T is contraction; therefore, by the Banach fixed-point theorem for contraction, there exists a unique solution to problem (13) and this completes the proof. $\square$

Remark 1. From Theorem 1, it follows that equations (14) and (15) have unique solutions.

Theorem 2. Let the following conditions be fulfilled

- (i) conditions (i)–(iii) hold;
- (ii) the functions $\underline{u}_m(s, t, r)$, $m \geq 1$, are defined by (17) and (18);
- (iii) the constant $L_G < 1$.

Then the series

$$\sum_{i=0}^{\infty} \underline{u}_i(s, t, r) \quad (19)$$

is convergent, and the sum of this series is the solution of equation (14).

Proof. Let the series in (19) be convergent. Then from the necessary condition for convergence of the series, it follows

$$\lim_{m \rightarrow \infty} \underline{u}_m(s, t, r) = 0$$

for any $(s, t, r) \in A \times [0, 1]$.

Denote

$$\underline{H}_m(s, t, r) = \frac{1}{m!} \left(\frac{\partial^m}{\partial p^m} G \left(\sum_{i=0}^{\infty} \underline{u}_i(s, t, r) p^i \right) \right) \bigg|_{p=0}.$$

From [30], if G is the contraction mapping and the series in (19) converges to $\underline{u}(s, t, r)$, then the series $\sum_{m=0}^{\infty} \underline{H}_m(s, t, r)$ is convergent to $G(\underline{u}(s, t, r))$.

By using the definition of operator L we have

$$\begin{aligned} \sum_{m=1}^n L(\underline{u}_m(s, t, r) - \chi_m \underline{u}_{m-1}(s, t, r)) &= \sum_{m=1}^n (\underline{u}_m(s, t, r) - \chi_m \underline{u}_{m-1}(s, t, r)) \\ &= \underline{u}_1(s, t, r) + (\underline{u}_2(s, t, r) - \underline{u}_1(s, t, r)) + \cdots + (\underline{u}_n(s, t, r) - \underline{u}_{n-1}(s, t, r)) = \underline{u}_n(s, t, r). \end{aligned}$$

Hence,

$$\sum_{m=1}^{\infty} L(\underline{u}_m(s, t, r) - \chi_m \underline{u}_{m-1}(s, t, r)) = \lim_{n \rightarrow \infty} \underline{u}_n(s, t, r) = 0.$$

From (7), we obtain

$$h \sum_{m=1}^{\infty} R_m(\tilde{\underline{u}}_{m-1}(s, t, r)) = \sum_{m=1}^{\infty} L(\underline{u}_m(s, t, r) - \chi_m \underline{u}_{m-1}(s, t, r)).$$

Since $h \neq 0$, it follows

$$\sum_{m=1}^{\infty} R_m(\tilde{\underline{u}}_{m-1}(s, t, r)) = 0.$$

Then, we obtain

$$\begin{aligned} 0 &= \sum_{m=1}^{\infty} R_m(\tilde{\underline{u}}_{m-1}(s, t, r)) = \sum_{m=1}^{\infty} \left(\underline{u}_{m-1}(s, t, r) - \frac{1 - \chi_m}{(m-1)!} \underline{g}(s, t, r) \right. \\ &\quad \left. - \int_c^t \int_a^s k(s, t, x, y) \left(\frac{1}{(m-1)!} \frac{\partial^{m-1}}{\partial p^{m-1}} G \left(\sum_{i=0}^{\infty} \underline{u}_i(x, y, r) p^i \right) p^i \right) \bigg|_{p=0} dx dy \right) \\ &= \sum_{m=1}^{\infty} \underline{u}_{m-1}(s, t, r) - \underline{g}(s, t, r) - \int_c^t \int_a^s k(s, t, x, y) \sum_{m=1}^{\infty} \underline{H}_{m-1}(x, y, r) dx dy. \end{aligned}$$

Hence,

$$0 = \underline{u}(s, t, r) - \underline{g}(s, t, r) - \int_c^t \int_a^s k(s, t, x, y) G(\underline{u}(x, y, r)) dx dy. \quad \square$$

Theorem 3. Let conditions (i)–(iii) be fulfilled. Then the series (19) is uniformly convergent in A .

Proof. We denote with $B = (C(A, \mathbf{R}), \|\cdot\|)$ the Banach space of all continuous functions on A . Let $0 \leq r \leq 1$ and $\underline{u}_0(.,., r)$ be the function of the space B . For $n = 0, 1, 2, \dots$, we denote the sequence of partial sum of the series (19)

$$\underline{S}_n(s, t, r) = \sum_{i=0}^n \underline{u}_i(s, t, r),$$

where $(s, t) \in A, r \in [0, 1]$. We will prove that the sequence $\{\underline{S}_n\}$ is a Cauchy sequence in B .

Then from condition (iii) we have

$$\begin{aligned} \|S_2 - S_1\| = \|\underline{u}_1\| &\leq |h| \left(\max_{(s,t) \in A} |\underline{u}_0(s, t, r)| + \max_{(s,t) \in A} |\underline{g}(s, t, r)| + \int_c^t \int_a^s |k(s, t, x, y)| \max_{(s,t) \in A} |G(\underline{u}_0(x, y, r))| dx dy \right) \\ &\leq |h| \left(\|\underline{u}_0\| + \|\underline{g}\| + \int_c^t \int_a^s M_k \max_{(s,t) \in A} |G(\underline{u}_0(x, y, r))| dx dy \right) \\ &\leq |h| (\|\underline{u}_0\| + \|\underline{g}\| + M_k \Delta \phi), \end{aligned}$$

where $\phi = \max_{(s,t) \in A} |G(\underline{u}_0(s, t, r))|$.

Let $m \geq 1$ and $n > m$. Then

$$\begin{aligned} \|\underline{S}_n - \underline{S}_m\| &= \max_{(s,t) \in A} |\underline{S}_n(s, t, r) - \underline{S}_m(s, t, r)| \\ &= \max_{(s,t) \in A} \left| \sum_{i=0}^n \underline{u}_i(s, t, r) - \sum_{i=0}^m \underline{u}_i(s, t, r) \right| \\ &= \max_{(s,t) \in A} \left| \sum_{i=m+1}^n \underline{u}_i(s, t, r) \right| \\ &= \max_{(s,t) \in A} \left| \sum_{i=m}^{n-1} \underline{u}_{i+1}(s, t, r) \right| \\ &= \max_{(s,t) \in A} \left| (1+h) \sum_{i=m}^{n-1} \underline{u}_i(s, t, r) - h \left(\sum_{i=m}^{n-1} \left(\int_c^t \int_a^s k(s, t, x, y) \underline{H}_i dx dy \right) \right) \right| \\ &= \max_{(s,t) \in A} \left| (1+h) \sum_{i=m}^{n-1} \underline{u}_i(s, t, r) - h \left(\int_c^t \int_a^s k(s, t, x, y) \sum_{i=m}^{n-1} \underline{H}_i dx dy \right) \right|. \end{aligned}$$

From [31], we have

$$\sum_{i=m}^{n-1} \underline{H}_i = G(\underline{S}_{n-1}) - G(\underline{S}_{m-1}).$$

Hence, from conditions (ii) and (iii) we obtain

$$\begin{aligned}\|\underline{S}_n - \underline{S}_m\| &\leq |1 + h|\|\underline{S}_{n-1} - \underline{S}_{m-1}\| + |h| \max_{(s,t) \in A} \left| \int_c^t \int_a^s |k(s, t, x, y)| |G(\underline{S}_{n-1}) - G(\underline{S}_{m-1})| dx dy \right| \\ &\leq |1 + h|\|\underline{S}_{n-1} - \underline{S}_{m-1}\| + |h| M_k L_G \Delta \|\underline{S}_{n-1} - \underline{S}_{m-1}\| \\ &= (|1 + h| + |h|\alpha) \|\underline{S}_{n-1} - \underline{S}_{m-1}\| = \beta_h \|\underline{S}_{n-1} - \underline{S}_{m-1}\|,\end{aligned}$$

where $\beta_h = |1 + h| + |h|\alpha$.

Let $n = m + 1$. Then

$$\|\underline{S}_{m+1} - \underline{S}_m\| \leq \beta_h \|\underline{S}_m - \underline{S}_{m-1}\| \leq \dots \leq \beta_h^m \|\underline{S}_1 - \underline{S}_0\| = \beta_h^m \|\underline{u}_1\|.$$

Using the triangle inequality we have,

$$\begin{aligned}\|\underline{S}_n - \underline{S}_m\| &\leq \|\underline{S}_{m+1} - \underline{S}_m\| + \|\underline{S}_{m+2} - \underline{S}_{m+1}\| + \dots + \|\underline{S}_n - \underline{S}_{n-1}\| \\ &\leq (\beta_h^m + \beta_h^{m+1} + \dots + \beta_h^{n-1}) \|\underline{u}_1\| \\ &\leq \beta_h^m (1 + \beta_h + \beta_h^2 + \dots + \beta_h^{n-m-1}) \|\underline{u}_1\| \\ &\leq \beta_h^m \frac{1 - \beta_h^{n-m}}{1 - \beta_h} \|\underline{u}_1\|.\end{aligned}$$

We choose the parameter h so that $\beta_h < 1$. This inequality is equivalent to the following $|1 + h| + |h|\alpha < 1$. Since $|h| \neq 0$ the last inequality is equivalent to

$$\alpha < \frac{1 - |1 + h|}{|h|}. \quad (20)$$

One can easily note that

$$\frac{1 - |1 + h|}{|h|} = \begin{cases} -1 - \frac{2}{h} & \text{for } h < -1, \\ 1 & \text{for } h \in [-1, 0), \\ -1 & \text{for } h > 0. \end{cases}$$

It implies that if condition $\alpha < 1$ is fulfilled, then we are able to choose the value of parameter h such that inequality (20) will be satisfied (for this aim it is enough to take any $h \in [-1, 0)$), which means that $\beta_h < 1$.

Since $0 < \beta_h < 1$ so, $1 - \beta_h^{n-m} \leq 1$, then

$$\|\underline{S}_n - \underline{S}_m\| \leq \frac{\beta_h^m}{1 - \beta_h} \|\underline{u}_1\|.$$

Since $\|\underline{u}_1\| < \infty$, then $\|\underline{S}_n - \underline{S}_m\| \rightarrow 0$ as $m \rightarrow \infty$, from which we conclude that $\{\underline{S}_n\}$ is a Cauchy sequence in B . Therefore, the series $\sum_{i=1}^{\infty} \underline{u}_i(s, t, r)$ converges. Similarly, we have $\{\bar{S}_n\}$ is a Cauchy sequence. $\square$

Theorem 4. Let conditions (i)–(iii) be fulfilled. Then we get the following estimation of error of the approximate solution:

$$\|\underline{u} - \underline{S}_n\| \leq \frac{\beta_h^n}{1 - \beta_h} |h| (\|\underline{u}_0\| + \|\underline{g}\| + M_k \Delta \phi),$$

where $\beta_h = |1 + h| + |h|\alpha$ and α is from condition (iii).

6 Numerical results

In this section, we give a numerical example to illustrate the obtained theoretical results.

Example. Let $A = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$. We consider the following two-dimensional nonlinear Volterra fuzzy integral equation:

$$u(s, t) = g(s, t) \oplus (FR) \int_c^t (FR) \int_a^s (tx + sy) \odot u^2(x, y) dx dy,$$

where $g(s, t, r) = (st(r+1) - \frac{1}{6}s^4t^4(r+1)^2, st(3-r) - \frac{1}{6}s^4t^4(3-r)^2)$ and the exact solution is $u_{\text{exact}}(s, t, r) = (st(r+1), st(3-r))$.

The parametric form of the equation is

$$\begin{aligned} \underline{u}(s, t, r) &= \left(st - \frac{1}{6}s^3t^3\right)(r+1) + \int_0^t \int_0^s (tx + sy) \underline{u}^2(x, y, r) dx dy, \\ \bar{u}(s, t, r) &= \left(st - \frac{1}{6}s^4t^4\right)(3-r) + \int_0^t \int_0^s (tx + sy) \bar{u}^2(x, y, r) dx dy. \end{aligned}$$

In this case, we have $M_k = \frac{1}{2}$ and $\alpha = \frac{1}{2}$. We choose the value of parameter h , so that $\beta_h < 1$. Hence, $h \in \left(-\frac{4}{3}, 0\right)$. Numerically determined, the optimal value of convergence control parameter is equal to -0.9956 .

By using HAM and the CAS “Wolfram Mathematica” and the proposed above method, we obtain

$$\begin{aligned} \underline{u}_0(s, t, r) &= \underline{g}(s, t, r) = st(r+1) - \frac{1}{6}s^4t^4(r+1)^2, \\ \underline{u}_1(s, t, r) &= h \left(\underline{u}_0(s, t, r) - \underline{g}(s, t, r) - \int_c^t \int_a^s (tx + sy) \underline{u}_0^2(x, y, r) dx dy \right), \\ \underline{u}_1(s, t, r) &= -\frac{h}{6} \left(s^4t^4(r+1)^2 - \frac{1}{7}s^7t^7(r+1)^3 + \frac{1}{270}s^{10}t^{10}(r+1)^4 \right), \\ \underline{u}_2(s, t, r) &= (1+h)\underline{u}_1(s, t, r) - h \int_c^t \int_a^s (tx + sy) 2\underline{u}_0(x, y, r) \underline{u}_1(x, y, r) dx dy, \\ \underline{u}_2(s, t, r) &= -\frac{(1+h)h}{2} \left(\frac{1}{3}s^4t^4(r+1)^2 - \frac{1}{7}s^7t^7(r-3)^3 + \frac{13}{21 \cdot 10 \cdot 9}s^{10}t^{10}(r+1)^4 \right) \\ &\quad + \frac{h^2}{15 \cdot 81} \left(\frac{1}{7}s^{13}t^{13}(r+1)^5 - \frac{1}{3 \cdot 10 \cdot 16}s^{16}t^{16}(r+1)^6 \right). \end{aligned}$$

Then the second partial sum is

$$\begin{aligned} \underline{S}_2(s, t, r) &= \underline{u}_0(s, t, r) + \underline{u}_1(s, t, r) + \underline{u}_2(s, t, r) \\ &= st(r+1) - \frac{1}{6}s^4t^4(r+1)^2 - \frac{h}{6} \left(s^4t^4(r+1)^2 - \frac{1}{7}s^7t^7(r+1)^3 + \frac{1}{270}s^{10}t^{10}(r+1)^4 \right) \\ &\quad - \frac{(1+h)h}{2} \left(\frac{1}{3}s^4t^4(r+1)^2 - \frac{1}{7}s^7t^7(r-3)^3 + \frac{13}{21 \cdot 10 \cdot 9}s^{10}t^{10}(r+1)^4 \right) \\ &\quad + \frac{h^2}{15 \cdot 81} \left(\frac{1}{7}s^{13}t^{13}(r+1)^5 - \frac{1}{3 \cdot 10 \cdot 16}s^{16}t^{16}(r+1)^6 \right). \end{aligned}$$

We show the error $\underline{\Delta}_2(s, t, 0.6) = |\underline{u}_{\text{exact}}(s, t, 0.6) - \underline{S}_2(s, t, 0.6)|$. The results are expressed in Table 1.

Table 1: Values of errors

s	h = -0.9956					h = -1.2				
	t = 0.2	0.3	0.4	0.5		t = 0.2	0.3	0.4	0.5	
0.1	1.329×10^{-12}	6.830×10^{-12}	2.219×10^{-11}	5.662×10^{-11}		2.730×10^{-9}	1.382×10^{-8}	4.368×10^{-8}	1.066×10^{-7}	
0.2	5.285×10^{-9}	2.676×10^{-8}	8.465×10^{-8}	2.069×10^{-7}		2.446×10^{-7}	1.238×10^{-6}	3.914×10^{-6}	9.556×10^{-6}	
0.3	2.676×10^{-8}	1.356×10^{-7}	4.298×10^{-7}	1.053×10^{-6}		1.238×10^{-6}	6.270×10^{-6}	1.981×10^{-5}	4.837×10^{-5}	
0.4	8.465×10^{-8}	4.298×10^{-7}	1.366×10^{-6}	3.367×10^{-6}		3.914×10^{-6}	1.981×10^{-5}	6.261×10^{-5}	1.528×10^{-4}	
s	h = -0.9					h = -0.7				
	t = 0.2	0.3	0.4	0.5		t = 0.2	0.3	0.4	0.5	
0.1	6.827×10^{-10}	3.456×10^{-9}	1.092×10^{-8}	2.669×10^{-8}		6.144×10^{-9}	3.110×10^{-8}	9.831×10^{-8}	2.401×10^{-7}	
0.2	1.190×10^{-7}	6.027×10^{-7}	1.905×10^{-6}	4.651×10^{-6}		3.506×10^{-7}	1.775×10^{-6}	5.609×10^{-6}	1.369×10^{-5}	
0.3	6.027×10^{-7}	3.051×10^{-6}	9.645×10^{-6}	2.355×10^{-5}		1.775×10^{-6}	8.986×10^{-6}	2.840×10^{-5}	6.934×10^{-5}	
0.4	1.905×10^{-6}	9.645×10^{-6}	3.049×10^{-5}	7.446×10^{-5}		5.609×10^{-6}	2.840×10^{-5}	8.976×10^{-5}	2.289×10^{-4}	

7 Conclusion

In this paper, HAM is used for solving the nonlinear fuzzy Volterra integral equations in two variables of the second kind. The solution of the discussed equation is obtained in the form of a series, the elements of which are iteratively determined. It is shown that if this series is convergent, its sum is the solution of the considered equation. The sufficient conditions for convergence of this series are received and then its sum is solution of the considered equation. The error of approximate solution, taken as the partial sum of generated series, is estimated. Example illustrating the use of the investigated method are presented as well.

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