Research Article

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On q -analogue of Janowski-type starlike functions with respect to symmetric points

<https://doi.org/10.1515/dema-2021-0008>

received October 18, 2020; accepted March 15, 2021

Abstract: The main objective of the present paper is to define a class of q -starlike functions with respect to symmetric points in circular domain. Some interesting results of these functions have been evaluated in this article. The sufficiency criteria in the form of convolutions are evaluated. Furthermore, other geometric properties such as coefficient bounds, distortion theorem, closure theorem and extreme point theorem are also obtained for these newly defined functions.

Keywords: holomorphic functions, Janowski functions, starlike functions, subordinations

MSC 2020: 30C45, 30C50

1 Introduction and preliminaries

The quantum calculus, or simply the q -calculus, is the generalization of classical calculus where the limit has been replaced by the parameter q . Jackson should be applauded for the systematic initiation of quantum calculus in his work in [1,2], by introducing the q -derivative and q -integral. The early work of q -calculus in the field of Geometric Function Theory was done by Ismail et al., see [3], by generalizing the class of starlike functions into q -analogue, known as the class of q -starlike functions. Another important development in this direction was the work of Anastassiou and Gal [4,5], who gave the q -generalizations of some complex operators, e.g., Picard and Gauss-Weierstrass singular integral operators. Following the same idea, Srivastava [6,7] gave a strong footing by giving some applications of q -calculus in this field by using q -analogue of hypergeometric functions. Other sets of valuable work were carried out by Aral and Gupta [8–10], by using the q -analogue of Baskakov-Durrmeyer operator. Their derivation of various geometric results along with their q -analogue is of most importance and worth mentioning. Some what along the same direction, various operators related to differentiation and integration have been generalized into q -calculus and with its help various classes of analytic functions have been introduced and studied. Using these ideas of q -calculus, Darus and Aldweby in [11,12] defined, with the help of convolution, the q -operators for analytic functions in their normalized form. Furthermore, they described the geometric structure of

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it for analytic functions using q -analogue of hypergeometric functions in the domain of compact disc. See more related work involving q -derivative [13–19].

Before we introduce and discuss our work, we start with some preliminaries which will be very helpful in understanding our main work.

In this article, $\mathcal{A}$ will represent the class of holomorphic (analytic) functions f in $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$, known as the open unit disc, which are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathcal{A}). \quad (1)$$

The family denoted by $\mathcal{S}$ are the family of functions which are in $\mathcal{A}$ and which are also univalent.

The well-known class of starlike functions denoted by $\mathcal{S}^*$ are those functions f in $\mathcal{A}$, which are of the form (1), and satisfy the conditions

$$f \in \mathcal{S} \quad \text{and} \quad \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > 0, \quad (z \in \mathcal{A}).$$

If f and g are two holomorphic functions in $\mathcal{A}$, then the function f is subordinate to the function g and is written mathematically as

$$f \prec g \quad \text{or} \quad f(z) \prec g(z),$$

if there occurs an holomorphic function v in $\mathcal{A}$, with conditions

$$v(0) = 0 \quad \text{and} \quad |v(z)| < 1,$$

implies that

$$f(z) = g(w(z)).$$

In addition, if g satisfies the condition of univalence in $\mathcal{A}$, then from [20,21], we have the relation

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathcal{A}) \subset g(\mathcal{A}).$$

The convolution of two functions f and g that are holomorphic in $\mathcal{A}$ and have the form (1), is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \mathcal{A}).$$

Sakaguchi [22], in year 1956, defined the class of starlike functions w.r.t. symmetrical points denoted by $\mathcal{S}^{**}$ of holomorphic univalent functions f in $\mathcal{A}$ if the following condition is satisfied:

$$\operatorname{Re} \frac{2zf'(z)}{f(z) - f(-z)} > 0, \quad (z \in \mathcal{A}). \quad (2)$$

The q -derivative (see [1,2]) is established for a function f as

$$d_q f(z) = \frac{f(z) - f(qz)}{z(1-q)}, \quad (z \neq 0, \quad 0 < q < 1). \quad (3)$$

With simple calculations and $n \in \mathbb{N} := \{1, 2, 3, \dots\}$ with $z \in \mathcal{A}$

$$d_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] a_n z^{n-1}, \quad (4)$$

and

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^{n-1} q^l, \quad [0, q] = 0.$$

Motivated by the work discussed at the beginning and [22], we now establish a class $S_q^{**}[A, B]$ of $\mathcal{A}$ as follows:

Let $-1 \leq B < A \leq 1$ and $q \in]0, 1[$. Then the function $f \in \mathcal{A}$ is in the class $S_q^{**}[A, B]$ if it satisfies

$$\frac{2zd_q f(z)}{f(z) - f(-z)} < \frac{1 + Az}{1 + Bz}, \quad (z \in \mathcal{A}), \quad (5)$$

where the notion “ $<$ ” denotes subordinations.

Or equivalently,

$$\left| \frac{\frac{2zd_q f(z)}{f(z) - f(-z)} - 1}{B \frac{2zd_q f(z)}{f(z) - f(-z)} - A} \right| < 1, \quad (z \in \mathcal{A}). \quad (6)$$

We see that for particular values of A, B and q we get some of the well-known classes, of which some are listed as follows:

1. For $q \rightarrow 1^-$ we get the class of starlike functions with respect to symmetric point associated with Janowski functions denoted by $S^{**}[A, B]$.
2. For $A = 1, B = -1$ and $q \rightarrow 1^-$ we get S^{**} , the class of starlike functions with respect to symmetric point.

1.1 Lemma

[23] Let $h(z)$ be holomorphic in $\mathbb{D}$ and has the form

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

and $k(z)$ is holomorphic and convex in $\mathbb{D}$ with series representation

$$k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n.$$

So if $h(z) < k(z)$, then $|c_n| \leq |k_n|$, for $n \in \mathbb{N} = \{1, 2, \dots\}$.

2 Analytic criteria

2.1 Theorem

Let $Hf(z) = f(z) - f(-z)$. If $f \in S_q^{**}(A, B)$, then $Hf \in S_q^{**}[A, B]$.

Proof. Since $f \in S_q^{**}[A, B]$ and $J(z) = \frac{1 + Az}{1 + Bz}$, then

$$\frac{2zd_q f(z)}{f(z) - f(-z)} < J(z)$$

and

$$\frac{2zd_q(-f)(z)}{f(z) - f(-z)} = \frac{2zd_q f(-z)}{f(-z) - f(z)} < J(-z) < J(z).$$

Thus, we have

$$\frac{2zd_q f(z)}{Hf(z)} \in J(\mathcal{A}) \quad \text{and} \quad \frac{2zd_q(-f)(z)}{Hf(z)} \in J(\mathcal{A}), \quad (z \in \mathcal{A}).$$

Since J is a convex function in $\mathcal{A}$, we have

$$\frac{1}{2} \frac{2zd_q f(z)}{Hf(z)} + \frac{1}{2} \frac{2zd_q(-f)(z)}{Hf(z)} = \frac{2zd_q(Hf)(z)}{Hf(z)} \in J(\mathcal{A}), \quad (z \in \mathcal{A}).$$

Or equivalently,

$$\frac{2zd_q(Hf)(z)}{Hf(z)} \in J(\mathcal{A}),$$

which implies that $Hf \in S_q^{**}[A, B]$. □

2.2 Theorem

Let $f \in \mathcal{A}$ be given by (1). Then the function $f \in S_q^{**}[A, B]$ if and only if

$$\frac{1}{z} \left[f(z) * \left(\frac{(B-A)e^{i\theta}z + (2 + (B+A)e^{i\theta})qz^2}{(1-z^2)(1-qz)} \right) \right] \neq 0, \quad (z \in \mathcal{A}). \quad (7)$$

Proof. Since $f \in S_q^{**}[A, B]$ is holomorphic in $\mathcal{A}$, so it implies that $f(z) \neq 0$ for all z in $\mathcal{A}^* = \mathcal{A} - \{0\}$ that is $\frac{1}{z}f(z) \neq 0$ for all $z \in \mathcal{A}$. From the definition of subordination, there occurs a holomorphic function $v(z)$ with $v(0) = 0$ and $|v(z)| < 1$, such that

$$\frac{2zd_q f(z)}{f(z) - f(-z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, \quad (z \in \mathcal{A}),$$

which is identical to

$$\frac{2zd_q f(z)}{f(z) - f(-z)} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}} \quad (8)$$

$$\Rightarrow zd_q f(z)(1 + Be^{i\theta}) - \frac{f(z) - f(-z)}{2}(1 + Ae^{i\theta}) \neq 0. \quad (9)$$

Now utilize the following convolution properties in (9):

$$f(z) * \frac{z}{1-z^2} = \frac{f(z) - f(-z)}{2} \quad \text{and} \quad f(z) * \frac{z}{(1-z)(1-qz)} = zd_q f(z).$$

We get

$$f(z) * \left(\frac{z(1 + Be^{i\theta})}{(1-z)(1-qz)} - \frac{z(1 + Ae^{i\theta})}{1-z^2} \right) \neq 0,$$

which gives

$$f(z) * \left(\frac{(B-A)e^{i\theta}z + (2 + (B+A)e^{i\theta})qz^2}{(1-z^2)(1-qz)} \right) \neq 0,$$

hence, the necessary condition

$$\frac{1}{z} \left[f(z) * \left(\frac{(B-A)e^{i\theta}z + (2 + (B+A)e^{i\theta})qz^2}{(1-z^2)(1-qz)} \right) \right] \neq 0.$$

Conversely, suppose that condition (7) holds, it follows that $\frac{1}{z}f(z) \neq 0$ for all $z \in \mathcal{A}$. Thus, the function $g(z) = \frac{2zd_q f(z)}{f(z) - f(-z)}$ is holomorphic in $\mathcal{A}$ and $g(0) = 1$. Hence, we have

$$\begin{aligned}
\frac{2zd_q f(z)}{f(z) - f(-z)} &= \frac{2z + 2 \sum_{n=2}^{\infty} [n, q] a_n z^n}{2z + \sum_{n=2}^{\infty} (1 - (-1)^n) a_n z^n}, \\
\left[\frac{2zd_q f(z)}{f(z) - f(-z)} \right]_{z=0} &= \left[\frac{1 + \sum_{n=2}^{\infty} [n, q] a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \frac{1 - (-1)^n}{2} a_n z^{n-1}} \right]_{z=0} = 1, \\
\frac{2zd_q f(z)}{f(z) - f(-z)} &\neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}}.
\end{aligned} \tag{10}$$

Suppose that

$$\mathcal{F}(z) = \frac{1 + Az}{1 + Bz}, \quad z \in \mathcal{A}.$$

Now from relation (10) it is clear that $\mathcal{F}(\partial\mathcal{A}) \cap g(\mathcal{A}) = \emptyset$. So, the simply connected domain $g(\mathcal{A})$ is contained in connected component of $\mathbb{C} \cap \mathcal{F}(\partial\mathcal{A})$. The univalence of “ g ” together with the fact $\mathcal{F}(0) = g(0) = 1$ shows that $g \prec \mathcal{F}$, which shows that $f \in S_q^{**}[A, B]$. $\square$

2.3 Theorem

Let $f \in \mathcal{A}$ of the form (1). Then the function $f \in S_q^{**}[A, B]$ if and only if it satisfies the condition

$$\sum_{n=2}^{\infty} \left\{ [n, q] (1 - B) - (1 + A) \frac{(1 - (-1)^n)}{2} \right\} |a_n| \leq B - A. \tag{11}$$

Proof. Let us presume that condition (11) holds. To show $f \in S_q^{**}[A, B]$, we only need to show the inequality (6). For this consider

$$\begin{aligned}
\left| \frac{\frac{2zd_q f(z)}{f(z) - f(-z)} - 1}{B \frac{2zd_q f(z)}{f(z) - f(-z)} - A} \right| &= \left| \frac{zd_q f(z) - \frac{f(z) - f(-z)}{2}}{Bzd_q f(z) - A \frac{f(z) - f(-z)}{2}} \right| = \left| \frac{\sum_{n=2}^{\infty} \left([n, q] - \frac{1 - (-1)^n}{2} \right)}{(B - A) + \sum_{n=2}^{\infty} \left(B[n, q] - A \frac{1 - (-1)^n}{2} \right)} \right| \\
&\leq \frac{\sum_{n=2}^{\infty} \left([n, q] - \frac{1 - (-1)^n}{2} \right)}{(B - A) - \sum_{n=2}^{\infty} \left(B[n, q] - A \frac{1 - (-1)^n}{2} \right)} < 1,
\end{aligned}$$

where we have used (11), which completes the direct part.

Conversely, let $f \in S_q^{**}[A, B]$ be of the form (1). Then from (6), we have

$$\left| \frac{\frac{2zd_q f(z)}{f(z) - f(-z)} - 1}{B \frac{2zd_q f(z)}{f(z) - f(-z)} - A} \right| = \left| \frac{\sum_{n=2}^{\infty} \left([n, q] - \frac{1 - (-1)^n}{2} \right)}{(B - A) + \sum_{n=2}^{\infty} \left(B[n, q] - A \frac{1 - (-1)^n}{2} \right)} \right| < 1.$$

Since $|\operatorname{Re} z| < |z|$, we have

$$\operatorname{Re} \left\{ \frac{\sum_{n=2}^{\infty} \left([n, q] - \frac{1 - (-1)^n}{2} \right)}{(B - A) + \sum_{n=2}^{\infty} \left(B[n, q] - A \frac{1 - (-1)^n}{2} \right)} \right\} < 1. \tag{12}$$

Now we choose values of z on real axis such that $\frac{2zd_q f(z)}{f(z) - f(-z)}$ is real. Upon clearing the denominator in (12) and letting $z \rightarrow 1^-$ through real values, we obtain the required inequality (11). $\square$

2.4 Theorem

For $f \in \mathcal{S}_q^{**}[A, B]$ and has series expansion (1). Then

$$|a_2| \leq \frac{A-B}{[2, q]}, \quad |a_3| \leq \frac{A-B}{[2, q]}, \quad |a_4| \leq \frac{(A-B)([2, q] + A - B)}{[2, q][4, q]}.$$

Proof. If $f \in \mathcal{A}$ is in the class $\mathcal{S}_q^{**}[A, B]$, then it satisfies

$$\frac{2zd_q f(z)}{f(z) - f(-z)} < \frac{1 + Az}{1 + Bz}.$$

Now let

$$h(z) = \frac{2zd_q f(z)}{f(z) - f(-z)}. \quad (13)$$

Since

$$\operatorname{Re} h(z) > 0,$$

so $h(z)$ is in the class P with its representation given by

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \quad (14)$$

Now as

$$h(z) < \frac{1 + Az}{1 + Bz}$$

and

$$\frac{1 + Az}{1 + Bz} = 1 + (A - B)z + \dots, \quad (15)$$

by using Lemma 1.1, we get

$$|c_n| \leq (A - B). \quad (16)$$

Thus,

$$\frac{2zd_q f(z)}{f(z) - f(-z)} = 1 + [2, q]a_2 z + [2, q]a_3 z^2 + ([4, q]a_4 - [2, q]a_2 a_3) + \dots. \quad (17)$$

Now comparing the coefficient of (17) with (14), we have

$$a_2 = \frac{c_1}{[2, q]}, \quad a_3 = \frac{c_2}{[2, q]}, \quad (18)$$

$$a_4 = \frac{1}{[4, q]}(c_3 + [2, q]a_2 a_3). \quad (19)$$

Using relations (16) to (18), we have

$$|a_2| \leq \frac{A-B}{[2, q]} \quad \text{and} \quad |a_3| \leq \frac{A-B}{[2, q]}. \quad (20)$$

Using triangular inequality to (19) and relations (16) to (19), we have

$$|a_4| \leq \frac{1}{[4, q]} \left((A - B) + [2, q] \frac{A - B}{[2, q]} \frac{A - B}{[2, q]} \right) \leq \frac{(A - B)([2, q] + A - B)}{[2, q][4, q]}. \quad \square$$

3 Closure theorem

Let the functions $f_k(z)$, $(k = 1, 2, 3, \dots, l)$ be defined by

$$f_k(z) = z + \sum_{n=2}^{\infty} a_{n,k} z^n, \quad (z \in \mathcal{A}). \quad (21)$$

3.1 Theorem

Let the functions $f_k(z)$, $(k = 1, 2, 3, \dots, l)$ defined by (21) be in the class $\mathcal{S}_q^{**}[A, B]$. Then the function $F \in \mathcal{S}_q^{**}[A, B]$, where

$$F(z) = \sum_{k=1}^l \lambda_k f_k(z); \quad \left(\lambda_k \geq 0, \sum_{k=1}^l \lambda_k = 1 \right). \quad (22)$$

Proof. From (22), we have

$$F(z) = z + \sum_{n=2}^{\infty} \left(\sum_{k=1}^l \lambda_k a_{n,k} \right) z^n.$$

By Theorem 2.3, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \left\{ [n, q](1-B) - (1+A) \frac{1-(-1)^n}{2} \right\} \left| \sum_{k=1}^l \lambda_k a_{n,k} \right| \\ &= \sum_{k=1}^l \lambda_k \left(\sum_{n=p+1}^{\infty} \left\{ [n, q](1-B) - (1+A) \frac{1-(-1)^n}{2} \right\} |a_{n,k}| \right) \leq \sum_{k=1}^l \lambda_k (B-A) = B-A. \end{aligned}$$

Hence, by Theorem 2.3, $F \in \mathcal{S}_q^{**}[A, B]$. $\square$

3.2 Theorem

The class $\mathcal{S}_q^{**}[A, B]$ is closed under convex combination.

Proof. Let the function $f_k(z)$ ($k = 1, 2$) given by (21) be in the class $\mathcal{S}_q^{**}[A, B]$. It is enough to show that

$$g(z) = \beta f_1(z) + (1-\beta)f_2(z), \quad 0 \leq \beta \leq 1,$$

is in the class $\mathcal{S}_q^{**}[A, B]$. Since for $0 \leq \beta \leq 1$,

$$g(z) = z + \sum_{n=2}^{\infty} [\beta a_{n,1} + (1-\beta)a_{n,2}] z^n,$$

so, by Theorem 2.3, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \left([n, q](1+B) - (1+A) \frac{1-(-1)^n}{2} \right) (\beta |a_{n,1}| + (1-\beta)|a_{n,2}|) = \beta \sum_{n=2}^{\infty} \left([n, q](1+B) - (1+A) \frac{1-(-1)^n}{2} \right) |a_{n,1}| \\ &+ (1-\beta) \sum_{n=p+1}^{\infty} \left([n, q](1+B) - (1+A) \frac{1-(-1)^n}{2} \right) |a_{n,2}| \leq \beta(A-B) + (1-\beta)(A-B) = (A-B). \end{aligned}$$

Hence, by Theorem 2.3, $g(z) \in \mathcal{S}_q^{**}[A, B]$. $\square$

4 Distortion theorem

4.1 Theorem

If $f \in S_q^{**}[A, B]$ and has the form (1), then for $|z| = r$, we have

$$r - \frac{B - A}{[2, q](1 - B)}r^2 \leq |f(z)| \leq r + \frac{B - A}{[2, q](1 - B)}r^2.$$

Proof. Consider

$$|f(z)| = \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \leq r + \sum_{n=2}^{\infty} |a_n| r^n \leq r + r^2 \sum_{n=2}^{\infty} |a_n| \leq r + \frac{B - A}{[2, q](1 - B)}r^2.$$

Similarly,

$$|f(z)| = \left| z + \sum_{n=2}^{\infty} a_n z^n \right| \geq r - \sum_{n=2}^{\infty} |a_n| r^n \geq r - r^2 \sum_{n=2}^{\infty} |a_n| \geq r - \frac{B - A}{[2, q](1 - B)}r^2.$$

Hence, the proof is completed. $\square$

4.2 Theorem

If $f \in S_q^{**}[A, B]$ and has the form (1), then for $|z| = r$, we have

$$r - \frac{B - A}{1 - B}r^2 \leq |d_q f(z)| \leq r + \frac{B - A}{1 - B}r^2.$$

Proof. The proof of this theorem is similar to Theorem 4.1, so the details are omitted. $\square$

5 Extreme point theorem

5.1 Theorem

Let $f_1(z) = z$ and

$$f_k(z) = z + \frac{B - A}{\left\{ [k, q](1 - B) - (1 + A) \frac{(1 - (-1)^k)}{2} \right\}} z^k, \quad z \in \mathcal{A}, \quad k \geq 2. \quad (23)$$

Then $f \in S_q^{**}[A, B]$ if and only if f can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \alpha_k f_k(z), \quad \alpha_k \geq 0, \quad (24)$$

and $\sum_{k=1}^{\infty} \alpha_k = 1$.

Proof. If a function f is of the form $f(z) = z + \sum_{k=1}^{\infty} \alpha_k f_k(z)$, $\alpha_k \geq 0$ and $\sum_{k=1}^{\infty} \alpha_k = 1$, then

$$\begin{aligned} & \sum_{k=2}^{\infty} \left(\frac{\left\{ [k, q](1-B) - (1+A) \frac{(1-(-1)^k)}{2} \right\}}{B-A} \right) |\alpha_k| \\ &= \sum_{k=2}^{\infty} \left(\frac{\left\{ [k, q](1-B) - (1+A) \frac{(1-(-1)^k)}{2} \right\}}{B-A} \right) \left(\frac{B-A}{\left\{ [k, q](1-B) - (1+A) \frac{(1-(-1)^k)}{2} \right\}} \right) \alpha_k \\ &< \sum_{k=2}^{\infty} \alpha_k = (1 - \alpha_1) < 1. \end{aligned}$$

Hence, by Theorem 2.3, $f \in S_q^{**}[A, B]$.

Conversely, if $f \in S_q^{**}[A, B]$, then we have

$$\alpha_k = \frac{\left\{ [k, q](1-B) - (1+A) \frac{(1-(-1)^k)}{2} \right\}}{B-A} |\alpha_k|, \quad k \geq 2,$$

and $\alpha_1 = 1 - \sum_{k=2}^{\infty} \alpha_k$. Then the function f is of the form (24), and this completes the proof. $\square$

6 Conclusion

In this article, a class of q -starlike functions with respect to symmetric points in circular domain is introduced. Various interesting properties of these functions have been investigated like sufficiency criteria in the form of convolutions, coefficient bounds, distortion theorem, closure theorem and extreme point theorem. These means and methods can be utilized for other important class of functions such as meromorphic functions, bi-univalent functions and Harmonic functions.

Acknowledgements: The authors would like to thank Associate Professor Dr. Wali Khan Mashwani from Institute of Numerical Sciences, Kohat University of Science and Technology, Kohat, Pakistan, for his help and support to improve the final version of this paper.

Author contributions: All authors contributed equally to and approved the final manuscript.

Conflict of interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

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