Research Article

Guoqiang Dang*

Meromorphic solutions of the $(2 + 1)$ - and the $(3 + 1)$ -dimensional BLMP equations and the $(2 + 1)$ -dimensional KMN equation

<https://doi.org/10.1515/dema-2021-0009>

received November 2, 2020; accepted April 7, 2021

Abstract: The complex method is systematic and powerful to build various kinds of exact meromorphic solutions for nonlinear partial differential equations on the complex plane $\mathbb{C}$. By using the complex method, abundant new exact meromorphic solutions to the $(2 + 1)$ -dimensional and the $(3 + 1)$ -dimensional Boiti-Leon-Manna-Pempinelli equations and the $(2 + 1)$ -dimension Kundu-Mukherjee-Naskar equation are investigated. Abundant new elliptic solutions, rational solutions and exponential solutions have been constructed.

Keywords: complex method, Boiti-Leon-Manna-Pempinelli equation, Kundu-Mukherjee-Naskar equation, Weierstrass elliptic function

MSC 2020: 30D35, 34A05

1 Introduction

Nonlinear partial differential equations (NLPDEs) vividly depict various natural laws in physics, engineering and other science, such as quantum mechanics, nonlinear optics, solid-state physics, fluid mechanics and so on. For the past few decades, with the aim of finding new solutions, lots of important and creative approaches have been established. For example, see [1–17]. Each method has its limitations, hence, there is no general approach to tackle each type of NLPDEs.

In recent years, high-dimensional NLPDEs have attracted much attention of researchers. From the significance and surprising properties of higher-dimensional differential equations, it is very important to investigate the analytical exact solutions. The starting points of our considerations are the most famous $(2 + 1)$ -dimensional and the $(3 + 1)$ -dimensional Boiti-Leon-Manna-Pempinelli (BLMP) equations which were given as follows:

$$u_{yt} + u_{xxx} - 3u_{xx}u_y - 3u_xu_{xy} = 0, \quad (1)$$

where $u = u(x, y, t)$, and

$$u_{yt} + u_{zt} + u_{xxx} + u_{xxx} - 3u_x(u_{xy} + u_{xz}) - 3u_{xx}(u_y + u_z) = 0, \quad (2)$$

where $u = u(x, y, z, t)$.

The $(2 + 1)$ -dimensional and $(3 + 1)$ -dimensional BLMP equations, being mathematical models of the incompressible fluid, which were investigated by employing the extended homoclinic test method, then several periodic solitary wave solutions and kink solutions were constructed in 2015 [18]. The transformed rational function method and the $\exp(-\Phi(\xi))$ methods were used to construct several different analytical solutions in 2017 [19].

* Corresponding author: Guoqiang Dang, School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, P. R. China, e-mail: dang@mail.bnu.edu.cn

By using traveling wave transformation

$$u(x, y, t) = u(\xi), \quad \xi = kx + ly + st + \tau,$$

where k, l, s, τ are constants, equation (1) reduces into the following ordinary differential equation (ODE):

$$k^3lu'''' + lsu'' - 6k^2u'u'' = 0. \quad (3)$$

Integrating equation (3) with respect to the variable ξ , we have

$$k^3lu''' + lsu' - 3k^2u'^2 + c = 0, \quad (4)$$

where c is a constant.

Now consider equation (2), by using traveling wave transformation

$$u(x, y, z, t) = u(\xi), \quad \xi = kx + ly + mz + st + \tau,$$

where k, l, m, s, τ are constants, equation (2) reduces into the following ODE:

$$k^3(l + m)u'''' + s(l + m)u'' - 6k^2(l + m)u'u'' = 0. \quad (5)$$

Integrating equation (5) with respect to the variable ξ , we have

$$k^3u''' + su' - 3k^2u'^2 + c = 0, \quad (6)$$

where c is a constant.

One of our destinations is to investigate equations (4) and (6) by utilizing the complex method proposed by Yuan and collaborators [20–23]. The complex method is a systematic and powerful tool for constructing traveling wave exact solutions for some certain partial differential equations. To the best of our knowledge, our research is the first to use the complex method on the BLMP equations and the Kundu-Mukherjee-Naskar (KMN) equation. We must mention here, in this paper, we investigate meromorphic solutions of ODEs on the complex plane $\mathbb{C}$.

Theorem 1. Assume that $lk \neq 0$, then equation (4) has the following three forms of meromorphic solutions:

1. The rational solutions:

$$u_{r1}(\xi) = \frac{-2kl}{\xi - \xi_0} + \frac{ls}{6k^2}(\xi - \xi_0) + c_1,$$

provided that $12ck^2 = -l^2s^2$, ξ_0, c_1 are arbitrary constants.

2. The exponential solutions:

$$u_{s1}(\xi) = -\frac{2lk\theta\eta_0}{e^{\theta(\xi-\xi_0)} - \eta_0} + \frac{(\theta^2k^3 + s)l}{6k^2}(\xi - \xi_0) + c_2,$$

provided that $12ck^2 = l^2(\theta^4k^6 - s^2)$, $\xi_0, \eta_0(\neq 0), c_2, \theta$ are arbitrary constants.

3. The elliptic solutions:

$$u_{e1}(\xi) = -2kl\zeta(\xi) - kl\frac{\wp'(\xi) + B}{\wp(\xi) - A} + \frac{ls}{6k^2}\xi + c_3,$$

provided that $12k^6l^2g_2 = l^2s^2 + 12ck^2$, $B^2 = 4a^3 - g_2A - g_3$, g_3, A, c_3 are arbitrary constants.

Theorem 2. Assume that $(l + m)k \neq 0$, then equation (6) has the following three forms of meromorphic solutions:

1. The rational solutions:

$$u_{r2}(\xi) = -\frac{2k}{\xi - \xi_0} + \frac{s}{6k^2}(\xi - \xi_0) + c_1,$$

provided that $12ck^2 = -s^2$, ξ_0, c_1 are arbitrary constants.

2. The exponential solutions:

$$u_{s2}(\xi) = -\frac{2k\theta\eta_0}{e^{\theta(\xi-\xi_0)} - \eta_0} + \frac{\theta^2k^3 + s}{6k^2}(\xi - \xi_0) + c_2,$$

provided that $12ck^2 = \theta^4k^6 - s^2$, $\xi_0, \eta_0 (\neq 0)$, c_2, θ are arbitrary constants.

3. The elliptic solutions:

$$u_{e2}(\xi) = -2k\zeta(\xi) - k\frac{\wp'(\xi) + B}{\wp(\xi) - A} + \frac{s}{6k^2}\xi + c_3,$$

provided that $12k^6g_2 = s^2 + 12ck^2$, $4a^3 - g_2A - g_3 = B^2$, g_3, A, c_3 are arbitrary constants.

Then we will study the $(2 + 1)$ -dimensional KMN equation [24–26]:

$$iq_t + aq_{xy} + ibq(q\bar{q}_x - \bar{q}q_x) = 0. \quad (7)$$

This equation depicts the oceanic rogue waves as well as hole waves [24] and optical wave propagation through coherently excited resonant waveguides [25]. The optical soliton solutions of the $(2 + 1)$ -dimensional KMN equation were intensively researched in [24] by the extended trial function method, and the first-order rogue wave solutions were studied by the Darboux transformation [26]. In [24], Ekici et al. applied the following reduction:

$$q(x, y, t) = u(\xi)e^{i\Phi(x,y,t)}, \quad \xi = A_1x + A_2y - vt + \tau, \quad \Phi(x, y, t) = -\kappa_1x - \kappa_2y + \omega t + \theta,$$

where $A_1, A_2, v, \tau, \kappa_1, \kappa_2, \omega, \theta$ are constants with respective physical means, and they reduced equation (7) into the ODE:

$$aA_1A_2u'' - (\omega + a\kappa_1\kappa_2)u - 2b\kappa_1u^3 = 0, \quad (8)$$

where $v = -a(\kappa_1A_2 + \kappa_2A_1)$.

Following their work, we obtain the following new results.

Theorem 3. Assume that $abA_1A_2\kappa_1 \neq 0$, then equation (8) has the following three forms of meromorphic solutions:

1. The exponential solutions:

$$u_{s3}(\xi) = \pm \frac{1}{2} \frac{\alpha\sqrt{aA_1A_2}}{\sqrt{b\kappa_1}} \frac{e^{\alpha(\xi-\xi_0)} + \eta_0}{e^{\alpha(\xi-\xi_0)} - \eta_0},$$

where $A_1A_2a\theta^2 + 2a\kappa_1\kappa_2 + 2\omega = 0$, $\xi_0, \eta_0 (\neq 0)$ are arbitrary constants.

2. The elliptic solutions:

$$u_{e3}(\xi) = \pm \frac{1}{2} \sqrt{\frac{aA_1A_2}{b\kappa_1}} \frac{\wp'(\xi, g_2, g_3)}{\wp(\xi, g_2, g_3) + A_0},$$

provided that

$$\begin{cases} A_0 = \frac{1}{6} \frac{a\kappa_1\kappa_2 + \omega}{aA_1A_2}, \\ b = \frac{(18A_1^2A_2^2a^2g_2 - 5a^2\kappa_1^2\kappa_2^2 - 10a\omega\kappa_1\kappa_2 - 5\omega^2)^2}{32400a^3A_1^3A_2^3\kappa_1}, \\ g_3 = \frac{(a\kappa_1\kappa_2 + \omega)(9A_1^2A_2^2a^2g_2 - a^2\kappa_1^2\kappa_2^2 - 2a\omega\kappa_1\kappa_2 - \omega^2)}{54A_1^3A_2^3a^3}, \end{cases}$$

where $\omega + a\kappa_1\kappa_2 \neq 0$, A_3, g_2 are arbitrary constants.

3. The Jacobi elliptic solutions:

$$u_{e3}(\xi) = \pm A_5 \operatorname{sn} \left(A_5 \left(\frac{\sqrt{-aA_1A_2b\kappa_1}\xi}{aA_1A_2} + A_4 \right), i \right),$$

where $\omega + a\kappa_1\kappa_2 = 0$, A_4, A_5 are arbitrary constants.

This paper is prepared as follows. In Section 2, we introduce the preliminary lemmas and the methodology. In Section 3, we present the proof of the theorems. In Section 4, the conclusions will be given.

2 Preliminaries

Definition 4. [27] A meromorphic function $f(z)$ belongs to the class W if $f(z)$ is a doubly periodic function (elliptic function), or a rational function of z , or a rational function of $\exp(\theta z)$ ($\theta \in \mathbb{C}$).

Definition 5. [21] Define differential monomial as

$$M_{(r_0, r_1, \dots, r_m)}[f(z)] \triangleq \prod_{i=0}^m [f^{(i)}(z)]^{r_i}.$$

$\deg M_{(r_0, r_1, \dots, r_m)}[f(z)] \triangleq \sum_{i=0}^m r_i$ is called the degree of $M_{(r_0, r_1, \dots, r_m)}[f(z)]$, where r_i are nonnegative integers.

Definition 6. [21] Define differential polynomial as

$$P(f, f', \dots, f^{(m)}) \triangleq \sum_{(r_0, r_1, \dots, r_m) \in \Lambda} a_{(r_0, r_1, \dots, r_m)} M_{(r_0, r_1, \dots, r_m)}[f(z)],$$

where $a_{(r_0, r_1, \dots, r_m)}$ are constants, and Λ is a finite index set. The degree of $P(f, f', \dots, f^{(m)})$ is defined by

$$\deg P(f, f', \dots, f^{(m)}) \triangleq \max_{r \in \Lambda} \{ \deg M_{(r_0, r_1, \dots, r_m)}[f(z)] \}.$$

Giving two complex constants v_1, v_2 with the property $\operatorname{Im} \frac{v_1}{v_2} > 0$, the period lattice $L \triangleq \{v \mid v = 2n_1v_1 + 2n_2v_2, n_1, n_2 \in \mathbb{Z}\}$, and

$$s_n \triangleq \sum_{v \in L \setminus \{0\}} \frac{1}{v^n}, \quad n \in \mathbb{N}, \quad n \geq 3.$$

Definition 7. The Weierstrass elliptic function [28] $\wp(z) \triangleq \wp(z, g_2, g_3)$ is a meromorphic function which admits differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

where $g_2 = 60s_4$, $g_3 = 140s_6$ and $g_2^3 - 27g_3^2 \neq 0$.

$\wp(z)$ satisfies the following addition formula:

$$\wp(z - z_1) = -\wp(z) - \wp(z_1) + \frac{1}{4} \left[\frac{\wp'(z) + \wp'(z_1)}{\wp(z) - \wp(z_1)} \right]^2, \quad (9)$$

and the Laurent series expansion at $z_0 = 0$

$$\wp(z) = \frac{1}{z^2} + \frac{g_2 z^2}{20} + \frac{g_3 z^4}{28} + O(|z|^6).$$

Furthermore, we have $\wp(z) = \wp(-z)$, $\wp'(-z) = -\wp'(z)$, $2\wp''(z) = 12\wp^2(z) - g_2$, $\wp'''(z) = 12\wp(z)\wp'(z)$.

Lemma 8. [29] *The Weierstrass elliptic functions $\wp(z)$ degenerate to the simply periodic functions*

$$\wp(z, 3d^2, d^3) = -\frac{d}{2} + \frac{3d}{2} \csc^2\left(\sqrt{\frac{3d}{2}} z\right), \quad (10)$$

provided that one root e_j is double ($g_2^3 - 27g_3^2 = 0$).

Proof. Since the order of the pole of $\wp(z, g_2, g_3)$ is 2, let $\wp(z, g_2, g_3) = \alpha \csc^2(\beta z) + \gamma$, then expand the Laurent series of both sides, then compare the coefficients of the like terms, we have $\alpha = \beta^2$, $g_2 = \frac{4}{3}\beta^4$, $g_3 = \frac{8}{27}\beta^6 = \left(\frac{2}{3}\beta^2\right)^3$. Let $d = \frac{2}{3}\beta^2$, then $\beta = \sqrt{\frac{3d}{2}}$, $\alpha = \frac{3d}{2}$, $g_3 = d^3$, $g_2 = 3d^2$, $\gamma = -\frac{\beta^2}{3} = -\frac{d}{2}$, then $\wp(z, 3d^2, d^3) = -\frac{d}{2} + \frac{3d}{2} \csc^2\left(\sqrt{\frac{3d}{2}} z\right)$. $\square$

Definition 9. The Weierstrass zeta function [28]

$$\zeta(z) := \frac{1}{z} + \sum_{v \neq 0} \left(\frac{1}{z-v} + \frac{1}{v} + \frac{z}{v^2} \right) \quad (11)$$

is a meromorphic function which satisfies the differential equation

$$\wp(z) = -\zeta'(z), \quad (12)$$

and the additional formula reads

$$\zeta(z + z_1) = \zeta(z) + \zeta(z_1) + \frac{1}{2} \frac{\wp'(z) + \wp'(z_1)}{\wp(z) - \wp(z_1)}. \quad (13)$$

Giving a complex ODE

$$f^n = P(f, f', \dots, f^{(m)}), \quad (14)$$

where P is a differential polynomial in $f, f', f'', \dots, f^{(m)}$ with constant coefficients, and f is a unknown meromorphic function.

Definition 10. [20] If there are at most p different Laurent series

$$f(z) = \sum_{j=-q}^{\infty} c_j z^j \quad (c_{-q} \neq 0, \quad q > 0, \quad p, q \in \mathbb{Z}) \quad (15)$$

solve equation (14), we call equation (14) satisfies the $\langle p, q \rangle$ condition.

Definition 11. [20] If there are only p differential principle parts

$$f(z) = \sum_{j=-q}^{-1} c_j z^j \quad (c_{-q} \neq 0, \quad q > 0), \quad (16)$$

can be determined, we call equation (14) satisfies the weak $\langle p, q \rangle$ condition.

Lemma 12. [30,31] *Giving the following k th order Briot-Bouquet equation:*

$$P(f^{(k)}, f) = 0, \quad (17)$$

where P is a polynomial in f and $f^{(k)}$ with constant coefficients. If f is a meromorphic solution of equation (17) which with at least a pole, then f belongs to W .

Lemma 13. [20] *Giving the following Briot-Bouquet equation:*

$$P(f^{(m)}, f) + bf^n = 0, \quad (18)$$

where $\deg P(f, f^{(m)}) < n$ and b is a constant. If equation (18) satisfies the weak $\langle p, q \rangle$ condition, then any meromorphic solution $f(z)$ belongs to class W .

Furthermore, any elliptic solution f with pole at origin is of the form:

$$f(z) = \sum_{i=1}^{l-1} \sum_{j=2}^q \frac{(-1)^j c_{-ij}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \left(\frac{1}{4} \left[\frac{\wp'(z) + B_i}{\wp(z) - A_i} \right]^2 - \wp(z) \right) + \sum_{i=1}^{l-1} \frac{c_{-i1}}{2} \frac{\wp'(z) + B_i}{\wp(z) - A_i} + \sum_{j=2}^q \frac{(-1)^j c_{-lj}}{(j-1)!} \frac{d^{j-2}}{dz^{j-2}} \wp(z) + c_0, \quad (19)$$

where $c_0 \in \mathbb{C}$, c_{-ij} come from (15), $B_i^2 = 4A_i^3 - g_2A_i - g_3$, provided that $\sum_{i=1}^l c_{-i1} = 0$.

Any rational solution $f(z)$ is of the form:

$$f(z) = \sum_{i=1}^l \sum_{j=1}^q \frac{c_{ij}}{(z - z_i)^j} + c_0, \quad (20)$$

with $l (\leq p)$ distinct poles which multiplicity is q .

Any exponential solution is a rational function of $\eta = \exp(\theta z)$ ($\theta \in \mathbb{C}$), and is of the form:

$$f(\eta) = \sum_{i=1}^l \sum_{j=1}^q \frac{c_{ij}}{(\eta - \eta_i)^j} + c_0. \quad (21)$$

In order to investigate the $(2+1)$ -dimensional and the $(3+1)$ -dimensional BLMP equations and the $(2+1)$ -dimensional KMN equation, we describe the broad outline of the complex method.

First, under the aforementioned conditions, we suppose that $u' = v$, then equations (4) and (6) reduce to

$$k^3lv'' + lsv - 3k^2v^2 + c = 0 \quad (22)$$

and

$$k^3v'' + sv - 3k^2v^2 + c = 0. \quad (23)$$

Plug the Laurent series (15) into equations (22), (23), (8) to check the weak $\langle p, q \rangle$ condition. Then prove that any meromorphic solution belongs to class W and construct all the rational, exponential and elliptic solutions for equations (22), (23), (8) according to (19)–(21).

3 Proofs

Proof of Theorem 1. Suppose that $lk \neq 0$. Substituting (15) into (22), it follows that $p = 1$, $q = 2$. Furthermore, $c_{-2} = 2kl$, $c_{-1} = 0$, $c_0 = \frac{ls}{6k^2}$. Therefore, equation (22) satisfies weak $\langle 1, 2 \rangle$ condition.

Take the virtue of Lemma 12, it follows that any meromorphic solution of equation (22) belongs to W . Then representations of each meromorphic solution of equation (22) will be constructed gradually.

By utilizing equation (20), we can assume that the forms of rational solutions of (22) with pole at origin are given by

$$f_1(\xi) = \frac{c_{-12}}{\xi^2} + \frac{c_{-11}}{\xi} + c_{10}. \quad (24)$$

Then plug (24) into equation (22), it follows that

$$c_{-12} = 2kl, \quad c_{-11} = 0, \quad c_{10} = \frac{ls}{6k^2}, \quad c = -\frac{l^2s^2}{12k^2}.$$

Figure 1: The propagation of solutions u_{r1} with $k = l = s = 1$, $\tau = 0$, $\xi_0 = 0$ when $t = -5$ in (a), $t = 0$ in (b), $t = 5$ in (c).

Therefore, the rational solutions of equation (22) are given by

$$v_{r1}(\xi) = \frac{2kl}{(\xi - \xi_0)^2} + \frac{ls}{6k^2},$$

provided that $c = \frac{-l^2s^2}{12k^2}$ and $\xi_0 \in \mathbb{C}$.

Therefore, equation (4) has the following rational solutions:

$$u_{r1}(\xi) = \int v_{r1}(\xi) d\xi = \frac{-2kl}{\xi - \xi_0} + \frac{ls}{6k^2}(\xi - \xi_0) + c_1,$$

provided that $c = \frac{-l^2s^2}{12k^2}$, ξ_0 , c_1 are arbitrary constants. The propagation of solutions u_{r1} is described in Figure 1.

We can see if $\xi = kx + ly + st + \tau \in \mathbb{R} \rightarrow \infty$ or $\xi = kx + ly + st + \tau \rightarrow 0$, then $u_{r1} \rightarrow \infty$.

By (21), we assume the form of the exponential solutions of equation (22) is decided by

$$f_2(\xi) = \frac{c_{-22}}{(\eta - \eta_0)^2} + \frac{c_{-21}}{(\eta - \eta_0)} + c_{20}, \quad (25)$$

where $\eta = e^{\theta\xi}$, $\theta \in \mathbb{C}$, $\eta_0 \neq 0$.

Then substituting (25) into equation (22), it follows that

$$c_{-22} = 2\theta^2kle^{2\theta\xi}, \quad c_{-21} = -2\theta^2kle^{\theta\xi}, \quad c_{20} = \frac{(\theta^2k^3 + s)l}{6k^2}, \quad c = \frac{l^2(\theta^4k^6 - s^2)}{12k^2}. \quad (26)$$

Therefore, all exponential solutions of equation (4) are given by

$$v_{s1}(\xi) = \frac{2\theta^2kle^{2\theta(\xi-\xi_0)}}{(e^{\theta(\xi-\xi_0)} - \eta_0)^2} - \frac{2\theta^2kle^{\theta(\xi-\xi_0)}}{e^{\theta(\xi-\xi_0)} - \eta_0} + \frac{(\theta^2k^3 + s)l}{6k^2} = \frac{2lk\eta_0\theta^2e^{\theta(\xi-\xi_0)}}{(e^{\theta(\xi-\xi_0)} - \eta_0)^2} + \frac{(\theta^2k^3 + s)l}{6k^2}, \quad (27)$$

provided that $c = \frac{l^2(\theta^4k^6 - s^2)}{12k^2}$, ξ_0 , $\eta_0 (\neq 0)$ are arbitrary constants.

Therefore, the exponential solutions of equation (4) are of the form:

$$u_{s1}(\xi) = \int v_{s1}(\xi) d\xi = -\frac{2lk\theta\eta_0}{e^{\theta(\xi-\xi_0)} - \eta_0} + \frac{(\theta^2k^3 + s)l}{6k^2}(\xi - \xi_0) + c_2, \quad (28)$$

provided that $c = \frac{l^2(\theta^4k^6 - s^2)}{12k^2}$, ξ_0 , $\eta_0 (\neq 0)$, c_2 are arbitrary constants. The propagation of solutions u_{s1} is shown in Figure 2. The analytic solutions u_{s1} are composed of a peakon-like soliton and a linear function.

Remark 1. Let $\eta_0 = -1$, (27) can be degenerated to the following solution:

$$v_{11}(\xi) = -\frac{kl\theta^2}{2} \operatorname{sech}^2\left(\frac{\theta}{2}\xi\right) + \frac{(\theta^2k^3 + s)l}{6k^2}. \quad (29)$$

Figure 2: The propagation of solutions u_{s1} with $k = l = s = 1$, $\tau = 0$, $\eta_0 = 1$, $\theta = 2$ when $t = -5$ in (a), $t = 0$ in (b), $t = 5$ in (c).

Then we get that

$$u_{11}(\xi) = \int v_{11}(\xi) d\xi = -kl\theta \tanh\left(\frac{\theta}{2}(\xi - \xi_0)\right) + \frac{(\theta^2 k^3 + s)l}{6k^2}(\xi - \xi_0) + c_2.$$

Let $\eta_0 = 1$, (27) can be degenerated to the following solution:

$$v_{12}(\xi) = \frac{kl\theta^2}{2} \operatorname{csch}^2\left(\frac{\theta}{2}\xi\right) + \frac{(\theta^2 k^3 + s)l}{6k^2}. \quad (30)$$

Then we get that

$$u_{12}(\xi) = \int v_{12}(\xi) d\xi = -kl\theta \coth\left(\frac{\theta}{2}(\xi - \xi_0)\right) + \frac{(\theta^2 k^3 + s)l}{6k^2}(\xi - \xi_0) + c_2.$$

By (19), we assume the forms of elliptic solutions of equation (22) are given by

$$f_3(\xi) = c_{-32}\wp(\xi) + c_{30}. \quad (31)$$

Put (31) into equation (22), it follows that

$$c_{-32} = 2kl, \quad c_{30} = \frac{ls}{6k^2}, \quad g_2 = \frac{l^2 s^2 + 12ck^2}{12k^6 l^2}.$$

Therefore, all elliptic solutions of equation (22) are given by

$$v_{e1}(\xi) = 2kl\wp(\xi - \xi_0) + \frac{ls}{6k^2}, \quad (32)$$

where $g_2 = \frac{l^2 s^2 + 12ck^2}{12k^6 l^2}$, g_3 is arbitrary.

Thus, the elliptic solutions of equation (4) are given by

$$u_{e1}(\xi) = \int v_{e1}(\xi) d\xi = -2kl\zeta(\xi - \xi_0) + \frac{ls}{6k^2}(\xi - \xi_0) + c_3, \quad (33)$$

where $g_2 = \frac{l^2 s^2 + 12ck^2}{12k^6 l^2}$, ξ_0 , g_3 , c_3 are arbitrary constants.

From equation (13), we can expand (33) to the following form:

$$u_{e1}(\xi) = -2kl\zeta(\xi) - \frac{kl}{2} \frac{\wp'(\xi) + B}{\wp(\xi) - A} + \frac{ls}{6k^2}\xi + c_3, \quad (34)$$

where $g_2 = \frac{l^2 s^2 + 12ck^2}{12k^6 l^2}$, $B^2 = 4a^3 - g_2 A - g_3$, g_3 , A , c_3 are constants. $\square$

The proof of Theorem 2 is similar to that of Theorem 1, it is therefore omitted.

Remark 2. From (32), we can assume that $g_2 = 3t^2$, $g_3 = t^3$, then $t = \pm \frac{\sqrt{l^2 s^2 + 12ck^2}}{6k^3 l}$. Then from Lemma 8, (32) can be degenerated to the following solution:

$$v_{13}(\xi) = 3klt \csc^2\left(\sqrt{\frac{3t}{2}} \xi\right) - klt + \frac{ls}{6k^2}. \quad (35)$$

It follows that

$$u_{13}(\xi) = -kl\sqrt{6t} \cot\left(\sqrt{\frac{3t}{2}}(\xi - \xi_0)\right) + \left(-klt + \frac{ls}{6k^2}\right)(\xi - \xi_0) + c_3. \quad (36)$$

Proof of Theorem 3. Suppose that $abA_1A_2\kappa_1 \neq 0$. Substituting equation (15) into equation (8), it follows that $p = 2$, $q = 1$, $c_{-1} = \pm \sqrt{\frac{aA_1A_2}{b\kappa_1}}$, $c_0 = 0$, $c_1 = \mp \frac{1}{6\kappa_1} \frac{a\kappa_1\kappa_2 + \omega}{\sqrt{\frac{aA_1A_2}{b\kappa_1}}}$. Hence, equation (22) satisfies weak $\langle 2, 1 \rangle$ condition.

Take the virtue of Lemma 12, it follows that any meromorphic solution of equation (8) belongs to W . Then representations of each meromorphic solution of equation (8) will be constructed gradually.

By utilizing (20), we can assume that the form of rational solutions of equation (22) with pole at origin is given by

$$h_1(\xi) = \frac{c_{-41}}{\xi} + c_{40}. \quad (37)$$

Then plug (37) into equation (8), combining the similar terms of z , it is trivial to check that there does not exist any rational solution that satisfies equation (8).

By (21), we assume the forms of the exponential solutions of equation (8) are decided by

$$h_2(z) = \frac{c_{-51}}{\eta - \eta_0} + c_{50} = \frac{c_{-51}}{e^{\theta\xi} - \eta_0} + c_{50}, \quad (38)$$

where $\eta = e^{\theta\xi}$, $\theta \in \mathbb{C}$, $\eta_0 \neq 0$.

Then substituting (38) into equation (8), it follows that

$$\begin{cases} c_{-51} = \pm \frac{\theta e^{\theta\xi}}{b\kappa_1} \sqrt{b\kappa_1 A_1 A_2 a}, \\ c_{50} = \mp \frac{\theta}{b\kappa_1} \sqrt{b\kappa_1 A_1 A_2 a}, \\ A_1 A_2 a \theta^2 + 2a\kappa_1 \kappa_2 + 2\omega = 0. \end{cases} \quad (39)$$

Therefore, all exponential solutions of equation (8) are given by

$$u_{s3}(\xi) = \pm \frac{1}{2} \frac{\theta \sqrt{aA_1A_2}}{\sqrt{b\kappa_1}} \frac{e^{\theta(\xi-\xi_0)} + \eta_0}{e^{\theta(\xi-\xi_0)} - \eta_0}, \quad (40)$$

where $A_1 A_2 a \theta^2 + 2a\kappa_1 \kappa_2 + 2\omega = 0$, $\xi_0, \eta_0 (\neq 0)$ are arbitrary constants.

By (19), we assume the form of elliptic solutions of equation (8) is given by

$$h_3(\xi) = \frac{c_{-61}}{2} \frac{\wp'(z) + B_0}{\wp(z) - A_0} + c_{60}. \quad (41)$$

Case 1. If $\omega + a\kappa_1\kappa_2 \neq 0$.

Put (41) into equation (8), it follows that

$$c_{-61} = \pm \frac{1}{2} \sqrt{\frac{aA_1A_2}{b\kappa_1}}, \quad c_{60} = 0.$$

Therefore, we have the following elliptic function solutions of equation (8):

$$u_{e3}(\xi) = \pm \frac{1}{2} \sqrt{\frac{aA_1A_2}{b\kappa_1}} \frac{\wp'(\xi, g_2, g_3)}{\wp(\xi, g_2, g_3) + A_0}, \quad (42)$$

provided that

$$\begin{cases} A_0 = \frac{1}{6} \frac{a\kappa_1\kappa_2 + \omega}{aA_1A_2}, \\ b = \frac{(18A_1^2A_2^2a^2g_2 - 5a^2\kappa_1^2\kappa_2^2 - 10a\omega\kappa_1\kappa_2 - 5\omega^2)^2}{32400a^3A_1^3A_2^3\kappa_1}, \\ g_3 = \frac{(a\kappa_1\kappa_2 + \omega)(9A_1^2A_2^2a^2g_2 - a^2\kappa_1^2\kappa_2^2 - 2a\omega\kappa_1\kappa_2 - \omega^2)}{54A_1^3A_2^3a^3}, \end{cases} \quad (43)$$

where A_3, g_2 are arbitrary constants.

Case 2. If $\omega + a\kappa_1\kappa_2 = 0$.

The following Jacobi elliptic solutions of equation (8) are easily derived by Maple:

$$u_{e3'}(\xi) = \pm A_5 \operatorname{sn} \left(A_5 \left(\frac{\sqrt{-aA_1A_2b\kappa_1}\xi}{aA_1A_2} + A_4 \right), i \right), \quad (44)$$

where A_4, A_5 are arbitrary constants. □

Remark 3. By Lemma 8, (42) can be degenerated to the following solutions:

$$u(\xi) = \pm \sqrt{\frac{aA_1A_2}{b\kappa_1}} \tanh(\theta\xi), \quad (45)$$

where $2A_1A_2B^2a + a\kappa_1\kappa_2 + \omega = 0$, θ is an arbitrary number.

4 Conclusion

Take the virtue of Theorems 1–3, and the inverse traveling wave transformation, the traveling wave solutions of the (2 + 1)-dimensional and (3 + 1)-dimensional BLMP equations and the (2 + 1)-dimensional KMN equation will be obtained instantly. The complex method has been applied to obtain all meromorphic solutions of the reduced BLMP equations and the reduced KMN equation. Furthermore, abundant whole new solutions have been introduced, containing elliptic solutions, exponential solutions and rational solutions. This paper shows a general approach for handling certain nonlinear higher order partial differential equations, for seeking simply periodic and elliptic solutions, especially solitary solutions or trigonometric function solutions.

Acknowledgements: The author is thankful to the referees for their invaluable comments and suggestions, which put the article in its present shape.

Conflict of interest: The author states no conflict of interest.

References

- [1] S. Zhang, C. Tian, and W. Qian, *Bilinearization and new multisoliton solutions for the (4+1)-dimensional Fokas equation*, *Pramana* **86** (2016), no. 6, 1259–1267, DOI: <https://doi.org/10.1007/s12043-015-1173-7>.
- [2] C. Rogers and W. F. Shadwick, *Bäcklund Transformations and Their Applications*, Academic Press, New York-London, 1982.

- [3] M. A. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, 1991.
- [4] M. Jimbo and T. Miwa, *Solitons and infinite-dimensional Lie algebras*, Publ. Res. Inst. Math. Sci. **19** (1983), no. 3, 943–1001, DOI: <https://doi.org/10.2977/prims/1195182017>.
- [5] C. Rogers and W. Schief, *Bäcklund and Darboux Transformations: Geometry and Modern Applications in Soliton Theory*, Cambridge University Press, Cambridge, 2002.
- [6] W. Malfliet, *Solitary wave solutions of nonlinear wave equations*, Am. J. Phys. **60** (1992), no. 7, 650–654, DOI: <https://doi.org/10.1119/1.17120>.
- [7] S. Liu, Z. Fu, S. Liu, and Q. Zhao, *Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations*, Phys. Lett. A **289** (2001), no. 1–2, 69–74, DOI: [https://doi.org/10.1016/S0375-9601\(01\)00580-1](https://doi.org/10.1016/S0375-9601(01)00580-1).
- [8] M. F. El-Sabbagh and A. T. Ali, *New generalized Jacobi elliptic function expansion method*, Commun. Nonlinear Sci. Numer. Simul. **13** (2008), no. 9, 1758–1766, DOI: <https://doi.org/10.1016/j.cnsns.2007.04.014>.
- [9] M. Wang, X. Li, and J. Zhang, *The G'/G-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics*, Phys. Lett. A **372** (2008), no. 4, 417–423, DOI: <https://doi.org/10.1016/j.physleta.2007.07.051>.
- [10] M. Wang, *Solitary wave solutions for variant Boussinesq equations*, Phys. Lett. A **199** (1995), no. 3–4, 169–172, DOI: [https://doi.org/10.1016/0375-9601\(95\)00092-H](https://doi.org/10.1016/0375-9601(95)00092-H).
- [11] J. He and X. Wu, *Exp-function method for nonlinear wave equations*, Chaos, Solitons Fractals **30** (2006), no. 3, 700–708, DOI: <https://doi.org/10.1016/j.chaos.2006.03.020>.
- [12] G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, 1994.
- [13] A. J. M. Jawad, M. D. Petković, and A. Biswas, *Modified simple equation method for nonlinear evolution equations*, Appl. Math. Comput. **217** (2010), no. 2, 869–877, DOI: <https://doi.org/10.1016/j.amc.2010.06.030>.
- [14] K. Hosseini, Z. Ayati, and R. Ansari, *New exact traveling wave solutions of the Tzitzéica type equations using a novel exponential rational function method*, Optik **148** (2017), 85–89, DOI: <https://doi.org/10.1016/j.ijleo.2017.08.030>.
- [15] M. Mirzazadeh, Y. Yıldırım, E. Yaşar, H. Triki, Q. Zhou, S. P. Moshokoa, et al., *Optical solitons and conservation law of Kundu-Eckhaus equation*, Optik **154** (2018), 551–557, DOI: <https://doi.org/10.1016/j.ijleo.2017.10.084>.
- [16] A. Biswas, Y. Yıldırım, E. Yaşar, Q. Zhou, S. P. Moshokoa, and M. Belic, *Optical solitons for Lakshmanan-Porsezian-Daniel model by modified simple equation method*, Optik **160** (2018), 24–32, DOI: <https://doi.org/10.1016/j.ijleo.2018.01.100>.
- [17] Y. S. Özkan, E. Yaşar, and A. R. Seadawy, *A third-order nonlinear Schrödinger equation: the exact solutions, group-invariant solutions and conservation laws*, J. Taibah Univ. Sci. **14** (2020), no. 1, 585–597, DOI: <https://doi.org/10.1080/16583655.2020.1760513>.
- [18] Y. Tang and W. Zai, *New periodic-wave solutions for $(2 + 1)$ - and $(3 + 1)$ -dimensional Boiti-Leon-Manna-Pempinelli equations*, Nonlinear Dyn. **81** (2015), 249–255, DOI: <https://doi.org/10.1007/s11071-015-1986-4>.
- [19] M. Kaplan, *Two different systematic techniques to find analytical solutions of the $(2 + 1)$ -dimensional Boiti-Leon-Manna-Pempinelli equation*, Chinese J. Phys. **56** (2018), no. 5, 2523–2530, DOI: <https://doi.org/10.1016/j.cjph.2018.06.005>.
- [20] W. Yuan, Y. Li, and J. Lin, *Meromorphic solutions of an auxiliary ordinary differential equation using complex method*, Math. Methods Appl. Sci. **36** (2013), no. 13, 1776–1782, DOI: <https://doi.org/10.1002/mma.2723>.
- [21] W. Yuan, Y. Wu, Q. Chen, and Y. Huang, *All meromorphic solutions for two forms of odd order algebraic differential equations and its applications*, Appl. Math. Comput. **240** (2014), 240–251, DOI: <https://doi.org/10.1016/j.amc.2014.04.099>.
- [22] Y. Huang, W. Yuan, and Y. Wu, *All traveling wave exact solutions of two kinds of nonlinear evolution equations*, Appl. Math. Comput. **235** (2014), 148–156, DOI: <https://doi.org/10.1016/j.amc.2014.02.071>.
- [23] Y. Gu, W. Yuan, N. Aminakbari, and J. Lin, *Meromorphic solutions of some algebraic differential equations related Painlevé equation IV and its applications*, Math. Methods Appl. Sci. **41** (2018), no. 10, 3832–3840, DOI: <https://doi.org/10.1002/mma.4869>.
- [24] M. Ekici, A. Sonmezoglu, A. Biswas, and M. R. Belic, *Optical solitons in $(2+1)$ -dimensions with Kundu-Mukherjee-Naskar equation by extended trial function scheme*, Chinese J. Phys. **57** (2019), 72–77, DOI: <https://doi.org/10.1016/j.cjph.2018.12.011>.
- [25] A. Biswas, Y. Yıldırım, E. Yaşar, Q. Zhou, S. P. Moshokoa, and A. Belic, *Optical soliton perturbation with quadratic-cubic nonlinearity using a couple of strategic algorithms*, Chinese J. Phys. **56** (2018), no. 5, 1990–1998, DOI: <https://doi.org/10.1016/j.cjph.2018.09.009>.
- [26] D. Qiu, Y. Zhang, and J. He, *The rogue wave solutions of a new $(2 + 1)$ -dimensional equation*, Commun. Nonlinear Sci. Numer. Simul. **30** (2016), no. 1–3, 307–315, DOI: <https://doi.org/10.1016/j.cnsns.2015.06.025>.
- [27] A. Eremenko, *Meromorphic traveling wave solutions of the Kuramoto-Sivashinsky equation*, J. Math. Phys. Anal. Geom. **2** (2006), no. 3, 278–286.
- [28] S. Lang, *Elliptic Functions*, Springer-Verlag, New York, 1987.
- [29] R. Conte and M. Musette, *Elliptic general analytic solutions*, Stud. Appl. Math. **123** (2009), no. 1, 63–81, DOI: <https://doi.org/10.1111/j.1467-9590.2009.00447.x>.
- [30] A. Eremenko, *Meromorphic solutions of equations of briot-bouquet type*, Teor. Funktsii Funktsional. Anal. Prilozhen **127** (1982), no. 38, 48–56.
- [31] A. Eremenko, L. Liao, and T. Ng, *Meromorphic solutions of higher order briotbouquet differential equations*, Math. Proc. Cambridge Philos. Soc. **146** (2009), no. 1, 197–206, DOI: <https://doi.org/10.1017/S030500410800176X>.