Research Article

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On some fixed point theorems for multivalued F -contractions in partial metric spaces

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Abstract: Altun et al. explored the existence of fixed points for multivalued F -contractions and proved some fixed point theorems in complete metric spaces. This paper extended the results of Altun et al. in partial metric spaces and proved fixed point theorems for multivalued F -contraction mappings. Some illustrative examples are provided to support our results. Moreover, an application for the existence of a solution of an integral equation is also enunciated, showing the materiality of the obtained results.

Keywords: fixed point theorems, multivalued F -contraction mappings, partial metric spaces

MSC 2020: 47H10, 54H25

1 Introduction and preliminaries

Metric fixed point theory has been the centre of extensive research for several researchers. Banach's theorem has been enriched, unified, extended and generalized by many researchers. Fixed point theory has become an important tool for solving many nonlinear problems related to science and engineering because of its applications. We refer the readers to some noteworthy papers for more details of this topic [1–7].

In 1969, the study of fixed points for multivalued mappings on complete metric spaces was introduced by Nadler [8]. He combined the ideas of multivalued mappings and contractions by providing the following theorem:

Theorem 1.1. [8] *Let (X, d) be a complete metric space and let $T : X \rightarrow CB(X)$ be a multivalued mapping satisfying*

$$H(Tx, Ty) \leq k(d(x, y)),$$

for all $x, y \in X$, where $k \in (0, 1)$ and $CB(X)$ denotes the collection of non-empty closed and bounded subsets of X . Then T has a fixed point $u \in X$ such that $u \in Tu$.

Recently, Wardowski [2] introduced the notion of F -contraction which is defined as:

Definition 1.2. [2] *Let (M, d) be a metric space. A mapping $T : M \rightarrow M$ is said to be an F -contraction on M , if there exists $\tau > 0$ such that for all $x, y \in M$,*

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$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \quad (1.1)$$

and $F : (0, +\infty) \rightarrow \mathbb{R}$ a mapping satisfying the following:

F1: F is strictly increasing, that is for all $x, y \in \mathbb{R}_+$ such that $x < y \Rightarrow F(x) < F(y)$.

F2: For each sequence $\{\alpha_n\}_{n \geq 1}$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$.

F3: There exists $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

The set of all functions satisfying the conditions (F1)–(F3) is denoted by Δ_F .

Remark 1. In future discussion, we will denote $(0, +\infty)$ by $\mathbb{R}_+$.

Following Nadler [8] and Wardowski [2], Altun et al. [3] introduced the concept of multivalued F -contraction mappings on complete metric spaces. They proved the following theorems:

Theorem 1.3. [3] *Let (X, d) be a complete metric space and $T : X \rightarrow K(X)$ be a multivalued F -contraction mapping, then T has a fixed point, where $K(X)$ is a compact subset of a metric space (X, d) .*

Remark 2. Let A be a compact subset of a metric space (X, d) and $x \in X$, then there exists $a \in A$ such that $d(x, a) = d(x, A)$.

Theorem 1.4. [3] *Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ be a multivalued F -contraction mapping. Suppose that $F \in \Delta_F$ also satisfies:*

$$F4 : F(\inf A) = \inf F(A) \quad \text{for all } A \subset (0, \infty) \text{ with } \inf A > 0. \quad (1.2)$$

Then T has a fixed point.

Acar et al. [9] extended the work of Altun et al. [3] and proved a fixed point theorem for generalized multivalued F -contraction mappings on complete metric spaces. On the other hand, the notion of metric space has been generalized in many directions. For example, Matthews [10] introduced the concept of partial metric spaces and proved a new fixed point theorem in partial metric setting.

Several researchers tried to extend the notion of multivalued F -contraction mappings to partial metric spaces in recent years. For instance, Paesano and Vetro [11] generalized the concept of multivalued F -contraction mappings in 0-complete partial metric spaces.

The following are some useful definitions and preliminaries required to establish the main results:

Matthews [10] defined partial metric space as follows:

Definition 1.5. [10] Let X be a non-empty set. A partial metric space is a pair (X, p) , where p is a function $p : X \times X \rightarrow \mathbb{R}_+$, called the partial metric, such that for all $x, y, z \in X$ the following axioms hold:

(P1) $x = y \Leftrightarrow p(x, y) = p(x, x) = p(y, y)$;

(P2) $p(x, x) \leq p(x, y)$;

(P3) $p(x, y) = p(y, x)$; and

(P4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Clearly, by (P1)–(P3), if $p(x, y) = 0$, then $x = y$. But, the converse is in general not true.

One example of partial metric spaces is a pair $([0, \infty), p)$ where $p(x, y) = \max\{x, y\}$ for all $x, y \in [0, \infty)$. More literature and examples of partial metric spaces may be found in [10, 12–16].

Each partial metric p on X generates a T_0 topology τ_p on X whose basis is the collection of all open p -balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$, and ε is a real number.

Let (X, p) be a partial metric space. Let A be any non-empty subset of the set X and x be an element of the set X . It is well known in [17] that $x \in \bar{A}$, where $\bar{A}$ is the closure of A , if and only if $p(x, A) = p(x, x)$. Also, the set A is said to be closed in (X, p) if and only if $A = \bar{A}$.

The following definition is of Matthews [10]:

Definition 1.6. [10] Let (X, p) be a partial metric space. Then,

- (i) A sequence $\{x_n\}$ in (X, p) is said to be convergent to $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$.
- (ii) A sequence $\{x_n\}$ in (X, p) is a Cauchy sequence if and only if $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists and is finite.
- (iii) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to the topology τ_p to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

Bukatin et al. [14] proved the following lemma which will be useful in our future discussion:

Lemma 1.7. [14] Let (X, p) be a partial metric space. Then the mapping $p^s : X \times X \rightarrow [0, \infty)$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y),$$

for all $x, y \in X$ defines a metric on X . Thus, associated with any partial metric space (X, p) and (X, d) is a metric space, where $d = p^s$, induced by p .

Bukatin et al. [14] also proved the following lemma:

Lemma 1.8. [14] Let (X, p) be a partial metric space. Then:

- (i) A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p^s) .
- (ii) A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.

In 1905, Pompeiu [18] defined the concept of distance between two closed sets, in the context of complex analysis, in his PhD thesis. Later in 1914, Hausdorff [19] considered all the basic concepts introduced by Pompeiu in his book *Grundzüge der Mengenlehre*, but in the general setting of a metric space, and adopted an alternative way to symmetrize the asymmetric distances $D(A, B)$ and $D(B, A)$, by defining what is currently denoted by $H(A, B)$ and commonly named Hausdorff metric as follows:

Definition 1.9. [8] Let (X, d) be a metric space and $CB(X)$ denotes the collection of all non-empty bounded closed subsets of X . For $A, B \in CB(X)$, define

$$H(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\},$$

where $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance of a point $x \in X$ from the set A . Here, H is a metric, called the Hausdorff-Pompeiu metric, on $CB(X)$ induced by the metric d . The metric space $(CB(X), H)$ is complete whenever (X, d) is a complete metric space.

Altun et al. [3] defined multivalued F -contraction as follows:

Definition 1.10. [3] Let (X, d) be a metric space and $T : X \rightarrow CB(X)$ be a mapping. Then T is said to be a multivalued F -contraction if $F \in \Delta_F$ and there exists $\tau > 0$ such that

$$H(Tx, Ty) > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leq F(d(x, y)). \quad (1.3)$$

Aydi et al. [20] provided the following definition and lemma:

Definition 1.11. [20] Let $CB^p(X)$ denote the collection of all non-empty bounded and closed subsets of a partial metric space (X, p) . For $A, B \in CB^p(X)$, define

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\},$$

where $p(x, A) = \inf\{p(x, a) : a \in A\}$, $\delta_p(A, B) = \sup\{p(a, B) : a \in A\}$ and $\delta_p(B, A) = \sup\{p(b, A) : b \in B\}$. Then, the mapping H_p is called the partial Hausdorff metric, on $CB^p(X)$ induced by the partial metric p .

Lemma 1.12. [20] Let (X, p) be a partial metric space. Let $A, B \in CB^p(X)$ and $q > 1$. Then for any $a \in A$, there exists $b \in B$ such that

$$p(a, b) \leq qH_p(A, B).$$

This paper intends to extend the notion of multivalued F -contraction mappings of Altun et al. [3] to a complete partial metric space.

2 Main results

We present the generalization of Definition 1.10 of Altun et al. [3] to partial metric spaces.

Definition 2.1. Let (X, p) be a partial metric space and $T : X \rightarrow CB^p(X)$ be a mapping. Then, T is said to be a multivalued F -contraction if $F \in \Delta_F$ and there exists $\tau > 0$ such that

$$H_p(Tx, Ty) > 0 \Rightarrow \tau + F(H_p(Tx, Ty)) \leq F(p(x, y)). \quad (2.1)$$

The following lemma is very useful in our results:

Lemma 2.2. Let X be a partial metric space and $K(X)$ a compact subset of X . Let $A \subseteq K(X)$, and define a function $f : A \rightarrow K(X)$, then the following statements are equivalent:

- (i) f is continuous.
- (ii) Since $K(X)$ is compact, then for any convergence subsequence $x_k \rightarrow c$, $f(x_k) \rightarrow f(c)$ for any point $c \in A$.

Proof.

$i \Rightarrow ii$

For any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(x, c) - p(c, c) < \delta$ and $p(x, c) - p(x, x) < \delta$ imply $p(f(x), f(c)) - p(f(c), f(c)) < \varepsilon$ and $p(f(x), f(c)) - p(f(x), f(x)) < \varepsilon$. Using the fact that sequence $x_{n_k} \rightarrow c$ to show that $f(x_{n_k}) \rightarrow f(c)$. Now $p(x_{n_k}, c) - p(c, c) < \delta$ and $p(x_{n_k}, c) - p(x_{n_k}, x_{n_k}) < \delta$ imply $p(f(x_{n_k}), f(c)) - p(f(c), f(c)) < \delta$ and $p(f(x_{n_k}), f(c)) - p(f(x_{n_k}), f(x_{n_k})) < \varepsilon$, which shows that $f(x_{n_k}) \rightarrow f(c)$.

$ii \Rightarrow i$

We know that $f(x_{n_k}) \rightarrow f(c)$, then we will show that f is continuous by contradiction. Suppose f is not continuous, then there exists $\varepsilon > 0$ such that $\delta > 0$, $p(x, c) - p(c, c) < \delta$ and $p(x, c) - p(x, x) < \delta$ imply $p(f(x), f(c)) - p(f(c), f(c)) \geq \varepsilon$ and $p(f(x), f(c)) - p(f(x), f(x)) \geq \varepsilon$. Choosing any $\delta = \frac{1}{x_{n_k}}$ for any $n \in \mathbb{N}$ such that $p(x_{n_k}, c) - p(c, c) < \frac{1}{x_{n_k}}$ and $p(x_{n_k}, c) - p(x_{n_k}, x_{n_k}) < \frac{1}{x_{n_k}}$ imply $p(f(x_{n_k}), f(c)) - p(f(c), f(c)) \geq \varepsilon$ and $p(f(x_{n_k}), f(c)) - p(f(x_{n_k}), f(x_{n_k})) \geq \varepsilon$, which shows that $x_{n_k} \rightarrow c$ while $f(x_{n_k}) \not\rightarrow f(c)$, and this contradicts the fact that $f(x_{n_k}) \rightarrow f(c)$. Hence, f is continuous. $\square$

Now, we will extend Theorem 1.3 to partial metric spaces:

Theorem 2.3. Let (X, p) be a complete partial metric space and let $T : X \rightarrow K(X)$ be a multivalued F -contraction, then T has a fixed point in X .

Proof. Let $x_0 \in X$ be an arbitrary point. Since Tx is non-empty for all $x \in X$, we can choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, this makes x_1 a fixed point of T and this completes the proof. Now, suppose $x_1 \notin Tx_1$. Since Tx_1 is closed $p(x_1, Tx_1) > 0$.

On the other hand, from $p(x_1, Tx_1) \leq H_p(Tx_0, Tx_1)$ and (F1) of Definition 1.2 we can have, $F(p(x_1, Tx_1)) \leq F(H_p(Tx_0, Tx_1))$.

From contractive condition (2.1) we have,

$$F(p(x_1, Tx_1)) \leq F(H_p(Tx_0, Tx_1)) \leq F(p(x_0, x_1)) - \tau. \quad (2.2)$$

Since Tx_1 is compact, then there exists $x_2 \in Tx_1$ such that $p(x_1, x_2) = p(x_1, Tx_1)$, with

$$F(p(x_1, x_2)) \leq F(H_p(Tx_0, Tx_1)) \leq F(p(x_1, x_0)) - \tau. \quad (2.3)$$

Continuing with the same process recursively, we obtain $\{x_n\} \in X$ such that $x_{n+1} \in Tx_n$, with

$$F(p(x_n, x_{n+1})) \leq F(p(x_n, x_{n-1})) - \tau, \quad \text{for all } n \in \mathbb{N}. \quad (2.4)$$

If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} \in Tx_{n_0}$, then x_{n_0} is a fixed point of T and this completes the proof. Now suppose that for every $n \in \mathbb{N}$, $x_n \notin Tx_n$. Denote by $d_n = p(x_n, x_{n+1})$ for all $n = (0, 1, 2, \dots)$. Then $d_n > 0$, for all $n \in \mathbb{N}$ and using (2.4) the following is true:

$$F(d_n) \leq F(d_{n-1}) - \tau \leq F(d_{n-2}) - 2\tau \leq \dots \leq F(d_0) - n\tau. \quad (2.5)$$

From (2.5) we obtain $\lim_{n \rightarrow \infty} F(d_n) = -\infty$ then by (F2) of Definition 1.2 we have $\lim_{n \rightarrow \infty} d_n = 0$. Then by (F3) of Definition 1.2 there exists $k \in (0, 1)$ such that, $\lim_{n \rightarrow \infty} d_n^k F(d_n) = 0$.

From (2.5) the following holds for all $n \in \mathbb{N}$:

$$d_n^k (F(d_n) - F(d_0)) \leq -d_n^k n\tau \leq 0. \quad (2.6)$$

Letting $n \rightarrow \infty$ in (2.6), we obtain

$$\lim_{n \rightarrow \infty} nd_n^k = 0. \quad (2.7)$$

From (2.7), there exists $n_1 \in \mathbb{N}$ such that, $nd_n^k \leq 1$, for all $n \geq n_1$ then we have,

$$d_n \leq \frac{1}{n^{\frac{1}{k}}}, \quad \text{for all } n \geq n_1. \quad (2.8)$$

Next, we will show that $\{x_n\}$ is a Cauchy sequence.

Consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$, then from (2.8) and by (P_3) of Definition 1.5 we have,

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \dots + p(x_{m-1}, x_m) - \sum_{j=n+1}^{m-1} p(x_j, x_j) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \dots + p(x_{m-1}, x_m) \\ &= d_n + d_{n+1} + \dots + d_{m-1} = \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ implies that $\lim_{n \rightarrow \infty} p(x_n, x_m) = 0$. By Lemma 1.7, we have for any $n, m \in \mathbb{N}$, $p^s(x_n, x_m) \leq 2p(x_n, x_m) \rightarrow 0$ as $n \rightarrow \infty$. This implies that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to p^s and hence converges by Lemma 1.8. Thus, there exists $v \in X$ such that $\lim_{n \rightarrow \infty} p^s(x_n, v) = 0$.

From contractive condition (2.1), for all $x, y \in X$ with $H(Tx, Ty) > 0$, we obtain $H(Tx, Ty) < p(x, y)$ and so $H(Tx, Ty) \leq p(x, y)$ for all $x, y \in X$. Thus,

$$p(x_{n+1}, Tv) \leq H_p(Tx_n, Tv) \leq p(x_n, v). \quad (2.9)$$

From (2.9) taking limit $n \rightarrow \infty$ we obtain $p(v, Tv) = 0$. This gives $v \in \bar{T}v = Tv$. Hence, v is a fixed point of T . $\square$

Next, we will present a fixed point theorem for multivalued F -contraction mapping satisfying (F1)–(F3) plus an additional condition (F4) imposed on the function F as defined in Theorem 1.4.

The following theorem extends Theorem 1.4 to partial metric spaces:

Theorem 2.4. *Let (X, p) be a complete partial metric space and let $T : X \rightarrow CB^p(X)$ be a multivalued F -contraction, if F also satisfies:*

$$F(\inf A) = \inf F(A) \quad \text{for all } A \subset (0, \infty) \text{ with } \inf A > 0, \quad (2.10)$$

then T has a fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and this completes the proof. Now let $x_1 \notin Tx_1$ then, since Tx_1 is closed, $p(x_1, Tx_1) > 0$.

Also from $p(x_1, Tx_1) \leq F(H_p(Tx_0, Tx_1))$ and by (F1) of Definition 1.2, we have $F(p(x_1, Tx_1)) \leq F(H_p(Tx_0, Tx_1))$. By (2.1) we can write,

$$F(p(x_1, Tx_1)) \leq F(H_p(Tx_0, Tx_1)) \leq F(p(x_0, x_1)) - \tau. \quad (2.11)$$

By (2.10) and $p(x_1, Tx_1) > 0$, we can write

$$F(p(x_1, Tx_1)) = \inf_{y \in Tx_1} F(p(x_1, y)).$$

Then, from (2.11), we have

$$\inf_{y \in Tx_1} F(p(x_1, y)) \leq F(p(x_1, x_0)) - \tau < F(p(x_1, x_0)) - \frac{\tau}{2}. \quad (2.12)$$

Then, from (2.12), there exists $x_2 \in Tx_1$ such that,

$$F(p(x_1, x_2)) \leq F(p(x_1, x_0)) - \frac{\tau}{2}.$$

If $x_2 \in Tx_2$, then the proof is complete, otherwise by the same manner we can choose $x_3 \in Tx_2$ such that,

$$F(p(x_2, x_3)) \leq F(p(x_2, x_1)) - \frac{\tau}{2}.$$

Proceeding recursively, we obtain a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ and

$$F(p(x_n, x_{n+1})) \leq F(p(x_n, x_{n-1})) - \frac{\tau}{2}, \quad \text{for all } n \in \mathbb{N}. \quad (2.13)$$

If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} \in Tx_{n_0}$, then x_{n_0} is a fixed point of T and this completes the proof. Now suppose that for every $n \in \mathbb{N}$, $x_n \notin Tx_n$. Denote by $b_n = p(x_n, x_{n+1})$ for all $n = 0, 1, 2, \dots$. Then $b_n > 0$, for all $n \in \mathbb{N}$ and using (2.13) the following is true:

$$F(b_n) \leq F(b_{n-1}) - \frac{\tau}{2} \leq F(b_{n-2}) - \tau \leq \dots \leq F(b_0) - \frac{n\tau}{2}. \quad (2.14)$$

From (2.14) we obtain $\lim_{n \rightarrow \infty} F(b_n) = -\infty$, then by (F2) of Definition 1.2 we have $\lim_{n \rightarrow \infty} b_n = 0$. Then by (F3) of Definition 1.2 there exists $k \in (0, 1)$ such that $\lim_{n \rightarrow \infty} b_n^k F(b_n) = 0$.

From (2.14) the following holds for all $n \in \mathbb{N}$,

$$b_n^k (F(b_n) - F(b_0)) \leq -\frac{n\tau}{2} b_n^k \leq 0. \quad (2.15)$$

Letting $n \rightarrow \infty$ in (2.15), we obtain

$$\lim_{n \rightarrow \infty} n b_n^k = 0. \quad (2.16)$$

From (2.16) there exists $n_1 \in \mathbb{N}$ such that $n b_n^k \leq 1$, for all $n \geq n_1$. Then we have

$$b_n \leq \frac{1}{n^{\frac{1}{k}}}, \quad \text{for all } n \geq n_1. \quad (2.17)$$

Next we will show that $\{x_n\}$ is a Cauchy sequence.

Consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$, then from (2.17) and by (P_3) of Definition 1.5 we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) - \sum_{j=n+1}^{m-1} p(x_j, x_j) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &= b_n + b_{n+1} + \cdots + b_{m-1} = \sum_{i=n}^{m-1} b_i \leq \sum_{i=n}^{\infty} b_i \leq \sum_{i=n}^{\infty} \frac{1}{i^k}. \end{aligned}$$

The convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^k}$ implies that

$$\lim_{n \rightarrow \infty} p(x_n, x_m) = 0. \quad (2.18)$$

By Lemma 1.7 we obtain that for any $n, m \in \mathbb{N}$, $p^s(x_n, x_m) \leq 2p(x_n, x_m) \rightarrow 0$ as $n \rightarrow \infty$. This implies that $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to p^s and hence converges by Lemma 1.8. Thus, there exists $u \in X$ such that

$$p(u, u) = \lim_{n \rightarrow \infty} p(x_n, u) = \lim_{n, m \rightarrow \infty} p(x_n, x_m). \quad (2.19)$$

Now we have to show that u is a fixed point of T .

By (2.1) of Definition 1.10 we can write $\tau + F(H_p(Tx_n, Tu)) \leq F(p(x_n, u))$. Therefore,

$$\lim_{n \rightarrow \infty} H_p(Tx_n, Tu) = 0. \quad (2.20)$$

Thus, $x_{n+1} \in Tx_n$ gives that

$$p(x_{n+1}, Tu) \leq \delta_p(Tx_n, Tu) \leq H_p(Tx_n, Tu).$$

From (2.20) we obtain that

$$\lim_{n \rightarrow \infty} p(Tx_{n+1}, Tu) = 0. \quad (2.21)$$

Then, by (P_3) of Definition 1.5, we have,

$$p(u, Tu) \leq p(u, x_{n+1}) + p(x_{n+1}, Tu) - p(x_{n+1}, x_{n+1}). \quad (2.22)$$

Taking the limit as $n \rightarrow \infty$ in (2.22), using Lemma 1.8 and from (2.21), we obtain $p(u, Tu) = 0$.

Therefore, from (2.19), $p(u, u) = 0$ this makes $p(u, u) = p(u, Tu) = 0$, which implies that $u \in \bar{T}u = Tu$. Hence, u is a fixed point of T . $\square$

We now present an illustrative example for Theorem 2.4.

Example 1. Let $X = [0, 1]$ endowed with partial metric defined by $p(x, y) = \frac{1}{4}|x - y| + \frac{1}{2} \max\{x, y\}$ for all $x, y \in X$. Clearly (X, p) is a complete partial metric space. Also define the mapping $T : X \rightarrow CB^p(X)$ by

$$T(x) = \begin{cases} \left\{ \frac{1}{8}, \frac{1}{4} \right\} & \text{if } x = 1, \\ \left\{ \frac{1}{4} \right\} & \text{if } x \neq 1. \end{cases}$$

Clearly T is multivalued mapping. Define the function $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ by $F(x) = \ln(x)$ for all $x \in \mathbb{R}^+$ and $\tau = \ln\left(\frac{8}{5}\right)$.

Now, we will show that T satisfies condition (2.1)

First, for $x \neq 1$ from (2.1) we write,

$$\begin{aligned}
\ln\left(\frac{8}{5}\right) + \ln(H_p(Tx, T1)) &= \ln\left(\frac{8}{5}\right) + \ln\left(\max\left\{\delta_p\left(\left\{\frac{1}{4}\right\}, \left\{\frac{1}{8}, \frac{1}{4}\right\}\right), \delta_p\left(\left\{\frac{1}{8}, \frac{1}{4}\right\}, \left\{\frac{1}{4}\right\}\right)\right\}\right) \\
&= \ln\left(\frac{8}{5}\right) + \ln\left(\max\left\{\inf\left\{\frac{1}{32} + \frac{1}{8}, \frac{1}{8}\right\}, \sup\left\{\frac{1}{32} + \frac{1}{8}, \frac{1}{8}\right\}\right\}\right) \\
&= \ln\left(\frac{8}{5}\right) + \ln\left(\max\left\{\frac{1}{8}, \frac{5}{32}\right\}\right) = \ln\left(\frac{8}{5}\right) + \ln\left(\frac{5}{32}\right) \leq \ln(p(x, 1)) \\
&= \ln\left(\frac{1}{4}\right) \leq \ln(p(x, 1)) \\
&= \ln\left(\frac{1}{4}\right) \leq \ln\left(\frac{1}{4}|x - 1| + \frac{1}{2}\right).
\end{aligned}$$

Thus, contractive condition (2.1) is satisfied for all $x, y \in X$. Hence, all hypotheses of Theorem 2.4 are satisfied, and thus $x = \frac{1}{4}$ is a fixed point T .

Next, we will provide another nontrivial example for Theorem 2.4.

Example 2. Let $X = \left\{x_n = 1 - \left(\frac{1}{2}\right)^n : n \in \mathbf{N}\right\}$ endowed with partial metric defined by $p(x, y) = |x - y|$. Clearly (X, p) is a complete partial metric space. Also define the mapping $T : X \rightarrow CB^p(X)$ by

$$Tx = \begin{cases} \{x_1\}, & x = x_1, \\ \{x_n, x_{n-1}\}, & x = x_n. \end{cases}$$

Let $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a function defined by $F(x) = \ln x + x$, for all $x \in \mathbb{R}^+$ and $\tau > 0$. On applying logarithm on both sides and simplifying, equation (2.21) takes the form:

$$\frac{H_p(Tx, Ty)}{p(x, y)} e^{H_p(Tx, Ty) - p(x, y)} \leq e^{-\tau}. \quad (2.23)$$

To see this, we consider the following cases:

Case I

For $m, n \in \mathbf{N}$, $H_p(Tx_m, Tx_n) > 0$ and $m > 2$ and $n = 1$, we have

$$\frac{H_p(Tx_m, Tx_1)}{p(x_m, x_1)} e^{H_p(Tx_m, Tx_1) - p(x_m, x_1)} \leq e^{-\tau}. \quad (2.24)$$

Now, we calculate the following metric:

$$\begin{aligned}
H_p(Tx_m, Tx_1) &= |x_{m-1} - x_1|, \\
p(x_m, x_1) &= |x_m - x_1|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{|x_{m-1} - x_1|}{|x_m - x_1|} e^{|x_{m-1} - x_1| - |x_m - x_1|} \leq e^{-\tau}, \\
&\frac{1 - \left(\frac{1}{2}\right)^{m-1}}{1 - \left(\frac{1}{2}\right)^m} e^{1 - \left(\frac{1}{2}\right)^{m-1} - \left(1 - \left(\frac{1}{2}\right)^m\right)} \leq e^{-\tau}, \\
&\frac{1 - \left(\frac{1}{2}\right)^{m-1}}{1 - \left(\frac{1}{2}\right)^m} e^{\left(\frac{1}{2}\right)^m - \left(\frac{1}{2}\right)^{m-1}} \leq e^{-\tau}.
\end{aligned}$$

Case II

For $m, n \in \mathbf{N}$, $H_p(Tx_m, Tx_n) > 0$ and $m > n > 1$, we obtain

$$\frac{H_p(Tx_{m-1}, Tx_{n-1})}{p(x_m, x_n)} e^{H_p(Tx_{m-1}, Tx_{n-1}) - p(x_m, x_n)} \leq e^{-\tau}. \quad (2.25)$$

Now, we calculate the following metric:

$$\begin{aligned} H_p(Tx_{m-1}, Tx_{n-1}) &= |x_{m-1} - x_{n-1}|, \\ p(x_m, x_n) &= |x_m - x_n|. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{|x_{m-1} - x_{n-1}|}{|x_m - x_n|} e^{|x_{m-1} - x_{n-1}| - |x_m - x_n|} &\leq e^{-\tau}, \\ \frac{\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{m-1}}{\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^m} e^{\left(\left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{m-1}\right) - \left(\left(\frac{1}{2}\right)^n - \left(\frac{1}{2}\right)^m\right)} &\leq e^{-\tau}, \end{aligned}$$

which is true.

Note that for cases I and II, all conditions of Theorem 2.4 and equation (2.21) are satisfied.

3 An application

As an application of the fixed point Theorem 2.3, proved in Section 2, we provide its application for the existence of solution of an integral equation. Consider the following integral equation:

$$g(t) = f(t) + \lambda \int_{t_0}^t K(t, s, g(s)) ds, \quad t \in [0, I], \quad (3.1)$$

where $K > 0$, $t, s \in I$ and $K : [a, b] \times [a, b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Let $X = C([a, b], \mathbb{R}^n)$ with usual sup-norm. Define $\|g\|_\tau = \sup_{t \in [a, b]} \{|g(t)|e^{-\tau t}\}$, where $\tau > 0$. $\|\cdot\|_\tau$ is a Banach space norm equivalent to maximum norm and $(X, \|\cdot\|_\tau)$ endowed with a metric d_τ defined by

$$d_\tau(g, h) = \sup_{t \in [a, b]} \{|g(t) - h(t)|e^{-\tau t}\},$$

for all $g, h \in X = C([a, b], \mathbb{R}^n)$.

One can see O'Regan and Petrusel [1] and Al-Rawashdeh et al. [21] for more details on it. For any increasing sequences $\{x_n\} \in X$ converging to $x \in X$, we have $\{x_n(t)\} \ll \{x(t)\}$ for all $t \in [a, b]$.

A standard approach in finding the solution of the integral equation is the use of the iteration procedure.

Theorem 3.1. Suppose that the following conditions hold:

- (i) $g(t, s)$ is continuous on the compact subset of X of $\mathbb{R}^2$;
- (ii) there exists $x_0 \in X$ such that $x_n \in Tx_{n-1}$;
- (iii) there exists a continuous function $q : [a, b] \times [a, b] \longrightarrow [a, b]$ such that

$$|K(t, s, g(s)) - K(t, s, h(s))| \leq q(g(s), h(s))|g(s) - h(s)|,$$

for each $t, s \in [a, b]$ and $g, h \in \mathbb{R}^2$. If $\alpha > 0$ such that $q(g(s), h(s)) \leq \alpha$ for all $s \in [a, b]$.

Then, the integral equation (3.1) has a solution.

Proof. Define a self-map $T : X \longrightarrow X$ by

$$Tg(t) = f(t) + \lambda \int_{t_0}^t K(t, s, g(s)) ds, \quad t \in [0, I]. \quad (3.2)$$

Now, by using conditions (i–iii), we obtain

$$\begin{aligned} |Tg(t) - Th(t)| &= \left| \lambda \int_a^b K(t, s, g(s)) ds - \lambda \int_a^b K(t, s, h(s)) ds \right| \\ &\leq \lambda \int_a^b |K(t, s, g(s)) - K(t, s, h(s))| ds \\ &\leq \int_a^b |q(g(s), h(s))| |g(s) - h(s)| ds \leq \alpha \|g - h\|. \end{aligned} \quad (3.3)$$

Let $x = g$ and $y = h$ in (2.21), we obtain

$$\begin{aligned} \tau + F(p(Tg, Th)) &\leq F(p(g, h)), \\ F(p(Tg, Th)) &\leq F(p(g, h)) - \tau. \end{aligned} \quad (3.4)$$

Passing natural logarithms in (3.4), we obtain

$$p(Tg, Th) \leq e^{-\tau} p(g, h).$$

If we choose $\alpha = e^{-\tau}$ in the above equation, we obtain

$$p(Tg, Th) \leq \alpha p(g, h). \quad (3.5)$$

Therefore, T is a contraction mapping and by virtue of Theorem 2.3, T has a fixed point. Hence, the integral equation (3.1) has a solution. $\square$

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