Research Article

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On almost e - $\mathcal{I}$ -continuous functions
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Abstract: This work is concerned with a new class of functions called almost e - $\mathcal{I}$ -continuous functions containing the class of almost e -continuous functions. This notion is stronger than almost $\delta\beta_{\mathcal{I}}$ -continuous functions and is weaker than both almost e -continuous functions and e - $\mathcal{I}$ -continuous functions. Relationships between this new class and other classes of functions are investigated and some characterizations of this new class of functions are studied.

Keywords: ideal topological space, e - $\mathcal{I}$ -open, semi*- $\mathcal{I}$ -open, pre*- $\mathcal{I}$ -open

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1 Introduction

semi*- $\mathcal{I}$ -open sets, pre*- $\mathcal{I}$ -open sets and e - $\mathcal{I}$ -open sets in ideal topological spaces were studied by [1–3], respectively. In this paper, further properties and some relationships of these sets are investigated. Some relationships of pre*- $\mathcal{I}$ -open sets, semi*- $\mathcal{I}$ -open sets and e - $\mathcal{I}$ -open sets in ideal topological spaces are discussed. Furthermore, decompositions of continuous functions have been introduced.

An ideal $\mathcal{I}$ on a nonempty set X is a nonempty collection of subsets of X , which satisfies the following conditions: $Q \in \mathcal{I}$ and $H \subset Q$ imply $H \in \mathcal{I}$; $Q \in \mathcal{I}$ and $H \in \mathcal{I}$ imply $Q \cup H \in \mathcal{I}$ [4]. Applications to various fields were further investigated by Jankovic and Hamlett [5]; Mukherjee et al. [6]; Arenas et al. [7]; Nasef and Mahmoud [8]; Modak and Noiri [9–11]; Zahran and Ghareeb [12], etc. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp(X)$ is the set of all subsets of X , a set operator $(.)^* : \wp(X) \rightarrow \wp(X)$, called the local function [4] of Q with respect to τ and $\mathcal{I}$ is defined as follows: for $Q \subseteq X$,

$$Q^*(\mathcal{I}, \tau) = \{x \in X \mid V \cap Q \notin \mathcal{I} \text{ for every } V \in \tau(x)\},$$

where $\tau(x) = \{V \in \tau \mid x \in V\}$. Furthermore, $Cl^*(Q) = Q \cup Q^*(\mathcal{I}, \tau)$ defines a Kuratowski closure operator for the topology τ^* , called the $\ast$ -topology, finer than τ . When there is no chance for confusion, we will simply write Q^* for $Q^*(\mathcal{I}, \tau)$. X^* is often a proper subset of X . By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $Q \subset X$, $Cl(Q)$ and $Int(Q)$ will denote the closure and interior of Q in (X, τ) , respectively.

A subset A of a space (X, τ) is said to be regular open (resp. regular closed) [13] if $A = Int(Cl(A))$ (resp. $A = Cl(Int(A))$). A set S is called δ -open [14] if for each $x \in S$, there exists a regular open set K such that $x \in K \subset S$. The union of all δ -open subsets of S is called δ -interior [14] of S in X and is denoted by $\delta int(S)$, S is δ -open if $\delta int(S) = S$. The complement of a δ -open set is called δ -closed [14]. $\delta cl(S) = \{x \mid U \in \mathcal{U}(x) \Rightarrow int(cl(U)) \cap S \neq \emptyset\}$, and a set S is δ -closed if and only if $S = \delta cl(S)$.

A subset Q of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be R - $\mathcal{I}$ -open (resp. R - $\mathcal{I}$ -closed) [15] if $Q = Int(Cl^*(Q))$ (resp. $Q = Cl(Int^*(Q))$). A point $x \in X$ is called a δ - $\mathcal{I}$ -cluster point of Q if $Int(Cl^*(V)) \cap Q \neq \emptyset$ for each open set V containing x . The family of all δ - $\mathcal{I}$ -cluster points of Q is called the δ - $\mathcal{I}$ -closure of Q and

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is denoted by $\delta Cl_I(Q)$. The set δ - I -interior of Q is the union of all R - I -open sets of X contained in Q and it is denoted by $\delta Int_I(Q)$. Q is said to be δ - I -closed (resp. δ - I -open) if $\delta Cl_I(Q) = Q$ (resp. $\delta Int_I(Q) = Q$) [15]. δ - I -open sets form a topology $\tau_{\delta-I}$ and that it is coarser than τ .

A subset Q of an ideal topological space (X, τ, I) is said to be e - I -open [3,16] if $Q \subset Cl(\delta Int_I(Q)) \cup Int(\delta Cl_I(Q))$. The complement of an e - I -open set is called an e - I -closed set [17,18]. The family of all e - I -open sets containing x of X will be denoted by $EOI(X, x)$. All of the above concepts may be defined with the help of grill, and some of them have been defined by Modak in [10].

The aim of this paper is to introduce one class of functions, namely, almost e - I -continuous functions containing the class of almost e -continuous functions by utilizing the notions of e - I -open sets due to Al-Omeri et al. [3]. We investigate several properties of this class. The class of almost e - I -continuity is a generalization of almost pre^* - I -continuity and almost $semi^*$ - I -continuity. At the same time, the class of almost e - I -continuity is a generalization of almost e -continuity.

This paper consists of eight sections. In Section 3, we introduce some new types of continuity such as e - I -continuous functions. In Section 4, we investigate not only some of their properties but also relationships between almost e - I -continuity and separation axioms. In Section 5, we introduce and study various types of continuity and connectedness and also almost e - I -continuity. In Section 6, we introduce strongly almost e - I -closed graphs and relationships between almost e - I -continuity and strongly almost e - I -closed graphs are obtained. Moreover, several characterizations and properties of strongly almost e - I -closed graphs are investigated. In Section 7, we investigate relationships between almost e - I -continuity and e - I -compactness. In Section 8, we deal with relationships among several functions, almost e - I -continuous, almost- I -continuous, almost $\delta\alpha$ - I -continuous, almost $semi^*$ - I -continuous, almost pre^* - I -continuous, almost $\delta\beta_I$ -continuous, almost- AI -continuous and continuous functions.

2 Preliminaries

Throughout the present paper, spaces always mean topological spaces and $f : (X, \tau, I) \longrightarrow (Y, \sigma)$ denotes a function f of an ideal topological space (X, τ, I) into a space (Y, σ) .

Definition 2.1. A subset U of an ideal topological space (X, τ, I) is said to be

1. $\delta\alpha$ - I -open [1] if $U \subset Int(Cl(\delta Int_I(U)))$.
2. $semi^*$ - I -open [1] if $U \subset Cl(\delta Int_I(U))$.
3. pre^* - I -open [2] if $U \subseteq Int(\delta Cl_I(U))$.
4. e - I -open [3] if $U \subset Cl(\delta Int_I(U)) \cup Int(\delta Cl_I(U))$.
5. $\delta\beta_I$ -open if $U \subseteq Cl(Int(\delta Cl_I(U)))$.
6. AI -set if $U = O \cap F$, where O is open and F is R - I -closed.
7. a -open [20] if $A \subset Int(Cl(\delta Int(A)))$.

Remark 2.2. Al-Omeri et al. [3] obtained Diagram I by using the notions in Definition 2.1. Furthermore, they showed via examples that the converses of implications are not true in general.

Diagram I

The e - $\mathcal{I}$ -interior (resp. $semi^*$ - $\mathcal{I}$ -interior [1], pre^* - $\mathcal{I}$ -interior [2]) of U , denoted by $Int_e^*(U)$ (resp. $S^*IInt(U)$, $P^*IInt(U)$), is defined by the union of all e - $\mathcal{I}$ -open [3] (resp. $semi^*$ - $\mathcal{I}$ -open [1], pre^* - $\mathcal{I}$ -open [2]) sets contained in U . For a subset U of an ideal topological space $(X, \tau, \mathcal{I})$, the intersection of all e - $\mathcal{I}$ -closed (resp. $semi^*$ - $\mathcal{I}$ -closed [1], pre^* - $\mathcal{I}$ -closed [2]) sets containing U is called the e - $\mathcal{I}$ -closure (resp. $semi^*$ - $\mathcal{I}$ -closure [1], pre^* - $\mathcal{I}$ -closure [2]) of U and is denoted by $Cl_e^*(U)$ (resp. $S^*ICl(U)$, $P^*ICl(U)$).

The family of all R - $\mathcal{I}$ -open (resp. $\delta\alpha$ - $\mathcal{I}$ -open, pre^* - $\mathcal{I}$ -open, $semi^*$ - $\mathcal{I}$ -open, e - $\mathcal{I}$ -open, $\delta\beta_{\mathcal{I}}$ -open) sets and ATR -sets of X will be denoted by $ROI(X)$ (resp. $\delta\alpha OI(X)$, $POI(X)$, $SOI(X)$, $EOI(X)$, $\beta OI(X)$) and $AOI(X)$. Similarly, the family of all R - $\mathcal{I}$ -closed (resp. e - $\mathcal{I}$ -closed) sets in X will be denoted by $RCI(X)$ (resp. $ECI(X)$).

Definition 2.3. A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is said to be almost- $\mathcal{I}$ -continuous [15] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subset Int(Cl^*(V))$.

For the function which is mentioned above, we will use the abbreviation: $a.\mathcal{I}.c$.

3 Almost e - $\mathcal{I}$ -continuous functions

We obtain some characterizations of almost e - $\mathcal{I}$ -continuous functions and give some properties of these functions.

Definition 3.1. A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is said to be almost $\delta\alpha$ - $\mathcal{I}$ -continuous (resp. almost pre^* - $\mathcal{I}$ -continuous, almost $semi^*$ - $\mathcal{I}$ -continuous, almost $\delta\beta_{\mathcal{I}}$ -continuous, almost ATR -continuous) if $f^{-1}(Q) \in \delta\alpha OI(X)$ (resp. $f^{-1}(Q) \in POI(X)$, $f^{-1}(Q) \in SOI(X)$, $f^{-1}(Q) \in \beta OI(X)$, $f^{-1}(Q) \in AOI(X)$) for every $Q \in RO\mathcal{J}(Y)$. For functions which are mentioned above, we will use the abbreviations: $a.\delta\alpha.c$, $a.p_{\mathcal{I}.c}$, $a.s_{\mathcal{I}.c}$, $a.\delta\beta.c$, $a.A_{TR}.c$, respectively.

We have Diagram II among the functions defined above.

The converses of implications are not true in general.

Diagram II

Definition 3.2. A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is said to be almost e - $\mathcal{I}$ -continuous (briefly $a.e.c.$) for each $x \in X$ if for each $Q \subseteq RO\mathcal{J}(Y)$ containing $f(x)$, there exists $H \in EOI(X)$ containing x such that $f(H) \subseteq Q$.

Theorem 3.3. For a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:

1. f is $a.e.c.$;
2. For each $x \in X$ and each Q containing $f(x)$, there exists $U \in EOI(X)$ containing x such that $f(U) \subseteq Int(\delta Cl_{\mathcal{J}}(Q))$;
3. $f^{-1}(V) \in ECI(X)$ for every $V \in RC\mathcal{J}(Y)$;
4. $f^{-1}(Q) \in EOI(X)$ for every $Q \in RO\mathcal{J}(Y)$.

Proof. The proof is obvious and so is thus omitted. □

Theorem 3.4. For a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:

1. f is $a.e.c.$;
2. $f(Cl_e^*(Q)) \subseteq \delta Cl_{\mathcal{J}}(f(Q))$ for every subset Q of X ;
3. $Cl_e^*(f^{-1}(H)) \subseteq f^{-1}(\delta Cl_{\mathcal{J}}(H))$ for every subset H of Y ;
4. $f^{-1}(T) \in ECI(X)$ for every δ - $\mathcal{J}$ -closed set subset T of Y ;
5. $f^{-1}(Q) \in EOI(X)$ for every δ - $\mathcal{J}$ -open set Q of Y .

Proof. (1) $\Rightarrow$ (2): Let Q be a subset of X . Since $\delta Cl_{\mathcal{J}}(f(Q))$ is a δ - $\mathcal{J}$ -closed set in Y , it is denoted by $\cap\{T_{\alpha} : f(Q) \subset T_{\alpha}, T_{\alpha} \in RCJ(Y), \alpha \in \Delta\}$, where Δ is an index set. By Theorem 3.3, and by Lemma 2.2.(3)+(5) in [15], we have $Q \subseteq f^{-1}(\delta Cl_{\mathcal{J}}(f(Q))) = \cap\{f^{-1}(T_{\alpha}) : \alpha \in \Delta\}$ and hence $Cl_e^*(Q) \subseteq f^{-1}(\delta Cl_{\mathcal{J}}(f(Q)))$. Hence, we obtain $f(Cl_e^*(Q)) \subseteq \delta Cl_{\mathcal{J}}(f(Q))$.

(2) $\Rightarrow$ (3): Let H be a subset of Y . We have $f(Cl_e^*(f^{-1}(H))) \subseteq \delta Cl_{\mathcal{J}}(f(f^{-1}(H))) \subseteq \delta Cl_{\mathcal{J}}(H)$ and hence $Cl_e^*(f^{-1}(H)) \subseteq f^{-1}(\delta Cl_{\mathcal{J}}(H))$.

(3) $\Rightarrow$ (4): Let T be any δ - $\mathcal{J}$ -closed set of Y . We have $Cl_e^*(f^{-1}(T)) \subseteq f^{-1}(\delta Cl_{\mathcal{J}}(T)) = f^{-1}(T)$ and $f^{-1}(T)$ is e - I -closed in X .

(4) $\Rightarrow$ (5): Let Q be any δ - $\mathcal{J}$ -open set of Y . According to (4), we have $f^{-1}(Y - Q) = X - f^{-1}(Q) \in ECI(X)$ and so $f^{-1}(Q) \in EOI(X)$.

(5) $\Rightarrow$ (1): Let Q be any R - I -open set of Y . Since Q is δ - $\mathcal{J}$ -open in Y , $f^{-1}(Q) \in EOI(X)$ and hence by Theorem 3.3, f is *a.e.c.* $\square$

Definition 3.5. [15] A function $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is said to be δ_I -continuous if for each $x \in X$ and each open neighborhood Q of $f(x)$, there exists an open neighborhood H of x such that $f(Int(Cl^*(H))) \subseteq Int(Cl^*(Q))$.

Theorem 3.6. [15] A function $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is δ_I -continuous if and only if for every R - I -open set Q of Y , $f^{-1}(Q)$ is δ - I -open in X .

Definition 3.7. [19] A function $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is said to be e - I -irresolute if $f^{-1}(H) \in EOI(X)$ for each $H \in EO\mathcal{J}(Y)$.

Definition 3.8. A function $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is said to be an R - I -map if $f^{-1}(Q) \in ROI(X)$ for every $H \in RO\mathcal{J}(Y)$.

Theorem 3.9. Let $f : (X, \tau, I) \longrightarrow (Y, \sigma_1, \mathcal{J})$ and $g : (Y, \sigma_1, \mathcal{J}) \longrightarrow (Z, \sigma_2, \mathcal{K})$ be functions. For the composition $g \circ f : (X, \tau, I) \longrightarrow (Z, \sigma_2, \mathcal{K})$, the following properties hold:

1. If f is *a.e.c.* and g is an R - I -map, then $g \circ f$ is *a.e.c.*;
2. If f is e - I -irresolute and g is *a.e.c.*, then $g \circ f$ is *a.e.c.*;
3. If f is *a.e.c.* and g is *a.I.c.*, then $g \circ f$ is *a.e.c.*;
4. If f is *a.e.c.* and g is δ_I -continuous, then $g \circ f$ is *a.e.c.*

Proof. The proof of (4) is obtained by using Theorems 3.6, 3.3 and 3.4. The proofs of (1)–(3) are obtained immediately as consequences of the definitions and it is thus omitted. $\square$

Let (X, τ, I) be an ideal topological space and B be a subset of X . If we denote by I/B the relative ideal on B , then $I/B = \{B \cap I : I \in \mathcal{I}\}$.

The following lemma is due to [3].

Lemma 3.10. Let (X, τ, I) be an ideal topological space and let $Q, U \subseteq X$. If Q is an e - I -open set and $U \in \tau$, then $Q \cap U \in EOI(U, \tau/U, I/U)$.

Theorem 3.11. If $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is *a.e.c.* and $U \in \tau$, then the restriction $f|_U : (U, \tau/U, I/U) \longrightarrow (Y, \sigma, \mathcal{J})$ is *a.e.c.*

Proof. Let Q be any R - $\mathcal{J}$ -open set of Y . By Theorem 3.3, we have $f^{-1}(Q) \in EOI(X)$ and hence $(f|_U)^{-1}(Q) = f^{-1}(Q) \cap U \in EOI(U)$ by Lemma 3.10. Thus, it follows from Theorem 3.3 that $f|_U$ is *a.e.c.* $\square$

4 Separation axioms

In this section, we obtain some relationships and several properties between separation axioms and almost e - $\mathcal{I}$ -continuous functions.

Definition 4.1. [19] An ideal topological space $(X, \tau, \mathcal{I})$ is said to be e - $\mathcal{I}$ - T_1 if for each pair of distinct points x and y in X , there exist e - $\mathcal{I}$ -open sets Q and H containing x and y , respectively, such that $y \notin Q$ and $x \notin H$.

Definition 4.2. [19] An ideal topological space $(X, \tau, \mathcal{I})$ is said to be e - $\mathcal{I}$ - T_2 if for each pair of distinct points x and y in X , there exist e - $\mathcal{I}$ -open sets Q and H containing x and y , respectively, such that $Q \cap H = \emptyset$.

Definition 4.3. An ideal topological space $(X, \tau, \mathcal{I})$ is said to be R - $\mathcal{I}$ - T_1 if for each pair of distinct points x and y in X , there exist R - $\mathcal{I}$ -open sets Q and H containing x and y , respectively, such that $y \notin Q$ and $x \notin H$.

Theorem 4.4. If $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is an a.e.c. injection and Y is R - $\mathcal{J}$ - T_1 , then X is e - $\mathcal{I}$ - T_1 .

Proof. Assume that Y is R - $\mathcal{J}$ - T_1 . For any distinct points x and y in X , there exist $Q, H \in RO\mathcal{J}(Y)$ such that $f(x) \in Q, f(y) \notin Q, f(x) \notin H$ and $f(y) \in H$. Since f is a.e.c., by Definition 3.2, there exist $K, L \in EO\mathcal{I}(X)$ such that $x \in K, y \in L, f(K) \subseteq Q$ and $f(L) \subseteq H$. Thus, we obtain $y \notin K, x \notin L$. This shows that X is e - $\mathcal{I}$ - T_1 . $\square$

Theorem 4.5. If $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is an a.e.c. injection and Y is T_2 , then X is e - $\mathcal{I}$ - T_2 .

Proof. For any pair of distinct points x and y in X , there exist disjoint open sets $Q, H \subseteq Y$ such that $f(x) \in Q, f(y) \in H$. Since f is a.e.c., there exist e - $\mathcal{I}$ -open sets $K, L \subseteq X$ containing x and y , respectively, such that $f(K) \subseteq \text{Int}(\delta Cl_{\mathcal{J}}(Q))$ and $f(L) \subseteq \text{Int}(\delta Cl_{\mathcal{J}}(H))$. Since Q and H are disjoint, we have $\text{Int}(\delta Cl_{\mathcal{J}}(H)) \cap \text{Int}(\delta Cl_{\mathcal{J}}(Q)) = \emptyset$, hence $K \cap L = \emptyset$. This shows that X is e - $\mathcal{I}$ - T_2 . $\square$

Definition 4.6. Let $(X, \tau, \mathcal{I})$ be an ideal topological space.

1. $\tau_e = \{Q \subseteq X : Q \cap H \in EO\mathcal{I}(X) \text{ whenever } H \in EO\mathcal{I}(X)\}$.
2. $(X, \tau, \mathcal{I})$ is called an E^* -space if $EO\mathcal{I}(X)$ is closed under finite intersection.
3. $(\tau_e)_s$ is the topology having $EO\mathcal{I}(X)$ as the subbase.

Lemma 4.7. If $(X, \tau, \mathcal{I})$ is an ideal topological space, then the following properties hold:

1. τ_e is a topology for X ;
2. $\tau \subset EO\mathcal{I}(X) \subset \tau_e \subset (\tau_e)_s$;
3. If $(X, \tau, \mathcal{I})$ is an E^* -space, then $\tau_e = EO\mathcal{I}(X) = (\tau_e)_s$;
4. $(\tau_e)_s$ is the smallest topology containing $EO\mathcal{I}(X)$.

Proof.

(1)

(i) It is obvious that $\emptyset, X \in \tau_e$.

(ii) Let $Q_1, Q_2 \in \tau_e$. Then $Q_1 \cap H, Q_2 \cap H \in EO\mathcal{I}(X)$ whenever $H \in EO\mathcal{I}(X)$. Hence, $(Q_1 \cap Q_2) \cap H = Q_1 \cap (Q_2 \cap H) \in EO\mathcal{I}(X)$ whenever $H \in EO\mathcal{I}(X)$. Therefore, $Q_1 \cap Q_2 \in \tau_e$.

(iii) Let $Q_\alpha \in \tau_e$ for every $\alpha \in \Delta$. Then $Q_\alpha \cap H \in EO\mathcal{I}(X)$ whenever $H \in EO\mathcal{I}(X)$ for every $\alpha \in \Delta$. Since $(\cup\{Q_\alpha : \alpha \in \Delta\}) \cap H = \cup\{(Q_\alpha \cap H) : \alpha \in \Delta\}$ and $EO\mathcal{I}(X)$ is closed under arbitrary union, $(\cup Q_\alpha) \cap H \in EO\mathcal{I}(X)$ and $\cup Q_\alpha \in \tau_e$.

(2) For every $U \in \tau, U \cap H \in EO\mathcal{I}(X)$ whenever $H \in EO\mathcal{I}(X)$ by Proposition 2.7 of [3]. Since $X \in EO\mathcal{I}(X)$, for every $Q \in \tau_e, Q = Q \cap X \in EO\mathcal{I}(X)$.

(3) and (4) The proofs are obvious. $\square$

Theorem 4.8. Let $(X, \tau, \mathcal{I})$ be an E^* -space. If $f, g : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ are a.e.c. and (Y, σ) is Hausdorff, then $\Delta = \{x \in X \mid f(x) = g(x)\}$ is e - $\mathcal{I}$ -closed in $(X, \tau, \mathcal{I})$.

Proof. Since $f, g : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ are *a.e.c.*, by Theorems 3.4 and 8.7 $f, g : (X, \tau_e, I) \longrightarrow (Y, \sigma_s, \mathcal{J})$ are continuous. Since $(Y, \sigma_{\delta\mathcal{J}}, \mathcal{J})$ is Hausdorff, $\Delta = \{x \in X \mid f(x) = g(x)\}$ is closed in (X, τ_e, I) and hence e - I -closed in (X, τ, I) . $\square$

Recall that a function $f : (X, \tau) \longrightarrow (Y, \sigma_s, \mathcal{J})$ is called weakly continuous [21] if for each $x \in X$ and each $V \in \sigma$ containing $f(x)$, there exists $U \in \tau$ containing x such that $f(U) \subset Cl(V)$. For weakly continuous functions, we will use the abbreviation w.c.

Theorem 4.9. *Let (X, τ, I) be an E^* -space and let $f, g : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ be functions and $(Y, \sigma, \mathcal{J})$ Hausdorff. If f is w.c. and g is *a.e.c.*, then $\Delta = \{x \in X \mid f(x) = g(x)\}$ is e - I -closed in X .*

Proof. Assume that $x \notin \Delta$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist open sets Q and H of Y such that $f(x) \in Q$, $g(x) \in H$ and $Q \cap H = \emptyset$ and hence $Cl(Q) \cap Int(\delta Cl_{\mathcal{J}}(H)) = \emptyset$. Since f is w.c., there exists an open set K containing x such that $f(K) \subseteq Cl(Q)$. Since g is *a.e.c.*, there exists $L \in EOI(X)$ containing x such that $g(L) \subseteq Int(\delta Cl_{\mathcal{J}}(H))$. Put $P = K \cap L$, then $P \in EOI(X)$ by Proposition 2.7 of [3] and $P \cap \Delta = \emptyset$. Therefore, we obtain $x \notin Cl_e^*(\Delta)$. This shows that Δ is e - I -closed in X . $\square$

5 Connectedness

In this section, we study the relationships between almost e - I -continuous functions and e - I -connectedness.

Definition 5.1. [19] An ideal topological space (X, τ, I) is said to be e - I -connected if X is not the union of two disjoint non empty e - I -open subsets of X .

This is a stronger form of connectedness and various types of stronger and weaker forms of connectedness have been discussed in [22–26]

Theorem 5.2. *If $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is an *a.e.c.* surjection and X is e - I -connected, then Y is connected.*

Proof. Suppose that Y is not a connected space. Then Y can be written as $Y = Q_0 \cup H_0$ such that Q_0 and H_0 are disjoint non empty open sets. Let $Q = Int(Cl^*(Q_0))$ and $H = Int(Cl^*(H_0))$. Then Q and H are disjoint nonempty R - I -open sets such that $Y = Q \cup H$. Since f is *a.e.c.*, then $f^{-1}(Q)$ and $f^{-1}(H)$ are e - I -open sets of X . We have $X = f^{-1}(Q) \cup f^{-1}(H)$ such that $f^{-1}(Q)$ and $f^{-1}(H)$ are non empty disjoint since f is surjective. This shows that X is not e - I -connected. Hence, Y is connected. $\square$

6 Strongly almost e - I -closed graphs

Definition 6.1. [19] The graph $G(f)$ of a function $f : (X, \tau, I) \longrightarrow (Y, \sigma)$ is said to be strongly almost e - I -closed if for each $(x, y) \in X \times Y - G(f)$, there exist $Q \in EOI(X, x)$ and a regular open set H in Y containing y such that $(Q \times H) \cap G(f) = \emptyset$.

Lemma 6.2. *A function $f : (X, \tau, I) \longrightarrow (Y, \sigma)$ has the strongly almost e - I -closed graph if and only if for each $x \in X$ and $y \in Y$ such that $f(x) \neq y$, there exist $Q \in EOI(X, x)$ and a regular open set H containing y such that $f(Q) \cap H = \emptyset$.*

Proof. The proof is obtained immediately from Definition 6.1. $\square$

Theorem 6.3. *If $f : (X, \tau, I) \longrightarrow (Y, \sigma)$ is w.c. and Y is Hausdorff, then $G(f)$ is strongly almost e - I -closed.*

Proof. Suppose that (x, y) is any point of $(X \times Y) - G(f)$. Then $y \neq f(x)$. But Y is Hausdorff and hence there exist open sets B_1 and B_2 in Y such that $y \in B_1$, $f(x) \in B_2$ and $B_1 \cap B_2 = \emptyset$. Since B_1 and B_2 are disjoint and open, we obtain $\text{Int}(Cl(B_1)) \cap Cl(B_2) = \emptyset$. Since f is w.c., then there exists $Q \in \tau$ containing x such that $f(Q) \subseteq Cl(B_2)$. Hence, $f(Q) \cap \text{Int}(Cl(B_1)) = \emptyset$. It follows from Lemma 6.2 that $G(f)$ is strongly almost e - I -closed. $\square$

Theorem 6.4. *If $f : (X, \tau, I) \longrightarrow (Y, \sigma)$ has a strongly almost e - I -closed graph $G(f)$ and f is injective, then X is e - I - T_1 .*

Proof. Let x and y be any two distinct points of X . Then we have $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly almost e - I -closed, there exist $Q \in EOI(X, x)$ and a regular open set H containing $f(y)$ such that $f(Q) \cap H = \emptyset$ by using Lemma 6.2. Therefore, $f^{-1}(H) \cap Q = \emptyset$ and hence we have $y \notin Q$. This shows that X is e - I - T_1 . $\square$

Definition 6.5. [19] A function $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is said to be pre e - I -open if the image of each e - I -open set of X is an e - $\mathcal{J}$ -open set in Y .

Theorem 6.6. *Let $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ have a strongly almost e - I -closed graph $G(f)$. If f is a surjective pre e - I -open function, then Y is e - $\mathcal{J}$ - T_2 .*

Proof. Let y_1 and y_2 be any distinct points of Y . Since f is surjection, $f(x) = y_1$ for some $x \in X$ and $(x, y_2) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly almost e - I -closed, by Lemma 6.2, there exist $Q \in EOI(X, x)$ and a regular open set H containing y_2 such that $f(Q) \cap H = \emptyset$. Since f is pre e - I -open, then $f(Q)$ is an e - $\mathcal{J}$ -open in Y such that $f(x) = y_1 \in f(Q)$. This implies that Y is e - $\mathcal{J}$ - T_2 . $\square$

7 Compactness

Recall that a subset Q of a space X is said to be e - I -compact [3] relative to X (resp. N -closed relative to X [27]) if every cover of Q by e - I -open (resp. regular open) sets of X has a finite subcover.

Theorem 7.1. *If $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is a.e.c. and Q is e - I -compact relative to X , then $f(Q)$ is N -closed relative to Y .*

Proof. Let $\{F_\alpha : \alpha \in \Delta\}$ be any cover of $f(Q)$ by regular open sets of Y . Since f is a.e.c., by Theorem 3.3 $\{f^{-1}(F_\alpha) : \alpha \in \Delta\}$ is a cover of Q by e - I -open sets of X . Since Q is e - I -compact relative to X , there exists a finite subset Δ_0 of Δ such that $Q \subseteq \cup \{f^{-1}(F_\alpha) : \alpha \in \Delta_0\}$. Therefore, we obtain $f(Q) \subseteq \cup \{F_\alpha : \alpha \in \Delta_0\}$. This shows that $f(Q)$ is N -closed relative to Y . $\square$

Corollary 7.2. *If $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is an a.e.c. surjection and X is e - I -compact, then Y is nearly compact.*

Theorem 7.3. *If $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ has a strongly almost e - I -closed graph, then $f(Q)$ is δ - I -closed in Y for each subset Q which is e - I -compact relative to X .*

Proof. Assume that $y \notin f(Q)$. Then $(x, y) \in (X \times Y) - G(f)$ for each $x \in Q$. Since $G(f)$ is the strongly almost e - I -closed graph, there exist $V_x \in EOI(X, x)$ containing x and a regular open set U_x of Y containing y such that $f(V_x) \cap U_x = \emptyset$. The family $\{V_x : x \in Q\}$ is a cover of Q by e - I -open sets of X . Since Q is e - I -compact

relative to X , there exists a finite subset Q_0 of Q such that $Q \subseteq \cup\{V_x : x \in Q_0\}$. Set $U = \cap\{U_x : x \in Q_0\}$. Then U is a regular-open set in Y containing y . Therefore, we have $f(Q) \cap U \subseteq \cup\{f(V_x) : x \in Q_0\} \cap U \subseteq \cup\{f(V_x) \cap U : x \in Q_0\} = \emptyset$. It follows that $y \notin \delta Cl_{\mathcal{J}} f(Q)$. Therefore, $f(Q)$ is δ - I -closed in Y . $\square$

8 Comparisons and examples

In this section, we investigate relationships between almost e - I -continuity and other related functions.

Proposition 8.1. *Let $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ be a function. Then, the following properties hold:*

1. *If f is $a.p_I.c.$, then it is $a.e.c.$;*
2. *If f is $a.s_I.c.$, then it is $a.e.c.$;*
3. *If f is $a.e.c.$, then it is $a.\delta\beta.c.$*

Proof. It follows immediately from Theorem 3.3, Remark 2.2 and Definition 3.1. $\square$

The converses of each statement in Proposition 8.1 need not be true as shown in the following examples, respectively.

Example 8.2. $X = Y = \{a, b, c, d\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \mathcal{J} = \{\emptyset, \{a\}\}$.

- (1) Let $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ be a function defined as follows: $f(a) = f(b) = f(c) = a$ and $f(d) = d$. Then, f is $a.p_I.c$ and hence $a.e.c$. However, f is not $a.s_I.c$ since there exists $\{a\} \in ROI(X)$ such that $f^{-1}(\{a\}) = \{a, b, c\}$ is not $semi^*I$ -open
- (2) Let $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ be a function defined as follows: $f(a) = f(b) = a$ and $f(c) = f(d) = c$. Then, f is $a.\delta\beta.c$ but is not $a.e.c$ since there exists $\{a\} \in ROI(X)$ such that $f^{-1}(\{a\}) = \{a, b\}$ is not e - I -open in X .

Example 8.3. Let $X = Y = \{a, b, c, d\}$, $\sigma = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$, $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \mathcal{J} = \{\emptyset, \{b\}\}$. Let $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ be a function defined as follows $f(a) = f(c) = b$, $f(b) = a$ and $f(d) = d$. Then, f is $a.s_I.c$ and hence $a.e.c$. However, f is not $a.p_I.c$ since there exists $\{b\} \in ROI(X)$ such that $f^{-1}(\{b\}) = \{a, c\}$ is not Pre^*I -open in X .

We recall that a space (X, σ, I) is said to be submaximal [28] if every dense subset of X is open in X . A space (X, τ, I) is said to be extremally disconnected [29] if the closure of every open set of X is open in X .

Definition 8.4. [30] A function $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ is said to be e -continuous if for each open set Q of (Y, σ) , $f^{-1}(Q)$ is e -open.

Theorem 8.5. *Let $f : (X, \tau, I) \longrightarrow (Y, \sigma, \mathcal{J})$ be a function such that X is submaximal and extremally disconnected. Then, the following properties are equivalent:*

1. f is $a.I.c$;
2. f is $a.\delta\alpha.c$;
3. f is $a.s_I.c$;
4. f is $a.p_I.c$;
5. f is $a.e.c$;
6. f is $a.\delta\beta.c$.

Proof. It is shown in Corollary 5.26 of [3] that if X is submaximal and extremally disconnected, then $POI(X, \tau) = SOI(X, \tau) = \delta\alpha OI(X, \tau) = \tau$. The proof is obvious by the fact. $\square$

The converses of the following implications are not true in general.

Diagram III

Lemma 8.6. [31] *If (X, τ) is resolvable, then τ_e is discrete.*

Theorem 8.7. *Let $(X, \tau, \mathcal{I})$ be an E^* -space. For a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

1. $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is *a.e.c.*;
2. $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma_s)$ is *e-I-continuous*;
3. $f : (X, \tau_e) \longrightarrow (Y, \sigma, \mathcal{J})$ is *a.I.c.*;
4. $f : (X, \tau_e) \longrightarrow (Y, \sigma_s)$ is *continuous*.

Proof. Every $Q \in \sigma_s$ is the union of regular open sets of (Y, σ) . Therefore, it is obvious that (1) and (3) are equivalent to (4) and also (1) and (2) are equivalent. $\square$

Recall that a space X is said to be resolvable [32] if there exists a dense subset Q of X such that $X - Q$ is also dense in X .

Theorem 8.8. *Let either every open subset of X be closed or $(X, \tau, \mathcal{I})$ be resolvable. For a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$, then the following properties are equivalent:*

1. f is *a.e.c.*;
2. f is *a.p_I.c.*;
3. $f : (X, 2^X) \longrightarrow (Y, \sigma, \mathcal{J})$ is *a.I.c.*

Proof. By Lemma 8.6, Theorem 8.7 and the former remarks, in either case, we have $\tau_e = 2^X$. $\square$

Theorem 8.9. *Let either X be a cofinite topological space, a countable topological space or the two point Sierpinski space. Then, for a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

1. f is *a. $\delta\beta$.c.*;
2. f is *a.s_I.c.*;
3. f is *a.I.c.*

Proof. This follows from Corollary 5.28 [3]. $\square$

Theorem 8.10. *For a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$, the following properties are equivalent:*

1. *If every subset of X is nowhere dense and f is a.e.c., then f is a.s_I.c.*;
2. *If every subset of X is $\delta\beta_I$ -open and closed, then a.e.c. and a. $\delta\beta$.c. are equivalent;*
3. *If X is an indiscrete space, then a.e.c. and a.p_I.c. are equivalent.*

Proof. This follows from Proposition 2.2. of [3]. $\square$

Recall a topological space $(X, \tau, \mathcal{I})$ is said to be semi-regular if for any open set Q of X and any $x \in Q$ there exists a regular open set H of X such that $x \in Q \subseteq H$.

Theorem 8.11. *If $f : (X, \tau) \longrightarrow (Y, \sigma, \mathcal{J})$ is an a.e.c. function and Y is semi-regular, then f is e-I-continuous.*

Proof. Let $x \in X$ and Q be an open set of Y containing $f(x)$. According to semi-regularity of Y , there exists a regular open set K of Y such that $f(x) \in K \subseteq Q$. Since f is *a.e.c.*, there exists $H \in EOI(X, x)$ such that $f(H) \subseteq K$. So, we have $f(H) \subseteq K \subseteq Q$. This shows that f is e - I -continuous. $\square$

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