Research Article

Khaldoun Al-Zoubi* and Shatha Alghueiri

On graded J_{gr}-classical 2-absorbing submodules of graded modules over graded commutative rings

https://doi.org/10.1515/dema-2021-0015
received December 23, 2020; accepted May 5, 2021

Abstract: Let G be an abelian group with identity e. Let R be a G-graded commutative ring with identity 1, and M be a graded R-module. In this paper, we introduce the concept of graded J_{gr}-classical 2-absorbing submodule as a generalization of a graded classical 2-absorbing submodule. We give some results concerning of these classes of graded submodules. A proper graded submodule C of M is called a graded J_{gr}-classical 2-absorbing submodule of M, if whenever $r_g, s_h, t_i \in h(R)$ and $x_j \in h(M)$ with $r_g s_h t_i x_j \in C$, then either $r_g s_h x_j \in C + J_{gr}(M)$ or $r_g t_i x_j \in C + J_{gr}(M)$ or $s_h t_i x_j \in C + J_{gr}(M)$, where $J_{gr}(M)$ is the graded Jacobson radical.

Keywords: graded J_{gr}-classical 2-absorbing submodule, graded classical 2-absorbing submodule, graded 2-absorbing submodule

MSC 2020: 13A02, 16W50

1 Introduction and preliminaries

Throughout this paper, all rings are commutative with identity and all modules are unitary. The concept of graded prime submodules was introduced and studied by many authors, see, for example, [1–4]. The concept of graded classical prime submodules as a generalization of graded prime submodules was introduced by Darani and Motmaen in [5,6] and studied in [7,8]. The concept of graded J_{gr}-classical prime submodule as a generalization of graded classical prime submodules was introduced and studied by Al-Zoubi and Alghueiri in [9]. The concept of graded 2-absorbing ideals as a generalization of graded prime ideals was introduced and studied by Al-Zoubi et al. and other authors, see [10,11]. The concept of graded 2-absorbing submodules as a generalization of graded prime submodules was introduced by Al-Zoubi and Abu-Dawwas in [12] and studied in [13,14]. The concept of graded classical 2-absorbing submodules as a generalization of graded 2-absorbing submodules was introduced and studied by Al-Zoubi and Al-Azaizeh in [6].

Here, we introduce the concept of graded J_{gr}-classical 2-absorbing submodule as a new generalization of a graded classical 2-absorbing submodule, on one hand, and a generalization of a graded J_{gr}-classical prime submodule, on the other hand. We investigate some basic properties of these classes of graded modules. For example, we give a characterization of graded J_{gr}-classical 2-absorbing submodule (see Theorem 2.10). We also study the behaviour of graded J_{gr}-classical 2-absorbing submodule under graded homomorphisms (see Theorem 2.11) and under localization (see Theorem 2.13).

*** Corresponding author: Khaldoun Al-Zoubi,** Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan, e-mail: kfzoubi@just.edu.jo
Shatha Alghueiri: Department of Mathematics and Statistics, Jordan University of Science and Technology, P.O. Box 3030, Irbid 22110, Jordan, e-mail: ghweiri64@gmail.com

First, we recall some basic properties of graded rings and modules, which will be used in the sequel. We refer to [15–18] for these basic properties and more information on graded rings and modules.

Let G be an abelian multiplicative group with identity e. By a G-graded ring, we mean a ring R together with direct sum decomposition (as abelian group) $R = \bigoplus_{g\in G} R_g$ with the property that $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The non-zero elements of R_h are said to be homogeneous of degree h, and all the homogeneous elements are denoted by $h(R)$, i.e., $h(R) = \cup_{h\in G} R_h$. If $t \in R$, then t can be written uniquely as $\sum_{g\in G} t_g$, where t_g is called a homogeneous component of t in R_g. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. An ideal P of R is said to be a graded ideal if $P = \bigoplus_{g\in G}(P \cap R_g) := \bigoplus_{g\in G} P_g$ (see [18]).

Let R be a G-graded ring and M be an R-module. Then, M is called *a G-graded R-module* if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M, such that, $M = \bigoplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Similarly, if an element of M belongs to $\cup_{g\in G} M_g = h(M)$, then it is called a homogeneous. Let R be a G-graded ring and M be a graded R-module. A submodule C of M is said to be *a graded submodule of M* if $C = \bigoplus_{g\in G}(C \cap M_g) := \bigoplus_{g\in G} C_g$. In this case, C_g is called the g-component of C (see [18]).

2 Results

Definition 2.1. Let R be a G-graded ring and M be a graded R-module. A proper graded submodule C of M is said to be *a graded J_{gr}-classical 2-absorbing submodule of M* if whenever $r_g, s_h, t_i \in h(R)$ and $x_j \in h(M)$ with $r_g s_h t_i x_j \in C$, then either $r_g s_h x_j \in C + J_{gr}(M)$ or $r_g t_i x_j \in C + J_{gr}(M)$ or $s_h t_i x_j \in C + J_{gr}(M)$.

Recall from [8] that a proper graded submodule C of a graded R-module M is said to be *a graded classical prime submodule of M* if whenever $r_g s_h x_i \in C$, where $r_g, s_h \in h(R)$ and $x_i \in h(M)$, then either $r_g x_i \in C$ or $s_h x_i \in C$.

It is clear that every graded classical prime submodule is a graded J_{gr}-classical 2-absorbing submodule. The next example shows that the converse is not true in general.

Example 2.2. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then, R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_8 \times \mathbb{Z}$. Then, M is a graded R-module with $M_0 = \mathbb{Z}_8 \times \mathbb{Z}$ and $M_1 = \{(0, 0)\}$. Now, consider the graded submodule $C = \langle \bar{4} \rangle \times \mathbb{Z}$ of M, then C is not a graded classical prime submodule since $2 \cdot 2 \cdot (\bar{1}, 1) \in C = \langle \bar{4} \rangle \times \mathbb{Z}$, where $2 \in R_0$ and $(\bar{1}, 1) \in M_0$, but $2 \cdot (\bar{1}, 1) \notin C = \langle \bar{4} \rangle \times \mathbb{Z}$. However, an easy computation shows that C is a graded J_{gr}-classical 2-absorbing submodule of M.

Recall from [9] that a proper graded submodule C of a graded R-module M is said to be *a graded J_{gr}-classical prime submodule of M*, if whenever $r_g s_h x_i \in C$, where $r_g, s_h \in h(R)$ and $x_i \in h(M)$, then either $r_g x_i \in C + J_{gr}(M)$ or $s_h x_i \in C + J_{gr}(M)$.

It is clear that every graded J_{gr}-classical prime submodule is a graded J_{gr}-classical 2-absorbing submodule. The next example shows that the converse is not true in general.

Example 2.3. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then, R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z} \times \mathbb{Z}$. Then, M is a G-graded R-module with $M_0 = \mathbb{Z} \times \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodule $C = 10\mathbb{Z} \times \{0\}$ of M, then C is not a graded J_{gr}-classical prime submodule since $2 \cdot 5 \cdot (1, 0) \in C = 10\mathbb{Z} \times \{0\}$, where $2, 5 \in R_0$ and $(1, 0) \in M_0$, but $2 \cdot (1, 0) \notin 10\mathbb{Z} \times \{0\} + J_{gr}(M)$ and $5 \cdot (1, 0) \notin 10\mathbb{Z} \times \{0\} + J_{gr}(M)$. However, an easy computation shows that C is a graded J_{gr}-classical 2-absorbing submodule of M.

Recall from [6] that a proper graded submodule C of a graded R-module M is said to be *a graded classical 2-absorbing submodule of M* if whenever $r_g, s_h, t_i \in h(R)$ and $x_j \in h(M)$ with $r_g s_h t_i x_j \in C$, then either $r_g s_h x_j \in C$ or $r_g t_i x_j \in C$ or $s_h t_i x_j \in C$.

It is clear that every graded classical 2-absorbing submodule is a graded J_{gr}-classical 2-absorbing submodule. The following example shows that the converse is not true in general.

Example 2.4. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$. Then, R is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_{32}$. Then, M is a graded R-module with $M_0 = \mathbb{Z}_{32}$ and $M_1 = \{0\}$. Now, consider the graded submodule $C = (\bar{16})$ of M. Then, C is not a graded classical 2-absorbing submodule of M since $2 \cdot 2 \cdot 2 \cdot \bar{2} \in (\bar{16})$, where $2 \in R_0$ and $\bar{2} \in M_0$, but $2 \cdot 2 \cdot \bar{2} \notin (\bar{16})$. However, an easy computation shows that C is a graded J_{gr}-classical 2-absorbing submodule of M.

Remark 2.5. Let R be a G-graded ring and M be a graded R-module.
(i) If $J_{gr}(M) = 0$, then every graded J_{gr}-classical 2-absorbing submodule of M is a graded classical 2-absorbing submodule of M.
(ii) If C is a graded J_{gr}-classical 2-absorbing submodule of M with $J_{gr}(M) \subseteq C$, then C is a graded classical 2-absorbing submodule of M.

Theorem 2.6. *Let R be a G-graded ring, M be a graded R-module and C, U be two graded submodules of M such that $C \subsetneq U$. If C is a graded J_{gr}-classical 2-absorbing submodule of M and $J_{gr}(M) \subseteq J_{gr}(U)$, then C is a graded J_{gr}-classical 2-absorbing submodule of U.*

Proof. Let $r_g, s_h, t_i \in h(R)$ and $u_j \in U \cap h(M)$ with $r_g s_h t_i u_j \in C$. Since C is a graded J_{gr}-classical 2-absorbing submodule of M, we get either $r_g s_h u_j \in C + J_{gr}(M)$ or $r_g t_i u_j \in C + J_{gr}(M)$ or $s_h t_i u_j \in C + J_{gr}(M)$. Hence, either $r_g s_h u_j \in C + J_{gr}(U)$ or $r_g t_i u_j \in C + J_{gr}(U)$ or $s_h t_i u_j \in C + J_{gr}(U)$ since $J_{gr}(M) \subseteq J_{gr}(U)$. Hence, C is a graded J_{gr}-classical 2-absorbing submodule of U. □

Recall from [10] that a proper graded ideal I of a G-graded ring R is said to be a graded 2-absorbing ideal if whenever $r_g, s_h, t_i \in h(R)$ with $r_g s_h t_i \in I$, which implies either $r_g s_h \in I$ or $r_g t_i \in I$ or $s_h t_i \in I$.

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module and C a proper graded submodule of M. If $(C + J_{gr}(M) :_R x_j)$ is a graded 2-absorbing ideal of R for each $x_j \in h(M)$, then C is a graded J_{gr}-classical 2-absorbing submodule of M.*

Proof. Let $r_g, s_h, t_i \in h(R)$ and $x_j \in h(M)$ such that $r_g s_h t_i x_j \in C$. This yields that $r_g s_h t_i \in (C + J_{gr}(M) :_R x_j)$. Then, either $r_g s_h \in (C + J_{gr}(M) :_R x_j)$ or $r_g t_i \in (C + J_{gr}(M) :_R x_j)$ or $s_h t_i \in (C + J_{gr}(M) :_R x_j)$ as $(C + J_{gr}(M) :_R x_j)$ is a graded 2-absorbing ideal of R. Thus, either $r_g s_h x_j \in C + J_{gr}(M)$ or $r_g t_i x_j \in C + J_{gr}(M)$ or $s_h t_i x_j \in C + J_{gr}(M)$. Therefore, C is a graded J_{gr}-classical 2-absorbing submodule of M. □

Theorem 2.8. *Let R be a G-graded ring, M a graded R-module, C a graded J_{gr}-classical 2-absorbing submodule of M and $K = \bigoplus_{i \in G} K_i$ a graded ideal of R. Then, for each $a_g, b_h \in h(R)$, $x_j \in h(M)$ and $i \in G$ with $a_g b_h K_i x_j \subseteq C$ and $a_g b_h x_j \notin C + J_{gr}(M)$, either $a_g K_i x_j \subseteq C + J_{gr}(M)$ or $b_h K_i x_j \subseteq C + J_{gr}(M)$.*

Proof. Let $a_g, b_h \in h(R)$, $x_j \in h(M)$ and $i \in G$ such that $a_g b_h K_i x_j \subseteq C$ and $a_g b_h x_j \notin C + J_{gr}(M)$. Assume that $a_g K_i x_j \nsubseteq C + J_{gr}(M)$ and $b_h K_i x_j \nsubseteq C + J_{gr}(M)$. Then, there exist $k_i, k_i' \in K_i$ such that $a_g k_i x_j \notin C + J_{gr}(M)$ and $b_h k_i' x_j \notin C + J_{gr}(M)$. By $a_g b_h k_i x_j \in C$, $a_g k_i x_j \notin C + J_{gr}(M)$ and $a_g b_h x_j \notin C + J_{gr}(M)$, we get $b_h k_i x_j \in C + J_{gr}(M)$ as C is a graded J_{gr}-classical 2-absorbing submodule of M. Similarly, by $a_g b_h k_i' x_j \in C$, we get $a_g k_i' x_j \in C + J_{gr}(M)$. By $k_i + k_i' \in K_i$, we get $a_g b_h (k_i + k_i') x_j \in C$. Hence, either $a_g (k_i + k_i') x_j \in C + J_{gr}(M)$ or $b_h (k_i + k_i') x_j \in C + J_{gr}(M)$ as C is a graded J_{gr}-classical 2-absorbing submodule of M. If $a_g (k_i + k_i') x_j \in C + J_{gr}(M)$, then $a_g k_i x_j \in C + J_{gr}(M)$ since $a_g k_i' x_j \in C + J_{gr}(M)$, a contradiction. Also, if $b_h (k_i + k_i') x_j \in C + J_{gr}(M)$, we get $b_h k_i' x_j \in C + J_{gr}(M)$ since $b_h k_i x_j \in C + J_{gr}(M)$, a contradiction. Therefore, either $a_g K_i x_j \subseteq C + J_{gr}(M)$ or $b_h K_i x_j \subseteq C + J_{gr}(M)$. □

Theorem 2.9. *Let R be a G-graded ring, M a graded R-module, C a graded J_{gr}-classical 2-absorbing submodule of M and $L = \bigoplus_{h \in G} L_h$, $K = \bigoplus_{i \in G} K_i$ be two graded ideals of R. Then, for each $a_g \in h(R)$, $x_j \in h(M)$ and $h, i \in G$ with $a_g L_h K_i x_j \subseteq C$, either $a_g L_h x_j \subseteq C + J_{gr}(M)$ or $a_g K_i x_j \subseteq C + J_{gr}(M)$ or $L_h K_i x_j \subseteq C + J_{gr}(M)$.*

Proof. Let $a_g \in h(R)$, $x_j \in h(M)$ and $h, i \in G$ such that $a_g L_h K_i x_j \subseteq C$, $a_g L_h x_j \nsubseteq C + J_{gr}(M)$ and $a_g K_i x_j \nsubseteq C + J_{gr}(M)$. Then, there exist $l'_h \in L_h$ and $k'_i \in K_i$ such that $a_g l'_h x_j \notin C + J_{gr}(M)$ and $a_g k'_i x_j \notin C + J_{gr}(M)$. We want to show that $L_h K_i x_j \subseteq C + J_{gr}(M)$. Let $l_h \in L_h$ and $k_i \in K_i$. Hence, $a_g l'_h K_i x_j \subseteq C$, $a_g l'_h x_j \notin C + J_{gr}(M)$ and $a_g K_i x_j \nsubseteq C + J_{gr}(M)$ implies that $l'_h K_i x_j \subseteq C + J_{gr}(M)$ by Theorem 2.8. Similarly, $a_g k'_i L_h x_j \subseteq C$, $a_g k'_i x_j \notin C + J_{gr}(M)$ and $a_g L_h x_j \nsubseteq C + J_{gr}(M)$ implies that $k'_i L_h x_j \subseteq C + J_{gr}(M)$ by Theorem 2.8. Since $l_h + l'_h \in L_h$ and $k_i + k'_i \in K_i$, we get $a_g(l_h + l'_h)(k_i + k'_i)x_j \in C$. This yields that either $a_g(l_h + l'_h)x_j \in C + J_{gr}(M)$ or $a_g(k_i + k'_i)x_j \in C + J_{gr}(M)$ or $(l_h + l'_h)(k_i + k'_i)x_j \in C + J_{gr}(M)$ as C is a graded J_{gr}-classical 2-absorbing submodule of M. If $a_g(l_h + l'_h)x_j \in C + J_{gr}(M)$, then $a_g l_h x_j \notin C + J_{gr}(M)$ since $a_g l'_h x_j \notin C + J_{gr}(M)$. Thus, $a_g l_h K_i x_j \subseteq C$, $a_g l_h x_j \notin C + J_{gr}(M)$ and $a_g K_i x_j \nsubseteq C + J_{gr}(M)$ imply that $l_h K_i x_j \subseteq C + J_{gr}(M)$ by Theorem 2.8, so $l_h k_i x_j \in C + J_{gr}(M)$. Similarly, if $a_g(k_i + k'_i)x_j \in C + J_{gr}(M)$, then $a_g k_i x_j \notin C + J_{gr}(M)$ since $a_g k'_i x_j \notin C + J_{gr}(M)$. Thus, $a_g k_i L_h x_j \subseteq C$, $a_g k_i x_j \notin C + J_{gr}(M)$ and $a_g L_h x_j \nsubseteq C + J_{gr}(M)$ imply that $k_i L_h x_j \subseteq C + J_{gr}(M)$ by Theorem 2.8, so $l_h k_i x_j \in C + J_{gr}(M)$. Also, if $(l_h + l'_h)(k_i + k'_i)x_j \in C + J_{gr}(M)$, then $l_h k_i x_j \in C + J_{gr}(M)$ since $l'_h K_i x_j \subseteq C + J_{gr}(M)$ and $k'_i L_h x_j \subseteq C + J_{gr}(M)$. Thus, $L_h K_i x_j \subseteq C + J_{gr}(M)$. □

The next theorem gives a characterization of graded J_{gr}-classical 2-absorbing submodules.

Theorem 2.10. *Let R be a G-graded ring, M a graded R-module, C a proper graded submodule of M and $U = \oplus_{g\in G} U_g$, $L = \oplus_{h\in G} L_h K = \oplus_{i\in G} K_i$ be graded ideals of R. Then, the following statements are equivalent:*

(i) *C is a graded J_{gr}-classical 2-absorbing submodule of M.*

(ii) *If whenever $x_j \in h(M)$ and $g, h, i \in G$ with $U_g L_h K_i x_j \subseteq C$, implies either $U_g L_h x_j \subseteq C + J_{gr}(M)$ or $U_g K_i x_j \subseteq C + J_{gr}(M)$ or $L_h K_i x_j \subseteq C + J_{gr}(M)$.*

Proof. (i) ⇒ (ii) Let $g, h, i \in G$ and $x_j \in h(M)$ such that $U_g L_h K_i x_j \subseteq C$ and $U_g L_h x_j \nsubseteq C + J_{gr}(M)$. For each $k_i \in K_i$, either $k_i U_g x_j \subseteq C + J_{gr}(M)$ or $k_i L_h x_j \subseteq C + J_{gr}(M)$ by Theorem 2.9. If $k_i U_g x_j \subseteq C + J_{gr}(M)$, for all $k_i \in K_i$ we are done. Similarly, if $k_i L_h x_j \subseteq C + J_{gr}(M)$, for all $k_i \in K_i$, we are done. Assume that there exist $k_i, k'_i \in K_i$ such that $k_i U_g x_j \nsubseteq C + J_{gr}(M)$ and $k'_i L_h x_j \nsubseteq C + J_{gr}(M)$, which yields that $k_i L_h x_j \subseteq C + J_{gr}(M)$ and $k'_i U_g x_j \subseteq C + J_{gr}(M)$. Since $k_i + k'_i \in K_i$, we get $(k_i + k'_i) U_g L_h x_j \subseteq N$. Then, either $(k_i + k'_i) U_g x_j \subseteq C + J_{gr}(M)$ or $(k_i + k'_i) L_h x_j \subseteq C + J_{gr}(M)$ by Theorem 2.9. If $(k_i + k'_i) U_g x_j \subseteq C + J_{gr}(M)$, we get $k_i U_g x_j \subseteq C + J_{gr}(M)$, which is a contradiction. Similarly, if $(k_i + k'_i) L_h x_j \subseteq C + J_{gr}(M)$, we get a contradiction. Therefore, either $U_g K_i x_j \subseteq C + J_{gr}(M)$ or $L_h K_i x_j \subseteq C + J_{gr}(M)$.

(ii) ⇒ (i) Let $r, s, t \in h(R)$ and $x \in h(M)$ with $rstx \in C$. Let U, L and K be ideals of R generated by the elements r, s, t, respectively, that is, $U = rR$, $L = sR$ and $K = tR$. So, $U = \oplus_{g\in G} rR_g$, $L = \oplus_{g\in G} sR_g$ and $K = \oplus_{g\in G} tR_g$ are graded ideals of R. Moreover, for every $g \in G$, $U_g = rR_g$, $L_g = sR_g$ and $K_g = tR_g$. In particular, $U_e = rR_e$, $L_e = sR_e$ and $K_e = tR_e$. Now, by our assumption, $U_e L_e K_e x \subseteq C$. Hence, either $U_e L_e x \subseteq C + J_{gr}(M)$ or $U_e K_e x \subseteq C + J_{gr}(M)$ or $L_e K_e x \subseteq C + J_{gr}(M)$. So, either $rsx = r1s1x \in rR_e sR_e x = U_e L_e x \subseteq C + J_{gr}(M)$ or $rtx = r1t1x \in rR_e tR_e x = U_e K_e x \subseteq C + J_{gr}(M)$ or $stx = s1t1x \in sR_e tR_e x = L_e K_e x \subseteq C + J_{gr}(M)$. Therefore, C is a graded J_{gr}-classical 2-absorbing submodule of M. □

Recall from [12] that a graded zero-divisor on a graded R-module M is an element $r_g \in h(R)$ for which there exists $x_h \in h(M)$, such that, $x_h \neq 0$ but $r_g x_h = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}Zdv_R(M)$.

The following result studies the behaviour of graded J_{gr}-classical 2-absorbing submodules under localization.

Theorem 2.11. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R.*

(i) *If C is a graded J_{gr}-classical 2-absorbing submodule of M with $(C :_R M) \cap S = \varnothing$, then $S^{-1}C$ is a graded J_{gr}-classical 2-absorbing submodule of $S^{-1}M$.*

(ii) *If $S^{-1}C$ is a graded J_{gr}-classical 2-absorbing submodule of $S^{-1}M$ with $S \cap G\text{-}Zdv_R(M/(C + J_{gr}(M))) = \varnothing$, then C is a graded J_{gr}-classical 2-absorbing submodule of M.*

Proof. (i) Since $(C :_R M) \cap S = \emptyset$, we get $S^{-1}C$ as a proper graded submodule of $S^{-1}M$. Assume that $\frac{a_{g_1}}{s_{h_1}}\frac{a_{g_2}}{s_{h_2}}\frac{a_{g_3}}{s_{h_3}}\frac{x_{g_4}}{s_{h_4}} \in S^{-1}C$, where $\frac{a_{g_1}}{s_{h_1}}, \frac{a_{g_2}}{s_{h_2}}, \frac{a_{g_3}}{s_{h_3}} \in h(S^{-1}R)$ and $\frac{x_{g_4}}{s_{h_4}} \in h(S^{-1}M)$. Then, there exists $b_{h_4} \in S$ such that $b_{h_4}a_{g_1}a_{g_2}a_{g_3}x_{g_4} \in C$. Since C is a graded J_{gr}-classical 2-absorbing submodule of M, we get $b_{h_4}a_{g_1}a_{g_2}x_{g_4} \in C + J_{gr}(M)$ or $b_{h_4}a_{g_1}a_{g_3}x_{g_4} \in C + J_{gr}(M)$ or $b_{h_4}a_{g_2}a_{g_3}x_{g_4} \in C + J_{gr}(M)$. It follows that either $\frac{a_{g_1}}{s_{h_1}}\frac{a_{g_2}}{s_{h_2}}\frac{x_{g_4}}{s_{h_4}} = \frac{b_{h_4}a_{g_1}a_{g_2}x_{g_4}}{b_{h_4}s_{h_1}s_{h_2}s_{h_4}} \in S^{-1}C + J_{gr}(S^{-1}M)$ or $\frac{a_{g_1}}{s_{h_1}}\frac{a_{g_3}}{s_{h_3}}\frac{x_{g_4}}{s_{h_4}} = \frac{b_{h_4}a_{g_1}a_{g_3}x_{g_4}}{b_{h_4}s_{h_1}s_{h_3}s_{h_4}} \in S^{-1}C + J_{gr}(S^{-1}M)$ or $\frac{a_{g_2}}{s_{h_2}}\frac{a_{g_3}}{s_{h_3}}\frac{x_{g_4}}{s_{h_4}} = \frac{b_{h_4}a_{g_2}a_{g_3}x_{g_4}}{b_{h_4}s_{h_2}s_{h_3}s_{h_4}} \in S^{-1}C + J_{gr}(S^{-1}M)$. Thus, $S^{-1}C$ is a graded J_{gr}-classical 2-absorbing submodule of $S^{-1}M$.

(ii) Assume that $a_{g_1}a_{g_2}a_{g_3}x_{g_4} \in C$, where $a_{g_1}, a_{g_2}, a_{g_3} \in h(R)$ and $x_{g_4} \in h(M)$. Hence, $\frac{a_{g_1}a_{g_2}a_{g_3}x_{g_4}}{1_e} = \frac{a_{g_1}}{1_e}\frac{a_{g_2}}{1_e}\frac{a_{g_3}}{1_e}\frac{x_{g_4}}{1_e} \in S^{-1}C$. Thus, $\frac{a_{g_1}}{1_e}\frac{a_{g_2}}{1_e}\frac{x_{g_4}}{1_e} \in S^{-1}C + J_{gr}(S^{-1}M)$ or $\frac{a_{g_1}}{1_e}\frac{a_{g_3}}{1_e}\frac{x_{g_4}}{1_e} \in S^{-1}C + J_{gr}(S^{-1}M)$ or $\frac{a_{g_2}}{1_e}\frac{a_{g_3}}{1_e}\frac{x_{g_4}}{1_e} \in S^{-1}C + J_{gr}(S^{-1}M)$ as $S^{-1}C$ is a graded J_{gr}-classical 2-absorbing submodule of $S^{-1}M$. If $\frac{a_{g_1}}{1_e}\frac{a_{g_2}}{1_e}\frac{x_{g_4}}{1_e} \in S^{-1}C + J_{gr}(S^{-1}M)$, then there exists $s_h \in S$ such that $s_h a_{g_1}a_{g_2}x_{g_4} \in C + J_{gr}(M)$. Hence, $a_{g_1}a_{g_2}x_{g_4} \in C + J_{gr}(M)$ since $S \cap G\text{-}Zdv_R(M/(C + J_{gr}(M))) = \emptyset$. Similarly, we can show that if $\frac{a_{g_1}}{1_e}\frac{a_{g_3}}{1_e}\frac{x_{g_4}}{1_e} \in S^{-1}C + J_{gr}(S^{-1}M)$, then $s_h a_{g_1}a_{g_3}x_{g_4} \in C + J_{gr}(M)$. Also, if $\frac{a_{g_2}}{1_e}\frac{a_{g_3}}{1_e}\frac{x_{g_4}}{1_e} \in S^{-1}C + J_{gr}(S^{-1}M)$, then $a_{g_2}a_{g_3}x_{g_4} \in C + J_{gr}(M)$. Thus, C is a graded J_{gr}-classical 2-absorbing submodule of M. ☐

Let M and S be two graded R-modules. A homomorphism of graded R-modules $f : M \to S$ is a homomorphism of R-modules that satisfy $f(M_g) \subseteq S_g$, for every $g \in G$, (see [18]).

Recall from [19] that a proper graded submodule C of a graded R-module M is said to be a gr-small submodule of M (for short $C \ll_g M$), if for every proper graded submodule K of M, we have $C + K \neq M$.

Theorem 2.12. [20, Theorem 2.12] *Let R be a G-graded ring and M, S be the graded R-modules.*

(i) *If $f : M \to S$ is a graded homomorphism, then $f(J_{gr}(M)) \subseteq J_{gr}(S)$.*

(ii) *If $f : M \to S$ is a graded epimorphism and $ker(f) \ll_g M$, then $f(J_{gr}(M)) = J_{gr}(S)$.*

Theorem 2.13. *Let R be a G-graded ring, M and S be two graded R-modules and $f : M \to S$ be a graded epimorphism.*

(i) *If C is a graded J_{gr}-classical 2-absorbing submodule of M with $ker(f) \subseteq C$, then $f(C)$ is a graded J_{gr}-classical 2-absorbing submodule of S.*

(ii) *If C' is a graded J_{gr}-classical 2-absorbing submodule of S with $ker(f) \ll_g M$, then $f^{-1}(C')$ is a graded J_{gr}-classical 2-absorbing submodule of M.*

Proof. (i) Let $a_g, a_h, a_i \in h(R)$ and $s_j \in h(S)$ with $a_g a_h a_i s_j \in f(C)$. Then, there exists $x_j \in h(M)$ such that $f(x_j) = s_j$ as f is a graded epimorphism. So, $a_g a_h a_i f(x_j) = f(a_g a_h a_i x_j) \in f(C)$. Hence, there exists $b_k \in C \cap h(M)$ such that $f(a_g a_h a_i x_j) = f(b_k)$. It follows that $a_g a_h a_i x_j - b_k \in ker(f) \subseteq C$, thus $a_g a_h a_i x_j \in C$. Since C is a graded J_{gr}-classical 2-absorbing submodule of M, we get either $a_g a_h x_j \in C + J_{gr}(M)$ or $a_g a_i x_j \in C + J_{gr}(M)$ or $a_h a_i x_j \in C + J_{gr}(M)$. By Theorem 2.12(i), we get either $a_g a_h s_j \in f(C) + J_{gr}(S)$ or $a_g a_i s_j \in f(C) + J_{gr}(S)$ or $a_h a_i s_j \in f(C) + J_{gr}(S)$. Therefore, $f(C)$ is a graded J_{gr}-classical 2-absorbing submodule of S.

(ii) Let $d_g, d_h, d_i \in h(R)$ and $b_j \in h(M)$ with $d_g d_h d_i b_j \in f^{-1}(C')$. So, $d_g d_h d_i f(b_j) \in C'$. Since C' is a graded J_{gr}-classical 2-absorbing submodule of S, we get either $d_g d_h f(b_j) = f(d_g d_h b_j) \in C' + J_{gr}(S)$ or $d_g d_i f(b_j) = f(d_g d_i b_j) \in C' + J_{gr}(S)$ or $d_h d_i f(b_j) = f(d_h d_i b_j) \in C' + J_{gr}(S)$. Since $ker(f) \ll_g M$, by Theorem 2.12(ii), we have $f(J_{gr}(M)) = J_{gr}(S)$. It follows that either $d_g d_h b_j \in f^{-1}(C') + J_{gr}(M)$ or $d_g d_i b_j \in f^{-1}(C') + J_{gr}(M)$ or $d_h d_i b_j \in f^{-1}(C') + J_{gr}(M)$. Therefore, $f^{-1}(C')$ is a graded J_{gr}-classical 2-absorbing submodule of M. ☐

Acknowledgements: The authors wish to thank sincerely the referees for their valuable comments and suggestions.

Conflict of interest: Authors state no conflict of interest.

References

[1] K. Al-Zoubi and R. Abu-Dawwas, *On graded quasi-prime submodules*, Kyungpook Math. J. **55** (2015), no. 2, 259–266, DOI: https://doi.org/10.5666/KMJ.2015.55.2.259.

[2] K. Al-Zoubi and F. Qarqaz, *An intersection condition for graded prime submodules in Gr-multiplication modules*, Math. Reports **20** (2018), no. 3, 329–336.

[3] S. E. Atani, *On graded prime submodules*, Chiang Mai J. Sci. **33** (2006), no. 1, 3–7.

[4] K. H. Oral, U. Tekir, and A. G. Agargun, *On graded prime and primary submodules*, Turk. J. Math. **35** (2011), 159–167, DOI: https://doi.org/10.3906/mat-0904-11.

[5] K. Al-Zoubi and M. Al-Azaizeh, *On graded classical 2-absorbing submodules of graded modules over graded commutative rings*, Rend. Istit. Mat. Univ. Trieste **50** (2018), 37–46, DOI: https://doi.org/10.13137/2464-8728/20578.

[6] A. Y. Darani and S. Motmaen, *Zariski topology on the spectrum of graded classical prime submodules*, Appl. Gen. Topol. **14** (2013), no. 2, 159–169, DOI: https://doi.org/10.4995/agt.2013.1586.

[7] K. Al-Zoubi and M. Jaradat, *The Zariski Topology on the graded classical prime spectrum of a graded module over a graded commutative ring*, Mat. Vesnik **70** (2018), no. 4, 303–313.

[8] K. Al-Zoubi, M. Jaradat, and R. Abu-Dawwas, *On graded classical prime and graded prime submodules*, Bull. Iranian Math. Soc. **41** (2015), no. 1, 217–225.

[9] K. Al-Zoubi and S. Alghueiri, *On graded J_{gr}-classical prime submodules*, submitted.

[10] K. Al-Zoubi, R. Abu-Dawwas, and S. Çeken, *On graded 2-absorbing and graded weakly 2-absorbing ideals*, Hacet. J. Math. Stat. **48** (2019), no. 3, 724–731, DOI: https://doi.org/10.15672/HJMS.2018.543.

[11] S. R. Naghani and H. F. Moghimi, *On graded 2-absorbing and graded weakly 2-absorbing ideals of a commutative ring*, Cankaya Univ. J. Sci. Eng. **13** (2016), no. 2, 11–17.

[12] K. Al-Zoubi and R. Abu-Dawwas, *On graded 2-absorbing and weakly graded 2-absorbing submodules*, J. Math. Sci. Adv. Appl. **28** (2014), 45–60.

[13] K. Al-Zoubi and M. Al-Azaizeh, *Some properties of graded 2-absorbing and graded weakly 2-absorbing submodules*, J. Nonlinear Sci. Appl. **12** (2019), no. 8, 503–508.

[14] K. Al-Zoubi, I. Al-Ayyoub, and M. Al-Dolat, *On graded 2-absorbing compactly packed modules*, Adv. Stud. Contemp. Math. (Kyungshang) **28** (2018), no. 3, 479–486.

[15] R. Hazrat, *Graded Rings and Graded Grothendieck Groups*, Cambridge University Press, Cambridge, 2016.

[16] C. Nastasescu and F. Van Oystaeyen, *Graded and Filtered Rings and Modules*, Lecture Notes in Mathematics, vol. 758, Springer-Verlag, Berlin-New York, 1979.

[17] C. Nastasescu and F. Van Oystaeyen, *Graded Ring Theory*, Mathematical Library 28, North-Holand, Amsterdam, 1982.

[18] C. Nastasescu and F. Van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Mathematics, vol. 1836, Springer-Verlag, Berlin-Heidelberg, 2004.

[19] K. Al-Zoubi and A. Al-Qderat, *Some properties of graded comultiplication modules*, Open Math. **15** (2017), 187–192, DOI: https://doi.org/10.1515/math-2017-0016.

[20] K. Al-Zoubi and S. Alghueiri, *On graded J_{gr}-semiprime submodules*, Ital. J. Pure Appl. Math. (to appear).